MATH327: StatMech and Thermo

Thursday, 23 April 2026

72 32 41

Something to consider

Pauli exclusion produces unique behaviour (including $\lim_{T\to 0}\langle E\rangle_{\mathsf{f}} > 0$) for low-temperature ideal gases of non-relativistic fermions

What physical systems could we expect to exhibit this behaviour?

Recap

Non-rel. fermion gas

$T \to 0$ Fermi energy $E_F = \mu \propto \rho_f^{2/3}$

$$\langle E\rangle_f = \frac{3}{5}\mu\langle N\rangle_f > 0$$

Plan

Confirm $\mu > 0$ for low T

$T \to 0$ pressure

physical examples

Ultra-rel. fermion gas

Start low-T Sommerfield expansion

$$\mu = \left.\frac{\partial E}{\partial N}\right|_{S,V}$$

For $T \to 0$ single micro-state $\qquad S = -\sum_i p_i \log p_i = -\log(1) = 0$

S unchanged for $\Delta N > 0$

Adding particles fills next energy levels $\sim E_F > 0$

$\therefore \Delta E \approx E_F(\Delta N) > 0 \rightarrow \mu > 0$

Next week will ~~con~~ check $\mu < 0$ for higher $T \gtrsim E_F$

(consistent classical limit)

$-\mu \gg T \gg E_F$

Pressure $P_f = -\frac{\partial}{\partial V} \langle E \rangle_F \Big|_{S_f} = -\frac{3}{5}\frac{\partial}{\partial V}\left(\mu \langle N \rangle_F\right)\Big|_{S_f}$

Single $T \to 0$ micro-state $\rightarrow S_f = \text{const.} = 0$

$\mu = E_F$

$$\boxed{\begin{aligned} P_f &= -\frac{3}{5}\frac{\partial}{\partial V}\left(\frac{\hbar^2}{2m}\left(3\pi^2 \frac{\langle N \rangle_F}{V}\right)^{2/3} \langle N \rangle_F\right) \\ &= +\frac{2}{3}\frac{\langle E \rangle_F}{V} = \frac{2}{5}\mu \frac{\langle N \rangle_F}{V} \\ &= \frac{\hbar^2}{5m}\left(3\pi^2\right)^{2/3} \rho_f^{5/3} \end{aligned}}$$

page 116

New feature: $P_f \neq 0$ as $T \to 0$

"Degeneracy pressure" from Pauli exclusion

unrelated to degenerate energy levels

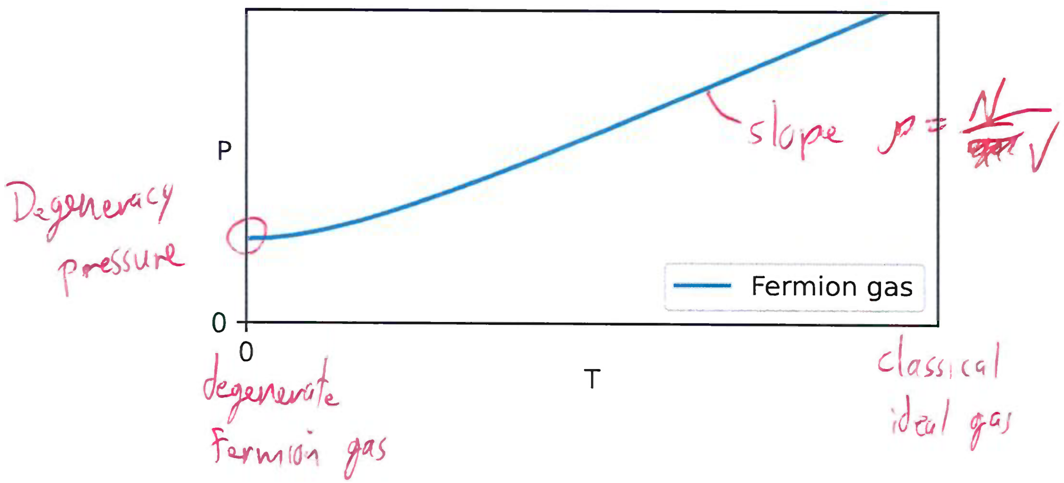
High-T classical limit: $P = \frac{N}{V} T = \rho T$

