Wed 29 Apr   88 55 18

Plan

Phonon gas $\to$ $c_V \sim T^3$ (Debye model)

Electron gas $\to$ $c_V \sim T$ (Sommerfeld expansion)

$\mu(T)$ for $T \gtrsim E_F$

Lattices

Phonon gas

Non-interacting quasi-particles $\to$ $\mu = 0$ Planck spectrum

$$\langle E \rangle \propto \int_0^{\omega_{max}} \frac{\omega^3}{e^{\beta\hbar\omega}-1}\, d\omega \qquad x = \beta\hbar\omega = \frac{\hbar\omega}{T}$$

$$\propto T^4 \int_0^{T_D/T} \frac{x^3}{e^x - 1}\, dx \qquad \text{Debye temp. } T_D \propto \sqrt[3]{N/V}$$

High T: $\frac{T_D}{T} \ll 1 \to 0 \leq x \ll 1 \to e^x - 1 \approx x$

$$\langle E \rangle \propto T^4 \int_0^{T_D/T} \frac{x^3}{x}\, dx \propto T^4 \left(\frac{T_D}{T}\right)^3 \propto T$$

$$c_V = \frac{\partial}{\partial T}\langle E \rangle = \text{constant} \checkmark$$

same as Einstein solid

Low T: $\frac{T_D}{T} \gg 1$ $\quad \int_0^{T_D/T} \frac{x^3}{e^x - 1}\, dx \approx \Gamma(4)\zeta(4) = \text{const.}$

$$\langle E \rangle \propto T^4 \to c_V \propto T^3 \checkmark$$

Electron gas

$x = \beta(E-\mu)$

$$\frac{\langle E\rangle_f}{g_0} = \int_0^\infty E^{3/2} F(E)\, dE$$

$$\approx \frac{2}{5}\int_{-\infty}^{\infty} \frac{e^x}{(e^x+1)^2} E^{5/2}\, dx$$

$$g_0 = \frac{3}{2}\frac{\langle N\rangle_f}{E_F^{3/2}}$$

$$\approx \frac{2}{5}\mu^{5/2} \cancel{I_0}\to 1 + T\mu^{3/2}\cancel{I_1}\to 0 + \frac{3}{4}T^2\mu^{1/2}\cancel{I_2}\to\frac{\pi^2}{3}$$

$$\langle E\rangle_f \approx \frac{3}{2}\frac{\langle N\rangle_f}{E_F^{3/2}}\left[\frac{2}{5}\mu^{5/2} + \frac{\pi^2}{4}\mu^{1/2}T^2\right]$$

$$\mu^{5/2} \approx E_F^{5/2}\left[1 - \frac{\pi^2 T^2}{8E_F^2}\right]^{5/3} \approx E_F^{5/2}\left[1 - \frac{5\pi^2T^2}{24E_F^2}\right]$$

$$\langle E\rangle_f \approx \frac{3}{5}\langle N\rangle_f E_F - \frac{\pi^2}{8}\frac{\langle N\rangle_f}{E_F}T^2 + \frac{3\pi^2}{8}\frac{\langle N\rangle_f}{E_F}T^2 + O\left(\frac{T^4}{E_F^3}\right)$$

$$\approx \frac{3}{5}\langle N\rangle_f E_F + \frac{\pi^2}{4}\frac{\langle N\rangle_f}{E_F}T^2$$

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$$C_V \approx \frac{\pi^2}{2}\frac{\langle N\rangle_f}{E_F}T \qquad a = \frac{\pi^2}{2(E_F/\langle N\rangle_f)} = \frac{\pi^2}{2\varepsilon_F}$$

What about $T \gtrsim E_F$ where $T \ll E_F$ Sommerfeld expansion unreliable?

