MATH327: StatMech and Thermo, Spring 2026

Tutorial — Phonons and electrons

This case study will be introduced in our 22 April tutorial, and you'll have the week until our next tutorial on 29 April to work on it. It improves upon Einstein's simple model of a solid, based on non-interacting oscillators with quantized energies, which predicts the heat capacity

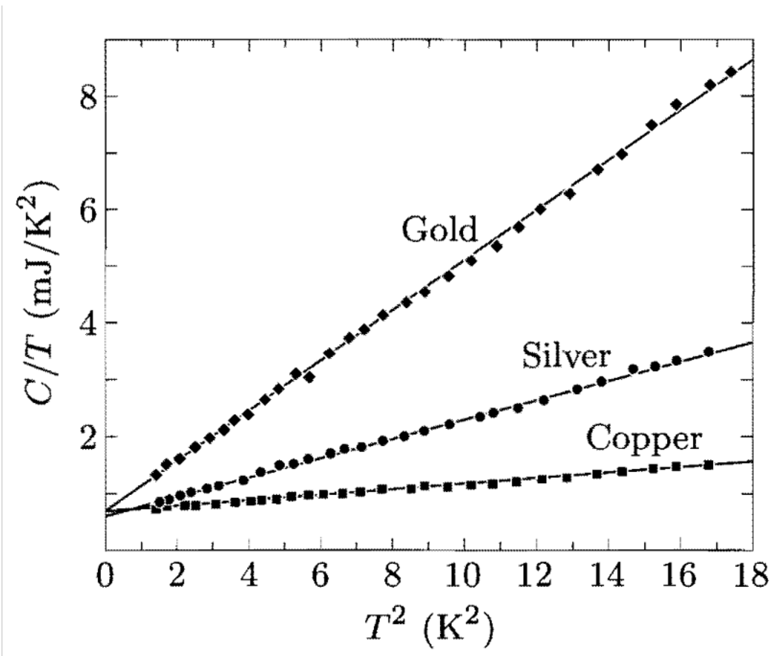
$$c_v = N\left(\frac{\hbar\omega}{T}\right)^2 \frac{e^{\hbar\omega/T}}{\left(e^{\hbar\omega/T}-1\right)^2} = \frac{Nx^2e^x}{\left(e^x-1\right)^2},$$

with $x \equiv \beta\hbar\omega = \hbar\omega/T$. As $T \to 0$, this vanishes exponentially rapidly,

$$\frac{c_v}{N} \approx \frac{\hbar^2\omega^2}{T^2 e^{\hbar\omega/T}} \qquad \text{for} \qquad T \ll \hbar\omega.$$

While the asymptotic $\lim_{T\to 0} c_v = 0$ is correct, the heat capacities of real materials vanish polynomially in T, as opposed to this exponential dependence. This is demonstrated by the figure below (from Schroeder's *Introduction to Thermal Physics*), which shows c_v/T varying linearly with T^2 at low temperatures $T \lesssim 4$ K, and (for these three metals) approaching a non-zero constant value at absolute zero:

$$\frac{c_v}{T} = \alpha + \gamma T^2 \qquad \Longrightarrow \qquad c_v = \alpha T + \gamma T^3.$$



These two terms in c_v arise from different physical sources. The γT^3 term appears upon addressing the limitations of the Einstein solid as a model of real materials. Recall that this model treats materials as lattices of atoms that are held in place by oscillators connecting them to their nearest neighbours. If an atom tries to move out of its position, it compresses the oscillator in the direction of its motion. This adds (quantized) energy to the oscillator, which exerts a force to push the atom back into place.

What the Einstein solid neglects is the equal and opposite force the atom exerts on the oscillator, which the oscillator passes on to the atom on its other side. This second atom is therefore shoved *out* of place, requiring intervention from the other oscillators it's connected to. Ultimately, we can expect this to result in patterns of correlated motion traveling long distances (relative to the atomic scale) through the solid. Thinking of the stadium wave or plants blowing in the wind can give us a rough mental picture of this behaviour.