Degeneracy pressure matters for $T \ll E_F \propto \rho^{2/3}$

$\rightarrow$ low T, high density or both

Metals have $\rho \sim \frac{N_A \text{ electrons}}{\text{c.c.}} \sim 10^{23}(10^2)^3$ electrons$/m^3$

10^{29}


Degeneracy pressure
P
0
0
T
Fermion gas
slope $p = \frac{N}{V}$
degenerate Fermion gas
classical ideal gas

$E_F \sim 10^4 K \sim 1\,eV \sim 10^{-19} J$

Room temperature $T \ll E_F = 10^4 K$ → degenerate electrons gas

Sun (on average) has similar $\rho \sim 10^{30}$ electrons/m^3 → $E_F \sim 10^5 K$

Core $T \sim 10^7 K \gg E_F$

Fusion of hydrogen and helium heats sun

Radiation pressure balance force of gravity, reducing ρ

After H and He "fuel" used up, less radiation pressure

→ higher ρ

→ White dwarf stars

Sun's mass with earth's radius ~ 100x smaller

$E_F \sim (10^6)^{2/3} E_F^{sun} \sim 10^4 \cdot 10^5 K \sim 10^9 K \gg T \lessapprox 10^7 K$

(Slowly cool to ~10^3 K after ~10^9 years)

∴ Degenerate electron gas pressure prevents further collapse

Binary ~~st~~ system

White dwarf captures matter from companion star

increases mass and density

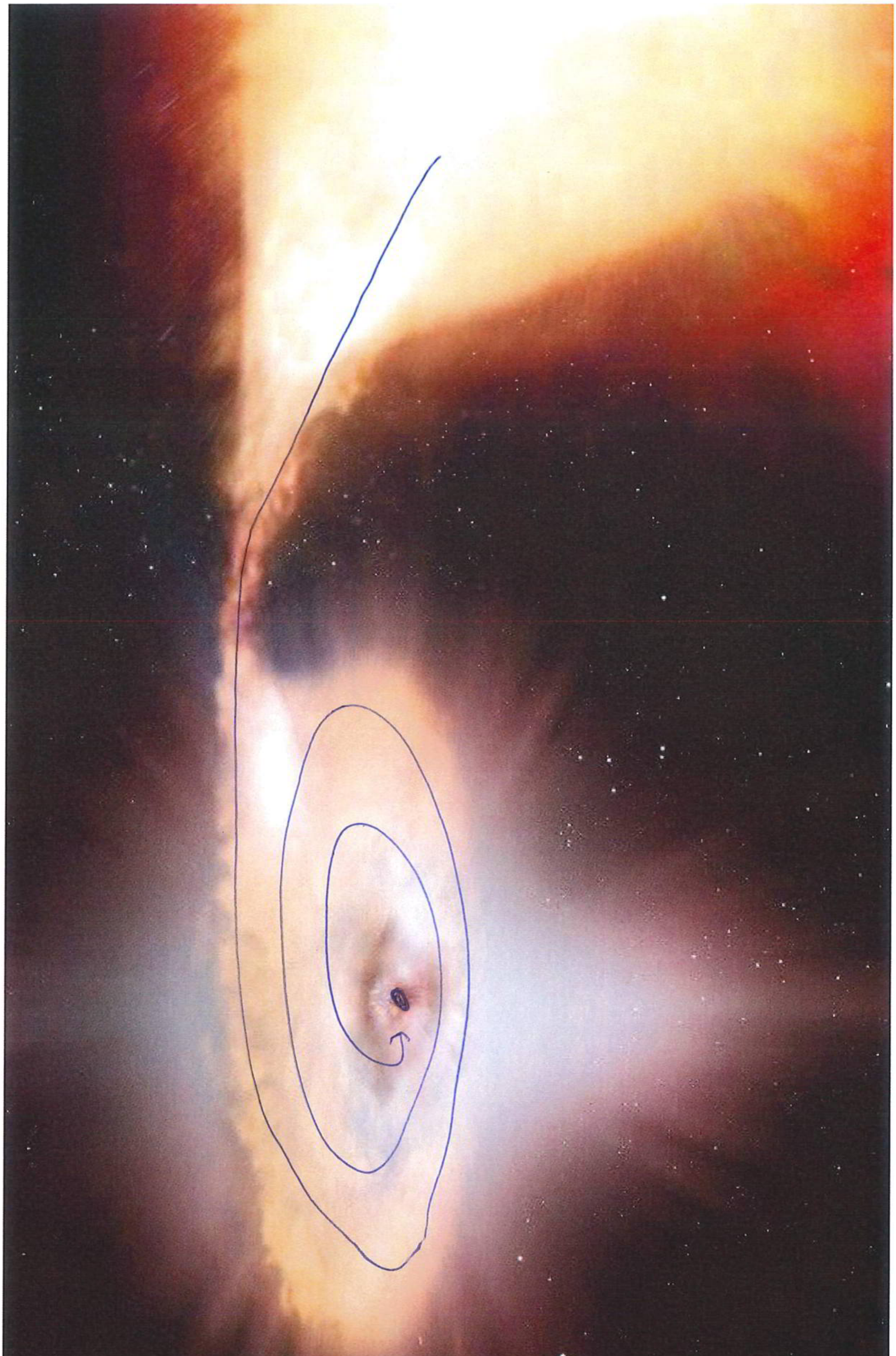
Chandrasekhar limit $M \sim 1.4\, M_{sun}$ → carbon and oxygen fusion