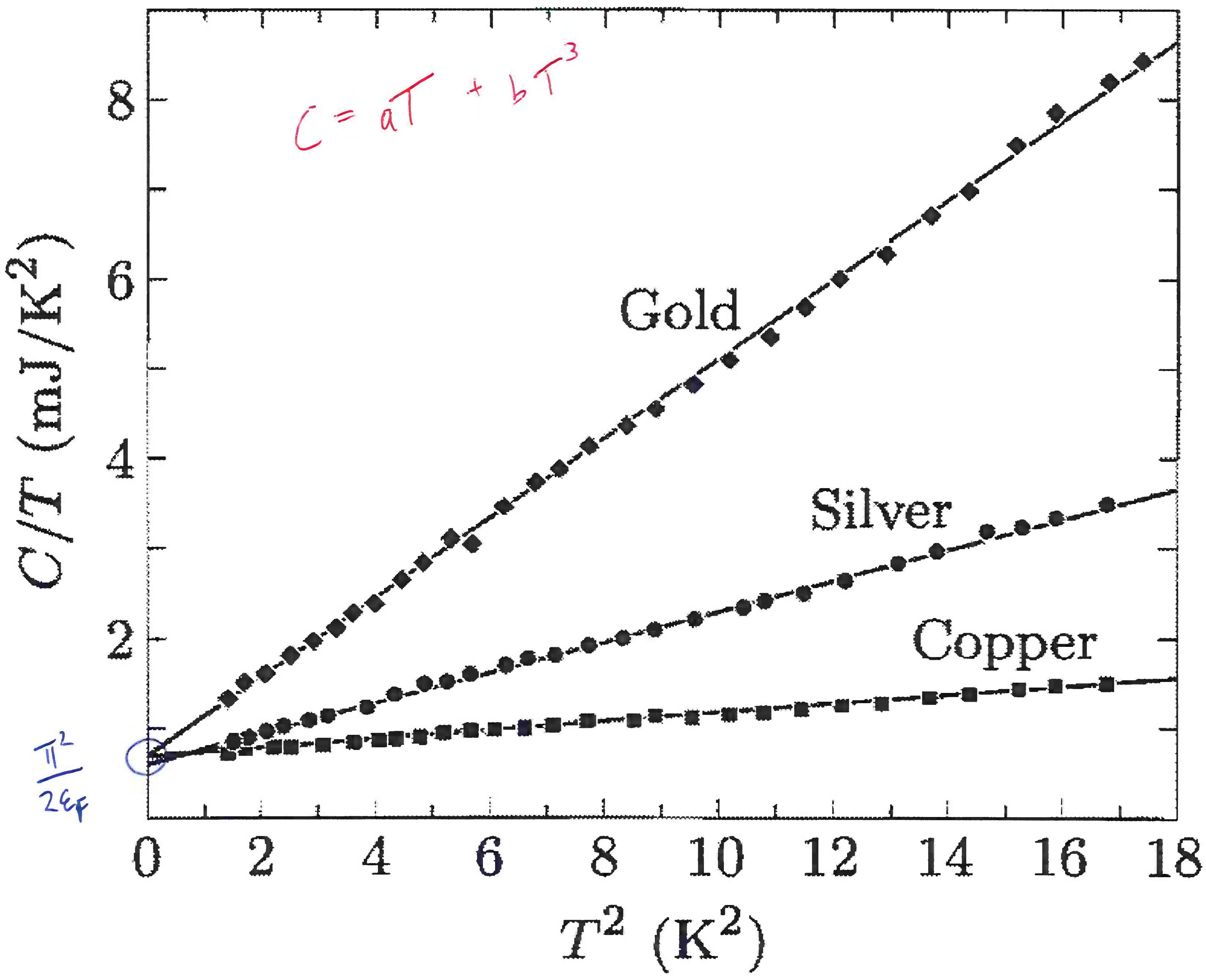
Need numerical solution of

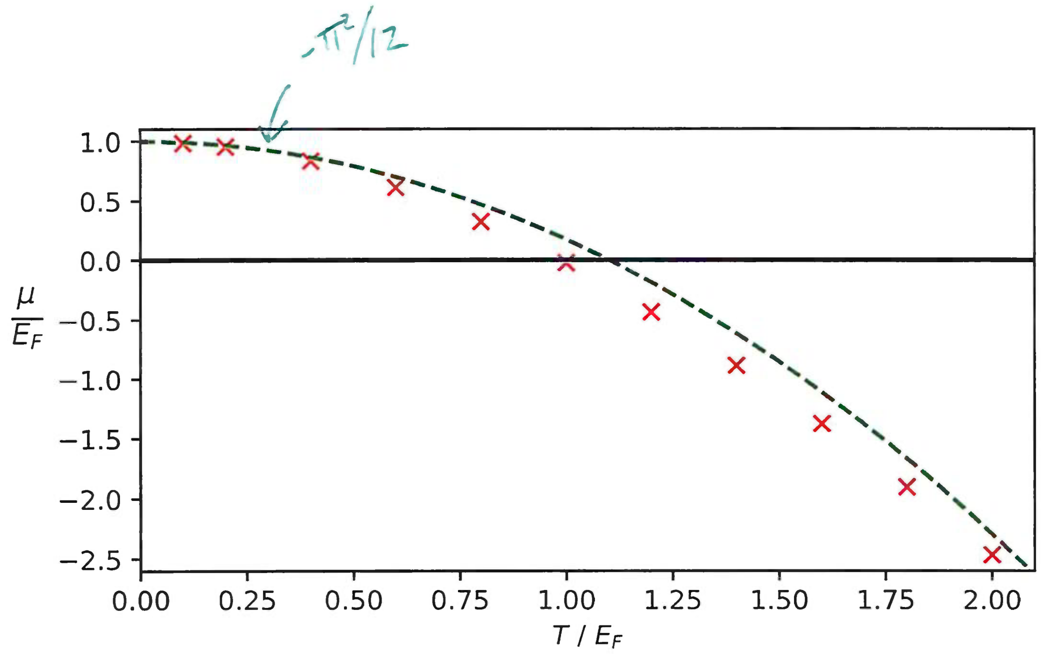
$$\cancel{\langle N\rangle_f} = \frac{3}{2}\frac{\cancel{\langle N\rangle_f}}{E_F^{3/2}}\int_0^\infty \frac{\sqrt{E}}{e^{\beta(E-\mu)}+1}\, dE$$

Use dim'less ratios $x = \frac{E}{T}$ $\quad t = \frac{T}{E_F}$ $\quad c = \frac{\mu}{E_F}$

$$\left(\frac{3}{2}t^{3/2}\int_0^\infty \frac{\sqrt{x}\,e^{-x}}{e^{-c/t} + e^{-x}}\, dx\right) - 1 = 0$$

Larger T makes μ more negative, approach classical $-\mu \gg T \gg E_F$


$C = aT + bT^3$
Gold
Silver
Copper
$\frac{\pi^2}{2\epsilon_F}$
C/T (mJ/K^2)
T^2 (K^2)
0 2 4 6 8 10 12 14 16 18
2 4 6 8



-π²/12
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
μ / E_F
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
T / E_F

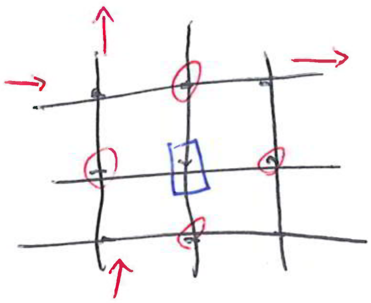
Ising model $E = -J \sum_{(jk)} s_j s_k - H \sum_n s_n$

n.n. pairs defined by lattice structure

1d lattice: 

$c=2$ n.n. per spin

"coordination number"

2d square: 

$c=4$

$\rightarrow 2d$ for d-dim'l simple cubic

Avoid edges with periodic boundary conditions (PBC)

$\rightarrow$ d-dim'l torus

Many important lattice structures

Fully connected lattice — exactly solvable (closed-form Z)

