MATH327: StatMech and Thermo

Tuesday, 28 April 2026

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Something to consider

The essence of statistical mechanics
is the emergence of macroscopic phenomena
from systems of **many** microscopic degrees of freedom

Will a given collection of particles
always lead to the same emergent phenomena?

Recap

Fermion gases

$T \to 0$ $F(E)$ simplification ~ e^- in metals

white dwarfs → SN Ia

Ultra-rel. (neutrino) gas EoS

Low-T Sommerfeld expansion $\frac{T}{E_F} \ll 1$

Plan

$\mu(T)$ for $\frac{T}{E_F} \ll 1$

Interacting systems

Phase transitions

Ising models

Lattices

Recall $\dfrac{\langle N\rangle_F}{g_0} = \dfrac{2}{3}\int_{\cancel{-\beta\mu}\to -\infty}^{\infty} \dfrac{e^x}{(e^x+1)^2} E^{3/2}\, dx \qquad x = \beta(E-\mu)$

$g_0 = V\dfrac{\sqrt{2m^3}}{\pi^2\hbar^3} \qquad E^{3/2} \approx \mu^{3/2}\left[1+\frac{3}{2}\left(\frac{xT}{\mu}\right)+\frac{3}{8}\left(\frac{xT}{\mu}\right)^2\right]$

$\mu \to E_F$ as $T \to 0$

$$\frac{\langle N\rangle_F}{g_0} \approx \frac{2}{3}\mu^{3/2}\underbrace{I_0}_{1} + T\mu^{1/2}\underbrace{I_1}_{0} + \frac{1}{4}T^2\mu^{-1/2}\underbrace{I_2}_{\pi^2/3}$$

$$I_0 = \int_{-\infty}^{\infty}\frac{e^x}{(e^x+1)^2}\,\underset{d(\beta E)}{dx} = \int_{-\infty}^{\infty}\frac{-dF}{dE}\,dE = -F(E)\Big|_{-\infty}^{\infty} = -0+1=1$$

$$I_1 = \int_{-\infty}^{\infty}\frac{x e^x}{(e^x+1)^2}dx = \int_{-\infty}^{\infty}\frac{x}{(e^x+1)(1+e^{-x})}dx = 0$$

$$I_2 = \int_{-\infty}^{\infty}\frac{x^2 e^x}{(e^x+1)^2}dx = 2\int_0^{\infty}\frac{x^2e^x}{(e^x+1)^2}dx$$

$\int u\,dv = uv - \int v\,du$

$u = x^2$

$dv = \dfrac{e^x}{(e^x+1)^2}dx$

$\Rightarrow v = \dfrac{-1}{e^x+1}$

$$I_2 = \underbrace{\frac{-2x^2}{e^x+1}\Big|_0^{\infty}}_{0} + 2\int_0^{\infty}\frac{2x}{e^x+1}dx$$

$$= 4\underbrace{\left(1-\tfrac{1}{2}\right)}_{2}\underbrace{\Gamma(2)}_{1}\underbrace{\zeta(2)}_{\pi^2/6} = \frac{\pi^2}{3}$$