Chain reaction → $T \sim 10^9 K$ in seconds

Radiation pressure → $\underbrace{\text{super~~nov~~nova}}_{\text{~5 billion times brighter than sun}}$ ("type Ia")

Regu~~m~~larity → "standard candle" measuring distance vs. time

→ accelerating expansion of Universe (2011 Nobel)



Ultra-rel. Fermion gas, briefly

Example: Neutrinos $\quad 0 < m_\nu \lesssim 10^{-37}\,\mathrm{kg} \sim \frac{m_e}{10^6}$

$\mu \simeq 0 \qquad E_\nu \simeq p_\nu c = \hbar\omega \qquad \omega \simeq \frac{2\pi c}{\lambda} = c\frac{\pi}{L}k \qquad k_{x,y,z} = 1,2,3,\ldots$

Just change signs vs. photon gas (2 spins vs. 2 pol.)

$$\Phi_\nu = -\frac{VT}{\pi^2 c^3}\int_0^\infty \omega^2 \log\left(1 + e^{-\beta\hbar\omega}\right) d\omega$$

$$\langle N\rangle_\nu = -\frac{\partial}{\partial\mu}\Phi_\nu\Big|_{\mu=0} = \frac{V\cancel{T}}{\pi^2 c^3}\int_0^\infty \frac{\omega^2\left(e^{-\beta\hbar\omega}e^{\beta\mu}\right)\cancel{\beta}}{1 + e^{-\beta\hbar\omega}e^{\beta\mu}}\,d\omega\Big|_{\mu=0}$$

$x = \beta\hbar\omega = \frac{\hbar\omega}{T}$

$$= \frac{V}{\pi^2 c^3}\left(\frac{T}{\hbar}\right)^3 \int_0^\infty \frac{x^2}{e^x+1}\,dx$$