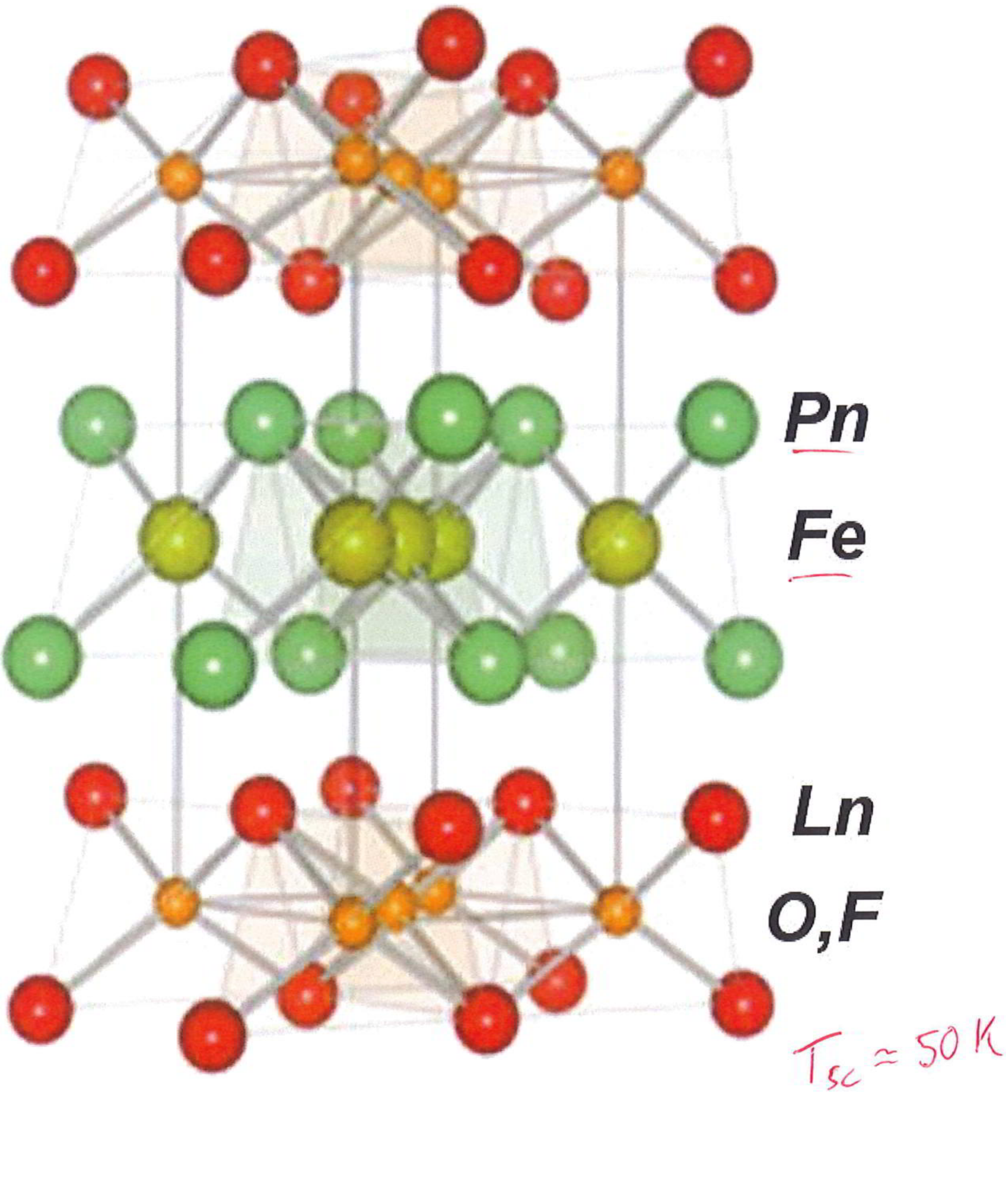
$$E = \frac{-J}{2N} \sum_{j \neq k} s_j s_k - H \sum_n s_n = f(m)$$

