MATH327: StatMech and Thermo, Spring 2026
Extra practice — Higher spin

The spin systems we have focused on in lectures and tutorials involve degrees of freedom that can align either parallel or anti-parallel to a background magnetic field. In nature, this behaviour characterizes "spin-$\frac{1}{2}$" particles, where 'spin' in this context is a quantized amount of intrinsic angular momentum. Electrons and protons are both spin-$\frac{1}{2}$ particles. Other values of spin are also realized in nature, especially in composite atoms and molecules. For example, the Higgs boson has spin 0 while the W and Z bosons[1] have spin 1 and lithium-7 (^{7}Li) nuclei have spin $\frac{3}{2}$.

A particle with spin J has $2J+1$ evenly spaced energy levels in a background field of strength $H > 0$, with $\frac{E}{2} = -JH$, $(-J+1)H$, $(-J+2)H$, $\cdots$, $(J-1)H$, JH. For $J = \frac{1}{2}$ this reduces to the familiar $E = \pm H$ that we analysed in lecture.

Working in the canonical ensemble with fixed N and $\beta = 1/T$, what are the partition function, internal energy and entropy for spin-1 particles? Compare distinguishable vs. indistinguishable particles. Do the same for spin-$\frac{3}{2}$ particles — or, if you are feeling ambitious, derive the expressions for arbitrary J (problem 5.30 in *Statistical and Thermal Physics* by Gould and Tobochnik).

If we were to write $E = -2\vec{S} \cdot \vec{H}$, then the energies considered above would correspond to spin vectors $\vec{S}$ aligned parallel or anti-parallel to the magnetic field, with increasing maximum magnitudes J. We can alternatively consider spins of fixed magnitude, $|\vec{S}| = \frac{1}{2}$, with N evenly spaced *orientations* with respect to the background field. With $E = -H\cos\theta$, the $N = 2$ case corresponds to $\theta = \{0, \pi\}$, while $\theta = \{0, 2\pi/3, 4\pi/3\}$ for $N = 3$ and $\theta = \{0, \frac{2\pi}{N}, 2\frac{2\pi}{N}, 3\frac{2\pi}{N}, \cdots, (N-1)\frac{2\pi}{N}\}$ in general. What are the partition function, internal energy and entropy for these systems, again considering both distinguishable and indistinguishable particles?

As an aside, the second set of systems described above are sometimes called "clock models". In the limit $N \to \infty$, where the spin can rotate continuously with respect to the background field, they produce the non-interacting limit of a system known as the XY model — the full interacting XY model features a famous 'topological' phase transition which was recognized by the 2016 Nobel Prize in Physics. For the first set of systems described above, a similar limit of continuous 'spin' $s \in \mathbb{R}$ can be related to free (non-interacting) scalar quantum field theory — the full interacting system is known as ϕ^4 theory and describes (with some caveats) the Higgs sector of the standard model of particle physics. Both the XY model and ϕ^4 theory can provide fun fourth-year MMath/MPhys projects.

[1] Photons also have spin 1, but a phenomenon known as gauge invariance leaves them with only the two polarizations we discussed before the break, rather than all $2J+1 = 3$ components.