$\left(1 - \frac{1}{2^2}\right)\Gamma(3)\zeta(3) = \frac{3}{2}\zeta(3)$

$$\langle N\rangle_\nu = \frac{3\zeta(3)}{2\pi^2\hbar^3 c^3} VT^3$$

Add factor of $\epsilon = \hbar\omega = xT$ for internal energy

$$\langle E\rangle_\nu = \frac{VT^4}{\pi^2\hbar^3 c^3}\int_0^\infty \frac{x^3}{e^x+1}\,dx \propto \langle N\rangle_\nu T$$

$\left(1 - \frac{1}{2^3}\right)\Gamma(4)\zeta(4) = \frac{7}{8}\,6\cdot\frac{\pi^4}{90}$

$$\langle E\rangle_\nu = \frac{(7/8)\cancel{\Gamma(4)}\zeta(4)}{(3/4)\cancel{\Gamma(3)}\zeta(3)}\langle N\rangle_\nu T = \frac{7}{2}\frac{\zeta(4)}{\zeta(3)}\langle N\rangle_\nu T = \frac{7\pi^4}{180\,\zeta(3)}\langle N\rangle_\nu T$$

~ 3.151

page 120

Equation of state involves pressure $P_\nu = -\frac{\partial}{\partial V}\langle E\rangle_\nu\Big|_{S_\nu}$

$$S_\nu = \frac{\langle E\rangle_\nu - \Phi_\nu}{T} \propto VT^3 \quad \text{constant if } T = b\,V^{-1/3}$$

$$\boxed{P_\nu = -\frac{7}{8}\left(\frac{\Gamma(4)\zeta(4)}{\pi^2\hbar^3c^3}\right)\frac{\partial}{\partial V}\left(b^4 V^{-1/3}\right) = \frac{1}{3V}\langle E\rangle_\nu}$$

page 121

$$\text{So}\quad P_\nu V = \frac{1}{3}\langle E\rangle_\nu = \frac{7}{6}\frac{\zeta(4)}{\zeta(3)}\langle N\rangle_\nu T = \underbrace{\frac{7\pi^4}{540\,\zeta(3)}}_{\sim 1.05}\langle N\rangle_\nu T$$

Compared to photon gas just extra factor of $\frac{7}{6} = \frac{1-1/2^3}{1-1/2^2}$

$T>0$ non-rel. fermion gas with full $F(E)$ $[\mu(T),\ c_v]$

Better notation:

$$\langle N\rangle_F = \int_0^\infty g(E)\,F(E)\,dE \qquad \langle E\rangle_F = \int_0^\infty E\,g(E)\,F(E)\,dE$$

"density of states" $g(E) = g_0\sqrt{E} = V\frac{\sqrt{2m^3}}{\pi^2\hbar^3}\sqrt{E}$

Counting energy levels per unit energy

$F(E) = \frac{1}{e^{\beta(E-\mu)}+1}$ is occupation prob.

$T>0 \rightarrow$ exp. suppressed occupation for $E>E_F$

$\therefore$ unoccupied energy levels with $E<E_F$

Will see $\mu(T)<E_F$ for $T>0$

$$\frac{\langle N\rangle_F}{g_0} = \int_0^\infty E^{1/2}\,F(E)\,dE \qquad \int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$u = F(E)$