Easier to approximate

Organize $\sum_i \rightarrow$ sum over magnetizations

$$Z = \sum_{m=-1}^{1} (\cdots) \rightarrow \int_{-1}^{1} (\cdots)\, dm$$

$$\rightarrow \int_{-\infty}^{\infty} (\cdots)\, dm$$


Pn
Fe
Ln
O,F
$T_{sc} \approx 50\,K$

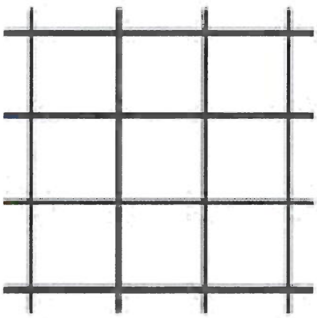
MATH327: StatMech and Thermo, Spring 2026

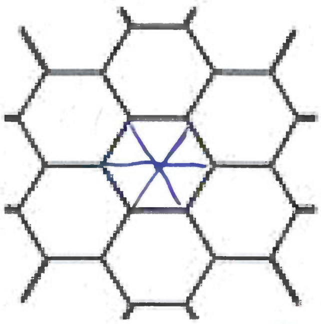
Tutorial — Lattices

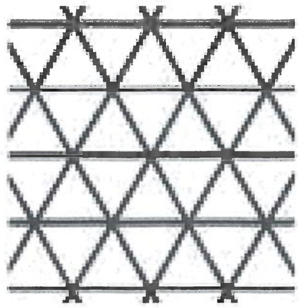
This case study will be introduced in our 29 April tutorial, and you'll have until our final tutorial on 6 May to work on it. Here we explore lattice structures different from the simple d-dimensional cubic lattices with periodic boundary conditions that we will focus on in lectures on the Ising model. These different lattice structures play important roles in both nature and mathematics. Some of the remarkable electronic properties of graphene, for example, are due to its two-dimensional honeycomb lattice structure, while more elaborate three-dimensional lattices are essential in the search for materials exhibiting high-temperature superconductivity.

The figures below show three simple two-dimensional lattices, each of which has a different coordination number C. In all cases we can impose periodic boundary conditions for simplicity. The square lattice has $C = 2d = 4$, and generalizes to simple cubic and hyper-cubic lattices in higher dimensions.

$C=4$
$\to 2d$



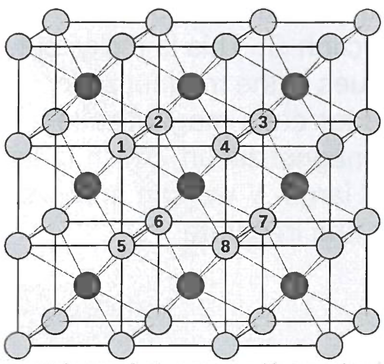




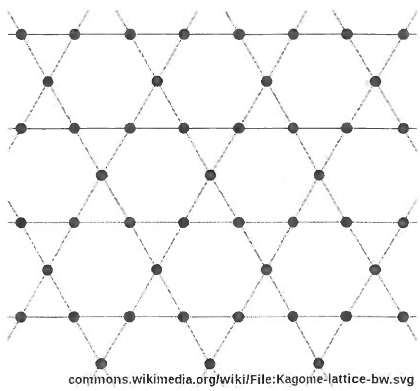
$C=6$
$\to 2(d+1)$

$C=3$
$\to d+1$

The honeycomb lattice of graphene has a smaller $C = d + 1 = 3$, and generalizes to 'hyper-diamond' lattices in higher dimensions. Finally, the triangular lattice effectively fills in the middle of each honeycomb cell, leading to coordination number $C = 2(d + 1) = 6$. Its higher-dimensional generalizations are known as A_d^* lattices, of which the simplest example is the three-dimensional body-centered cubic lattice shown below. Also shown below is the 2d 'kagome' lattice, which has the same $C = 4$ as the square lattice, illustrating that the coordination number may not be sufficient to completely characterize a lattice.



msestudent.com/body-centered-cubic-bcc-unit-cell


commons.wikimedia.org/wiki/File:Kagome-lattice-bw.svg

$C=4$
(only two-dim'l)