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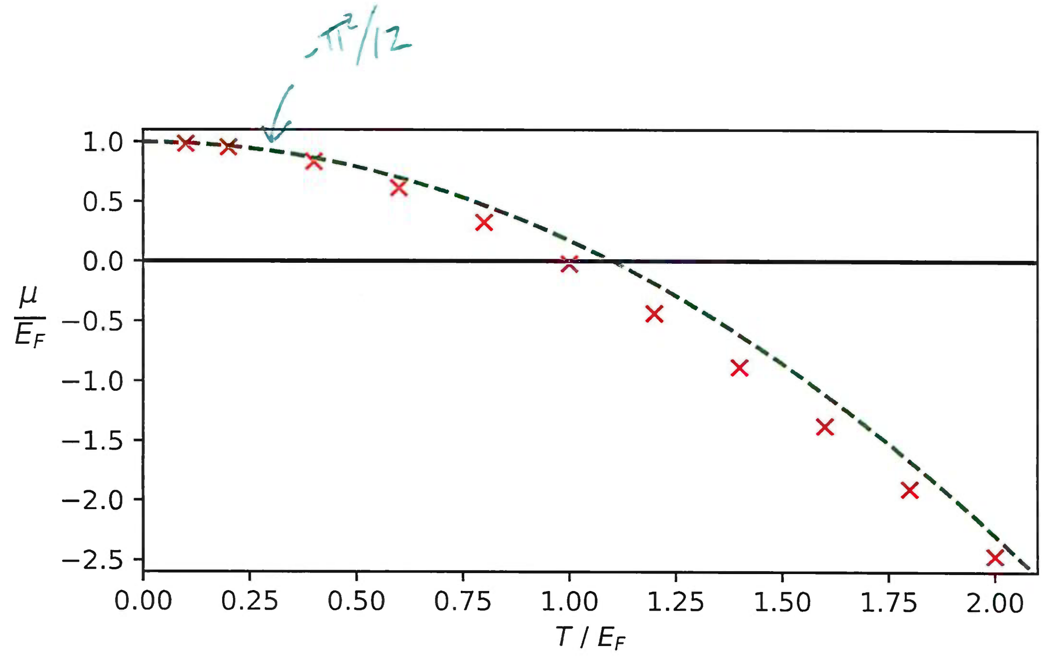
Result: $\langle N\rangle_F \approx g_0\left[\dfrac{2}{3}\mu^{3/2} + \dfrac{\pi^2T^2}{12\mu^{1/2}}\right]$

relate to $E_F = \dfrac{\hbar^2}{2m}(3\pi^2\rho_F)^{2/3}$

$$E_F^{3/2} = \frac{3\pi^2\hbar^3}{2\sqrt{2m^3}}\frac{\langle N\rangle_F}{V}$$

g_0



$-\pi^2/12$
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
$\frac{\mu}{E_F}$
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
T / E_F

$$\cancel{\langle N\rangle_F} \simeq \frac{3}{2}\frac{\cancel{\langle N\rangle_F}}{E_F^{3/2}}\left[\frac{2}{3}\mu^{3/2} + \frac{\pi^2 T^2}{12\,\mu^{1/2}}\right]$$

$$1 \simeq \left(\frac{\mu}{E_F}\right)^{3/2} + \frac{\pi^2 T^2}{8\,E_F^{3/2}\,\mu^{1/2}}$$

$$\frac{\mu}{E_F} \simeq \left[1 - \frac{\pi^2 T^2}{8\,E_F^{3/2}\,\mu^{1/2}}\right]^{2/3} \simeq 1 - \frac{\pi^2}{12}\left(\frac{T}{E_F}\right)^2$$

$E_F \simeq \mu$ for low T

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$\therefore$ $T > 0 \rightarrow \mu < E_F$, quadratic approach to $T \rightarrow 0$ limit

Next, $\langle E\rangle_F \rightarrow$ tutorial

Turn to interacting systems

Much harder to analyse but needed for phenomena like phase transitions

Phases are different emergent behaviour from same particles

H_2O

Early Universe

Superconductivity

What makes a system interacting?

Consider N spins in d-dim'l lattice (dist'able) $\quad T = 1/\beta$

H $\quad \uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow\downarrow$ $\quad s_1 = 1$, $s_8 = -1$

H
$$\begin{array}{ccc}\uparrow & \uparrow & \uparrow \\ \uparrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \downarrow\end{array}$$
$s_{1,3} = 1$, $s_{3,3} = -1$