page 122

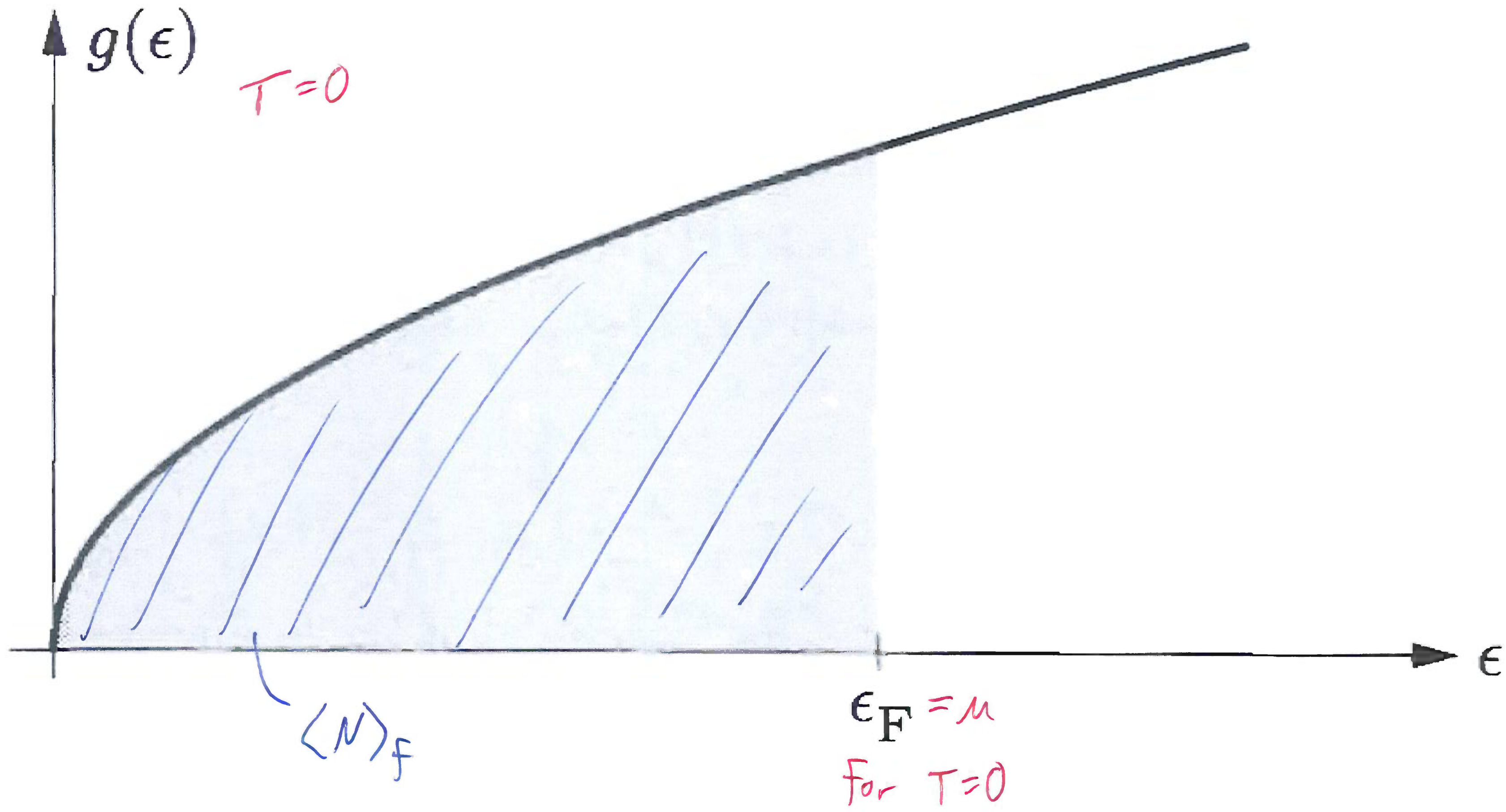
$dv = E^{1/2}dE \rightarrow \frac{2}{3}E^{3/2} = v$

$$\frac{\langle N\rangle_F}{g_0} = \underbrace{\frac{2}{3}E^{3/2}F(E)\Big|_0^\infty}_{0} - \frac{2}{3}\int_0^\infty E^{3/2}\left(\frac{dF}{dE}\right)dE$$

