MATH327: StatMech and Thermo, Spring 2026

Tutorial — Probabilities

This case study is officially for our 28 January tutorial, but at that point we won't yet have had enough lecture time to introduce all the concepts it involves. We'll spend the 28th completing our definition of probability spaces and introducing the roulette wheel (through page 13 of the lecture notes). At that point you can start working on the first two parts below. In our 29 January lecture we'll cover the central limit theorem, and you can move on to the final two parts. We'll review the model solutions next week, either during or after the first computer lab.

Consider a simple game of roulette in which we just place bets on the colour of the pocket where the ball ends up (assuming a fair wheel). If we bet correctly we get back twice the money we put in; otherwise we lose our money. We define our (potentially negative) *gain* to be the amount we receive minus the amount we spend on bets.

- Suppose we place £5 bets on 'black' for each of N spins of the roulette wheel. What are the probabilities and gains of winning and of losing for any single one of those spins? Letting $W = 0, \cdots, N$ be the number of spins where we win, what is our total gain G_W as a function of (N, W)?

- The number of different ways we could win W times out of N attempts (in any order) is given by the binomial coefficient

  $$\binom{N}{W} = \frac{N!}{W!\,(N-W)!},$$

  with $0! = 1$. Setting $N = 5$, what are the six probabilities p_0 through p_5 that we win $W = 0, \cdots, 5$ times? What is the general p_W for any (N, W)?

- Now let's apply the central limit theorem. What are the mean gain and its variance for a single spin of the wheel? What is the resulting probability distribution $p(g)$ given by the central limit theorem for the gain $g \in \mathbb{R}$ after $N \gg 1$ spins?

- In order to compare the **distribution** $p(g)$ against the probabilities p_W considered above, we need to integrate $p(g)$ over appropriate intervals as discussed in lecture (cf. page 16 of the lecture notes). Natural intervals to consider are

  $$P_{\text{integ}}(G_W) \equiv \int_{G_W-\Delta G/2}^{G_W+\Delta G/2} p(g)\ dg,$$

  where $\Delta G = G_{W+1} - G_W$ is a constant you can read off from your work so far. These numerical integrations are not convenient to do by hand, but can easily be performed by Python, R, MATLAB, etc. Alternatively, we can simplify further by approximating $p(g)$ as a constant within each interval, which would give us

  $$P_{\text{const}}(G_W) \equiv p(G_W)\ \Delta G.$$

  Keeping $N = 5$, what are the six P_{integ} and the six P_{const}? How do they compare to the exact p_W? How does the comparison change as N increases?