MATH327: StatMech and Thermo, Spring 2026

Extra practice — Rotating particles

The classical ideal gases we considered in lectures and tutorials involve ‘point particles’ with no structure or shape. It is also interesting to consider rod-shaped particles such as the diatomic molecules nitrogen (N_2) and oxygen (O_2), which together account for roughly 96–99% of the atmosphere at sea level, depending on humidity. If we set up our coordinate system so that this rod is oriented along the z-axis, then it can rotate around (any linear combination of) the x- and y-axes (with the same moment of inertia I, by symmetry). The energy of each particle is therefore

$$E(\vec{p}, L_x, L_y) = \frac{p^2}{2m} + \frac{L_x^2}{2I} + \frac{L_y^2}{2I}.$$

Here L_x and L_y are the components of angular momentum in these two directions, so that the single-particle partition function is

$$Z_1(T) = \int \exp\left[-E(\vec{p}, L_x, L_y)/T\right]\, d^3p\, d^2L.$$

Explicitly calculate the canonical partition function, internal energy and entropy for such an ideal gas of N indistinguishable diatomic molecules, and compare your result to what you would expect from the concept of equipartition introduced in the model solutions for the first homework assignment.

If we are truly interested in real, physical N_2 and O_2 molecules, then we should be a bit more careful about quantum mechanics, which discretizes the allowed energies as $E_J = J(J+1)\frac{\hbar^2}{2I}$ with $J = 0, 1, 2, \cdots$. In this ‘rigid rotor’ expression, there are $2J+1$ degenerate energy levels for each value of J — a single ground state with $E_0 = 0$, then three energy levels with $E_1 = 2\epsilon$ and five with $E_2 = 6\epsilon$, where $\epsilon \equiv \hbar^2/(2I)$. For high temperatures (relative to $\hbar$), we can employ classical statistics and analyse the single-particle partition function

$$Z_1(T) \approx \int (2J+1) \exp\left[-p^2/(2mT)\right] \exp\left[-J(J+1)\hbar^2/(2IT)\right]\, d^3p\, dJ,$$

where we approximate the sum over $J \geq 0$ by an integral.

Recompute the N-particle canonical partition function, internal energy and entropy from this starting point and compare the results to those you found above.