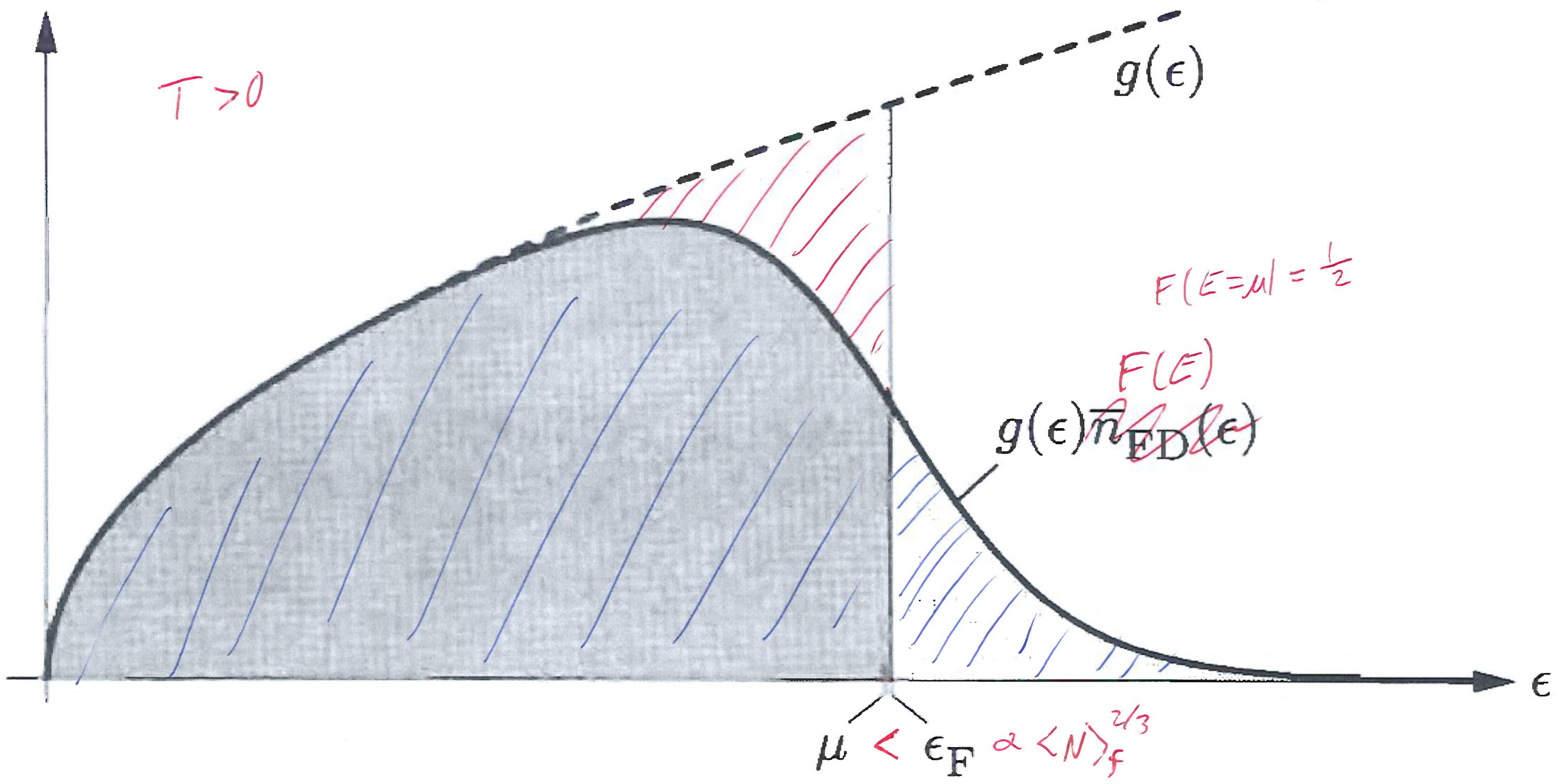
$$\frac{-d}{dE}\left(e^{\beta(E-\mu)}+1\right)^{-1} = \frac{e^{\beta(E-\mu)}\beta}{\left(e^{\beta(E-\mu)}+1\right)^2} = \frac{\beta e^x}{\left(e^x+1\right)^2}$$

