MATH327: StatMech and Thermo, Spring 2026

Extra practice — Particle number fluctuations

Consider the fugacity expansion of the grand-canonical partition function (Eq. 82 in the lecture notes):

$$Z_g(T, \mu) = \sum_{N=0}^{\infty} \xi^N \, Z_N(T).$$

Here $\xi = e^{\beta\mu} = e^{\mu/T}$ is the fugacity and $Z_N(T)$ is the N-particle canonical partition function, which is independent of ξ. Recall that $\Phi(T, \mu) = -T \log Z_g(T, \mu)$ is the corresponding grand-canonical potential.

(a) Derive a relation between the average particle number $\langle N \rangle$ and the derivative $\frac{\partial}{\partial \log \xi} \Phi$.

(b) Derive a relation between $\left\langle (N - \langle N \rangle)^2 \right\rangle$ and $\left(\frac{\partial}{\partial \log \xi} \right)^2 \Phi$.

(c) Specialize to the case of Maxwell–Boltzmann statistics, for which the fugacity expansion simplifies to $Z_g^{\mathsf{MB}}(T, \mu) = \exp[\xi Z_1(T)]$, where Z_1 is the single-particle partition function. Use the relations you have derived to determine $\langle N \rangle$ and $\left\langle (N - \langle N \rangle)^2 \right\rangle$ for this case, and compute

$$\frac{\sqrt{\left\langle (N - \langle N \rangle)^2 \right\rangle}}{\langle N \rangle}.$$

It may be useful to note $\frac{\partial}{\partial \log \xi} = \xi \frac{\partial}{\partial \xi}$.

The final result should indicate that the relative fluctuations in the particle number become negligible for sufficiently large $\langle N \rangle$. In other words, when $\langle N \rangle$ is large it can be treated as approximately constant, making the canonical and grand-canonical ensembles effectively equivalent. This is a general result, and not specific to the case of Maxwell–Boltzmann statistics.