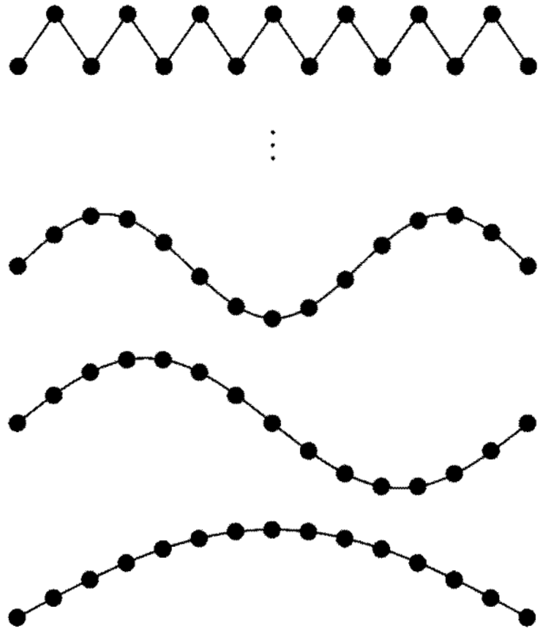
Such a picture makes it clear that the Einstein-solid approach of randomly assigning units of energy to oscillators throughout the material is at best a crude approximation. Remarkably, it is still possible to more realistically model the collective motion of many atoms in terms of non-interacting degrees of freedom. Taking inspiration from having considered photons as quantized electromagnetic waves, we describe the propagating waves of coherent atomic motion in terms of **phonons**.[1] Analysing solids in terms of non-interacting phonons produces the **Debye** model (named after Peter Debye), which provides an important foundation for modern solid-state physics. As an aside, the concept of phonons was only formalized in 1932, about twenty years after Debye introduced this model as a refinement of the Einstein solid. Phonons are an example of a *quasi-particle* — a collective excitation of many degrees of freedom that behaves approximately like a non-interacting particle.

The **first task** is to analyse the high- and low-temperature behaviour of the heat capacity for an ideal gas of phonons. To approach this task, you can revisit the photon gas considered in Section 8.3 of the lecture notes, accounting for these similarities and differences between photons and phonons:

- Both photons and phonons are massless (ultra-relativistic) bosons with chemical potential $\mu \approx 0$.
- Phonons travel at the (material-dependent) speed of sound c_s rather than at the much larger speed of light.
- Phonons have three polarizations compared to the photons' two, but you can feel free to neglect constant factors of this sort and consider only the functional form of the heat capacity at high and low temperatures.
- Most significantly, phonons possess a minimum wavelength set by the distance between the atoms in the solid, as illustrated by the figure below (also from Schroeder's *Introduction to Thermal Physics*). This corresponds to a maximum frequency, $\omega_{\text{max}} \propto \sqrt[3]{N}$ for N atoms in three dimensions.
- While there is also a minimum frequency set by the size of the solid, even tiny solids are so large compared to the atomic scale that this minimum frequency can be set to zero.

[1] The word 'phonon' (like 'telephone') is based on the Greek term $\phi\omega\nu o\varsigma$ ("phonos"), meaning "sound" — the behaviour we have described is essentially how acoustic waves carry sound.

In the end, you should find the same $T \to \infty$ limit as for the Einstein solid, while for low temperatures you should find $c_v \propto T^3 \to 0$, correcting the Einstein solid's exponential T-dependence.



We still need to explain the origin of the linear αT term in the experimental data shown above. To do so, we can note that this is most relevant at very low temperatures $\frac{T}{T_D} \ll 1 \implies \left(\frac{T}{T_D}\right)^3 \ll \frac{T}{T_D}$. ($T_D$ is the *Debye temperature*, and although it is important in solid-state physics, here we are just using it as an unspecified reference scale in order to consider dimensionless ratios.) At these low temperatures, there is not enough thermal energy for any phonons to form (or oscillators to oscillate) — the lattice of atoms is effectively frozen. But ideal gases of non-relativistic fermions retain non-zero energy even as $T \to 0$. This provides a hint about what's going on: The linear heat capacity at very low temperatures comes not from the atoms in the solid, but from a low-temperature gas of electrons.

The **second task** is to determine the low-temperature behaviour of the heat capacity predicted by an ideal, non-relativistic electron gas. This requires going beyond the approximation of the Fermi function $F(E)$ as a step function in Section 8.5 of the lecture notes. Instead, integrate $\langle E \rangle_\text{f} \propto \int_0^\infty F(E) E^{3/2} \, dE$ by parts, then show that the boundary term vanishes while the remaining integrand $\propto E^{5/2}$ is sharply peaked around $E \approx \mu$. Expanding that $E^{5/2}$ factor in a Taylor series around $E = \mu$ produces simpler integrals — by evaluating them and taking the derivative $\frac{\partial}{\partial T} \langle E \rangle_\text{f}$ you should find $c_v \propto T$ at leading order. Again feel free to neglect constant factors and focus on the functional form.