$x = \beta(E-\mu)$

$dx = \beta dE$



$g(\epsilon)$
$T=0$
ϵ
$\langle N \rangle_f$
$\epsilon_F = \mu$
for $T=0$



T >0
g(ϵ)
F(E=μ) = 1/2
F(E)
g(ϵ)n̄FD(ϵ)
ϵ
μ < ϵF ∝ ⟨N⟩f^2/3

$$\frac{\langle N\rangle_F}{g_0} = \frac{2}{3}\int_{-\beta\mu}^{\infty} \frac{e^x}{(e^x+1)^2}\, E^{3/2}\, dx$$

depends on x

sharply peaked around $x=0$ $\quad E=\mu$

Peak allows approximations

1) Extend $\int_{-\beta\mu}^{\infty} \rightarrow \int_{-\infty}^{\infty}$ since $\mu>0$ for low T

$\sim$ large β

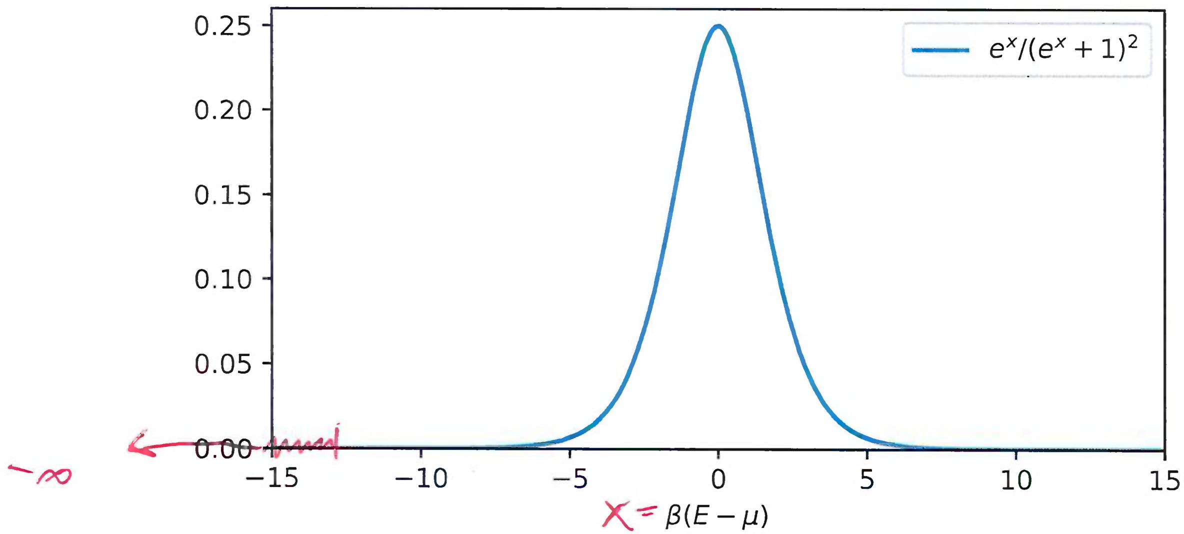
2) Expand
$$E^{3/2} \approx \mu^{3/2} + (E-\mu)\frac{\partial}{\partial E}E^{3/2}\Big|_{E=\mu} + \frac{1}{2}(E-\mu)^2\frac{\partial^2}{\partial E^2}E^{3/2}\Big|_{E=\mu}$$

$$= \mu^{3/2} + \frac{3}{2}(E-\mu)\mu^{1/2} + \frac{3}{8}(E-\mu)^2\mu^{-1/2}$$

$$= \mu^{3/2}\left[1 + \frac{3}{2}\left(\frac{xT}{\mu}\right) + \frac{3}{8}\left(\frac{xT}{\mu}\right)^2\right]$$

Sommerfeld expansion for $\frac{T}{\mu} \ll 1$

$\left(\frac{T}{E_F} \ll 1\right)$



0.25
0.20
0.15
0.10
0.05
0.00
−15
−10
−5
0
5
10
15
$e^x/(e^x+1)^2$
$x = \beta(E-\mu)$
$-\infty$