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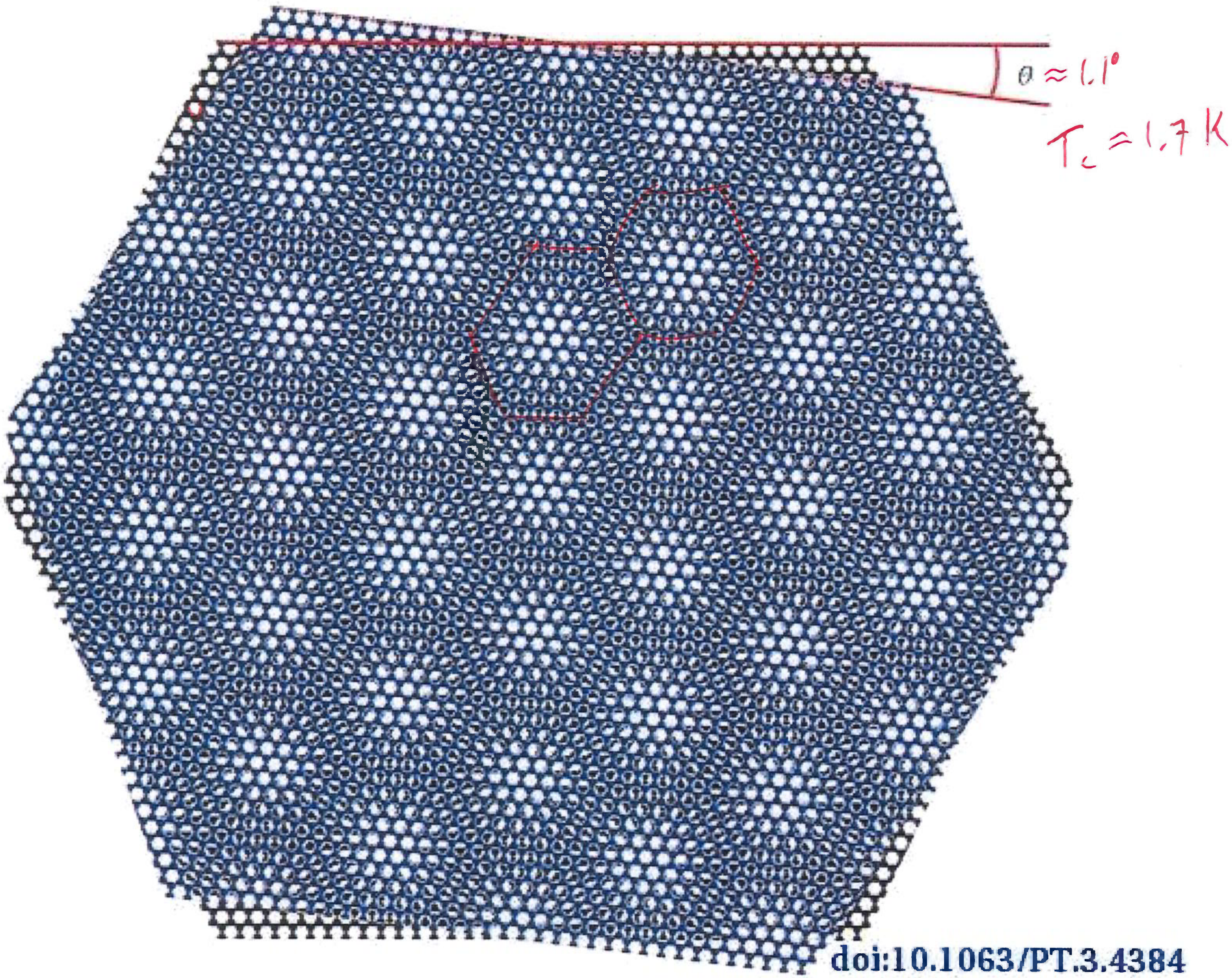
Micro-states w_i defined by $\{s_n\}$, $s_n = \pm 1$



doi:10.1063/PT.3.4384

Non-interacting $E_i = -H \sum_{n=1}^{N} s_n = \sum_n \varepsilon_n$ (factorization)

$\rightarrow$ very simple $Z_N = Z_1^N = \left(2\cosh(\beta H)\right)^N$

More interesting $E_i = -\sum_{(jk)} s_j s_k - H \sum_n s_n$

all pairs of nearest-neighbour (n.n.) spins

Definition:

Consider change ΔE_j from alteration to jth particle

Non-interacting iff. ΔE_j independent of all particles $k \neq j$

Example: Flip spin $s_j \rightarrow -s_j$

Simple $E = -H\left(s_j + \sum_{k\neq j} s_k\right)$

$\rightarrow \Delta E_j = 2 H s_j$ independent of s_k $k \neq j$

$\therefore$ non-interacting ✓

Interesting $E = -s_j \sum_{k\in(jk)} s_k - \sum_{(mk)\not\ni j} s_m s_k - H \sum_n s_n$

$\Delta E_j = 2 s_j \left(H + \sum_{k\in(jk)} s_k\right)$

page 129 Depends on s_k with $k \neq j$ $\rightarrow$ interacting ✓

Ising model