MATH327: StatMech and Thermo, Spring 2026

Second homework assignment

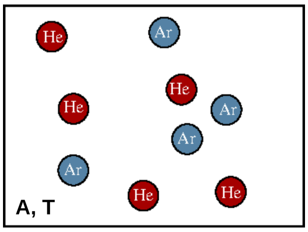
Instructions

Complete all five questions below and submit your solutions by file upload on Canvas. Clear and neat presentations of your workings and the logic behind them will contribute to your mark. Use of resources beyond the module materials must be explicitly referenced in your submissions. This assignment is **due by 17:00 on Friday, 24 April 2026**. Anonymous marking is turned on, model solutions will be posted on Friday, 8 May 2026, and I will aim to return feedback by Monday, 11 May 2026.

You should already be familiar with the Department's academic integrity guidance for 2025–2026, which states that by submitting solutions to this assessment you affirm that you have read and understood the Academic Integrity Policy detailed in Appendix L of the Code of Practice on Assessment, and that you have successfully passed the Academic Integrity Tutorial and Quiz in the course of your studies. You also affirm that the work you are submitting is your own and you have not copied material from another source nor committed plagiarism nor commissioned all or part of the work (including unacceptable proof-reading) nor fabricated, falsified or embellished data when completing your work. You also affirm that you have not colluded with any other student in the preparation and production of your work. You also affirm that any use of generative artificial intelligence (AI) software in relation to the assessment is in accordance with these instructions and the University Guidance. You also affirm that you have acted honestly, ethically and professionally in the preparation and production of your work. Marks achieved on this assessment remain provisional until they are ratified by the Board of Examiners in June 2026.

Question 1: Mixed ideal gases

Consider a mixture of two classical ideal gases in thermodynamic equilibrium, like that illustrated below. The mixture is confined to a two-dimensional surface of area $A = L^2$, with temperature T. Let N_1 and N_2 be the particle numbers of the two gases. Within each gas the particles are indistinguishable, but particles of one gas are distinguishable from particles of the other gas. In particular, they have different masses m_1 and m_2, implying different thermal de Broglie wavelengths λ_1 and λ_2.



(a) First consider the case of fixed particle numbers. Calculate the partition function Z and the Helmholtz free energy of the $(N_1 + N_2)$-particle mixture, approximating $\log(N_i!) \approx N_i \log N_i - N_i$.

[4 marks]

(b) Calculate the internal energy $\langle E \rangle$ and the entropy S of the mixture. Determine the condition of constant entropy.

[4 marks]

(c) Calculate the pressure P of the mixture, which is now the force per unit length on the boundary of the area A, and is given by

$$P = -\left.\frac{\partial \langle E \rangle}{\partial A}\right|_S . \tag{1}$$

Relate your result to the pressures P_1 and P_2 of each gas in isolation (as illustrated below).

[4 marks]

(d) Now allow the particle numbers to fluctuate, with distinct chemical potentials μ_1 and μ_2 for each type of particle. Calculate the grand-canonical partition function Z_g and the grand-canonical potential of the mixture.

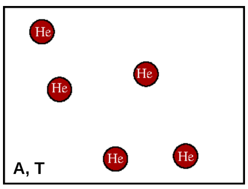
[4 marks]

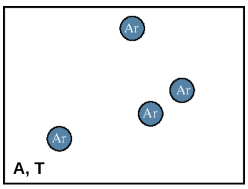
(e) Calculate the average particle number $\langle N \rangle = \langle N_1 \rangle + \langle N_2 \rangle$.

[4 marks]

(f) Re-calculate the internal energy $\langle E \rangle$ now that the particle numbers are allowed to fluctuate — explicitly evaluate the relevant derivatives to ensure you obtain what you expect; don't just write down an answer by inspection.

[4 marks]





Question 2: Atmospheric pressure

Consider a horizontal slab of air with vertical thickness dz. The pressure (force per unit area) $P(z)$ supporting this slab from below must balance $P(z+dz)$ from above plus the weight per unit area of the slab itself. For simplicity, we can treat the air in the slab as N molecules each with an average mass of $m \approx 4.811 \times 10^{-26}$ kg. We can also take the gravitational acceleration $g \approx 9.8$ m s^{-2} to be independent of the altitude z.

(a) Using the ideal gas law, show

$$\frac{\partial P}{\partial z} = f(T)\, P,$$

and obtain an expression for $f(T)$.

Hint: When applying the ideal gas law, you can assume dz is small enough for the pressure within the slab to be approximately constant.

[6 marks]

(b) This equation is easy to solve for $P(z)$ if we assume that the temperature of the air, T, is independent of the altitude z. This is not necessarily a great assumption. To test it, compute the relative atmospheric pressure at the top of Mount Everest (8850 m) compared to Liverpool (70 m), taking $T = 10$℃ and using the unit conversion factor $k_B \approx 1.381 \times 10^{-23}$ J·K^{-1}. The true value is approximately 30.6%.

[4 marks]

Question 3: Photon gas

The grand-canonical potential for a gas of massless bosons confined to a two-dimensional surface of area A, with temperature $T = 1/\beta$ and chemical potential μ, is

$$\Phi(T,\mu) = \frac{AT}{\pi c^2} \int_0^\infty \omega \, \log\left[1 - e^{-(\hbar\omega - \mu)/T}\right] d\omega\,.$$

Consider the case of vanishing chemical potential, $\mu = 0$.

(a) Calculate the internal energy $\langle E \rangle$ and the entropy S.

Hint: You can look up derivatives or integrals, citing any sources you use.

[8 marks]

(b) Calculate $\langle N \rangle$ and show $\langle N \rangle \propto S$.

[6 marks]

(c) Calculate the pressure P, which is again the force per unit length, Eq. 1.

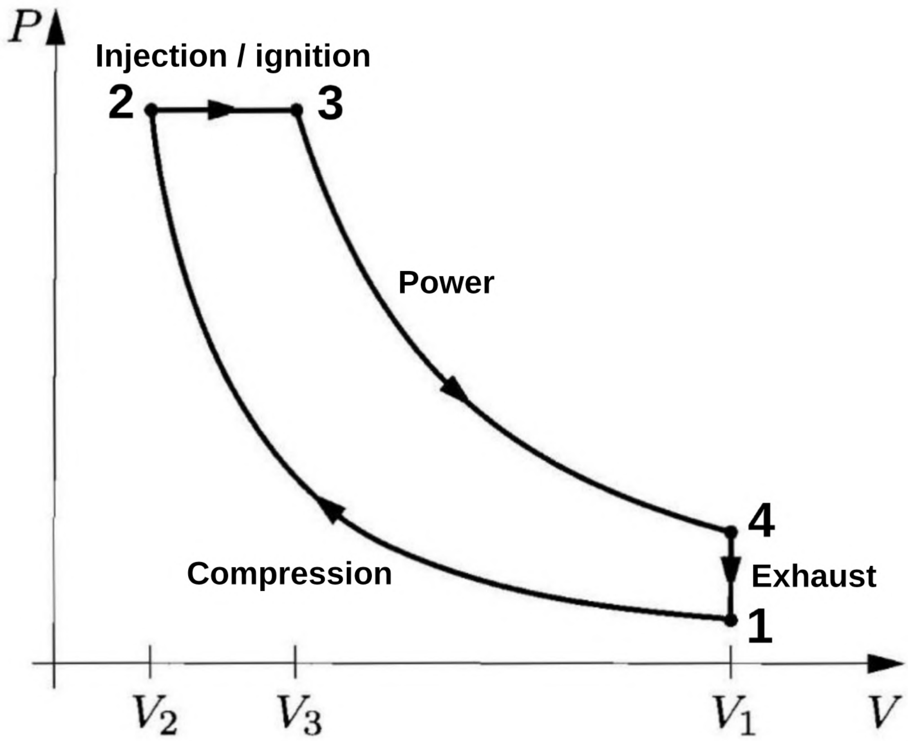
[5 marks]

(d) Show that the equation of state is given by $PA = \kappa \langle N \rangle T$ and calculate the constant κ.

[3 marks]

Question 4: Diesel cycle

Consider the Diesel cycle defined by the PV diagram shown below, in which the 'compression' stage $1 \to 2$ and the 'power' stage $3 \to 4$ are both adiabatic, while the pressure is constant during the 'injection/ignition' stage $2 \to 3$, and the volume is constant during the 'exhaust' stage $4 \to 1$. The compression ratio is $r \equiv V_1/V_2 > 1$ and the cutoff ratio is $C \equiv V_3/V_2 > 1$, with $C < r$.



(a) Compute all four of W_{out}, W_{in}, Q_{out} and Q_{in},

[8 marks]

(b) Show that the Diesel cycle's efficiency is

$$\eta_D = 1 - \frac{f(C)}{r^{2/3}}$$

and determine the function $f(C)$ that depends only on the cutoff ratio.

[6 marks]

(c) Show $f(C) > 1$ for $C > 1$. This indicates the Diesel cycle is less efficient than the Otto cycle with the same compression ratio r.

[6 marks]

(d) With $C = 3$, compute the minimum compression ratio needed in order for the Diesel cycle to be more efficient than the Otto cycle with $r = 8$.

[4 marks]

Question 5: Magnetization

Consider a classical system of N distinguishable, non-interacting 'spins' in a lattice at temperature $T = 1/\beta$, where the value s_n of each spin can vary *continuously* in the range $-1 \leq s_n \leq 1$. In a background magnetic field of strength $H > 0$, the internal energy of the system is $E = -H \sum_{n=1}^{N} s_n$.

(a) By integrating over the continuous s_n, calculate the canonical partition function Z and the Helmholtz free energy F of the system, both as functions of βH.

[6 marks]

(b) The derivative of the Helmholtz free energy with respect to the magnetic field defines the magnetization

$$\langle m \rangle = -\frac{1}{N} \frac{\partial F}{\partial H}.$$

Assuming finite $H > 0$, calculate $\langle m \rangle$ for this system as a function of βH, and determine its low- and high-temperature limits, $\lim_{T \to 0} \langle m \rangle$ and $\lim_{T \to \infty} \langle m \rangle$.

[6 marks]

(c) Calculate the leading T-dependent correction to **each** of the low-temperature and high-temperature limits of $\langle m \rangle$ from the previous part.

[8 marks]