MATH327: StatMech and Thermo

Tuesday, 21 April 2026

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Something to consider

Moving from gases of photons to gases of fermions,
what effects do you expect Pauli exclusion will have?

For what temperatures and chemical potentials
do you expect these effects to be significant?

Recap

Photon gas Planck spectrum vs. sun and CMB
equation of state $PV \sim \langle N \rangle T$

Non-rel. fermion gas $\Phi_F = -VT\left(\frac{\sqrt{2m^3}}{\pi^2 \hbar^3}\right)\int_0^\infty \log\left(1+e^{-\beta(E-\mu)}\right)\sqrt{E}\,dE$

Plan

$T \to 0$ fermion gas

Fermi energy

Degeneracy pressure

Start with particle density

$$\frac{\langle N \rangle_F}{V} = \frac{-\partial}{\partial \mu}\left(\frac{\Phi_F}{V}\right) = \frac{\sqrt{2m^3}}{\pi^2 \hbar^3}\cancel{T}\int_0^\infty \frac{e^{-\beta(E-\mu)}\cancel{\beta}}{1+e^{-\beta(E-\mu)}}\sqrt{E}\,dE$$

$$= \frac{\sqrt{2m^3}}{\pi^2 \hbar^3}\int_0^\infty \frac{1}{e^{\beta(E-\mu)}+1}\sqrt{E}\,dE$$

$$= \frac{\sqrt{2m^3}}{\pi^2 \hbar^3}\int_0^\infty F(E)\sqrt{E}\,dE$$

$$F(E) = \frac{1}{e^{\beta(E-\mu)}+1} \sim \langle n_\ell \rangle_{FD}$$ is <u>Fermi function</u>

Assume $\mu > 0$

Threshold at $F(E=\mu) = \frac{1}{2}$ for all T

$E > \mu \to$ exponentially suppressed $F(E) \to 0$

$E < \mu \to$ exponentially approach $F(E) \to 1$

Lower $T \to$ larger $\beta \to$ faster approach to limits

Simplification: Approximate $F(E)$ as a step func.

$$F(E) \approx \begin{cases} 1 & 0 \le E < \mu \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\langle N \rangle_F}{V} = \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\infty F(E)\sqrt{E}\, dE \to \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \underbrace{\int_0^\mu \sqrt{E}\, dE}_{\frac{2}{3}\mu^{3/2}}$$

$$= \frac{(2m\mu)^{3/2}}{3\pi^2 \hbar^3}$$

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Leading order of expansion in powers of $\frac{T}{\mu} \ll 1$

(Sommerfeld)

Physical picture: All energy levels with $E_\ell \le \mu$ occupied $n_\ell = 1$

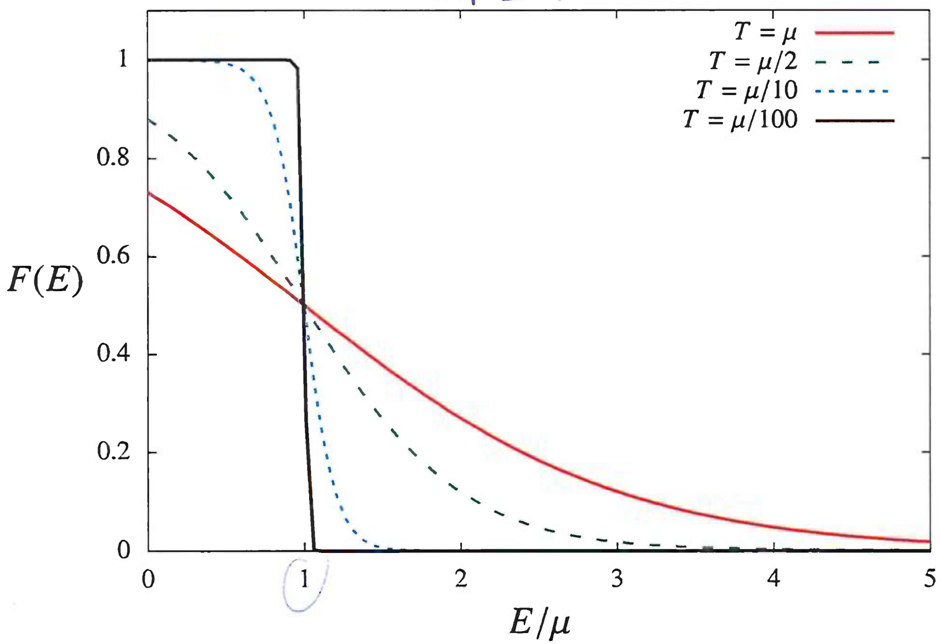
$E_\ell \propto k^2 \to \langle N \rangle_F$ fill octant of sphere with radius $\sqrt{\mu}$

$\propto \mu^{3/2}$

For $T \to 0$, max energy is <u>Fermi energy</u>

$$E_F = \mu = \frac{1}{2m}\left(3\pi^2 \hbar^3 \frac{\langle N \rangle_F}{V}\right)^{2/3} = \frac{\hbar^2}{2m}\left(3\pi^2 \rho_F\right)^{2/3}$$

$$F(E) = \frac{1}{\exp\left[\frac{\mu}{T}\left(\frac{E}{\mu}-1\right)\right]+1} = \left(\left[\exp\left(\frac{E}{\mu}-1\right)\right]^{\mu/T}+1\right)^{-1}$$



$\frac{\mu}{T} = 1$

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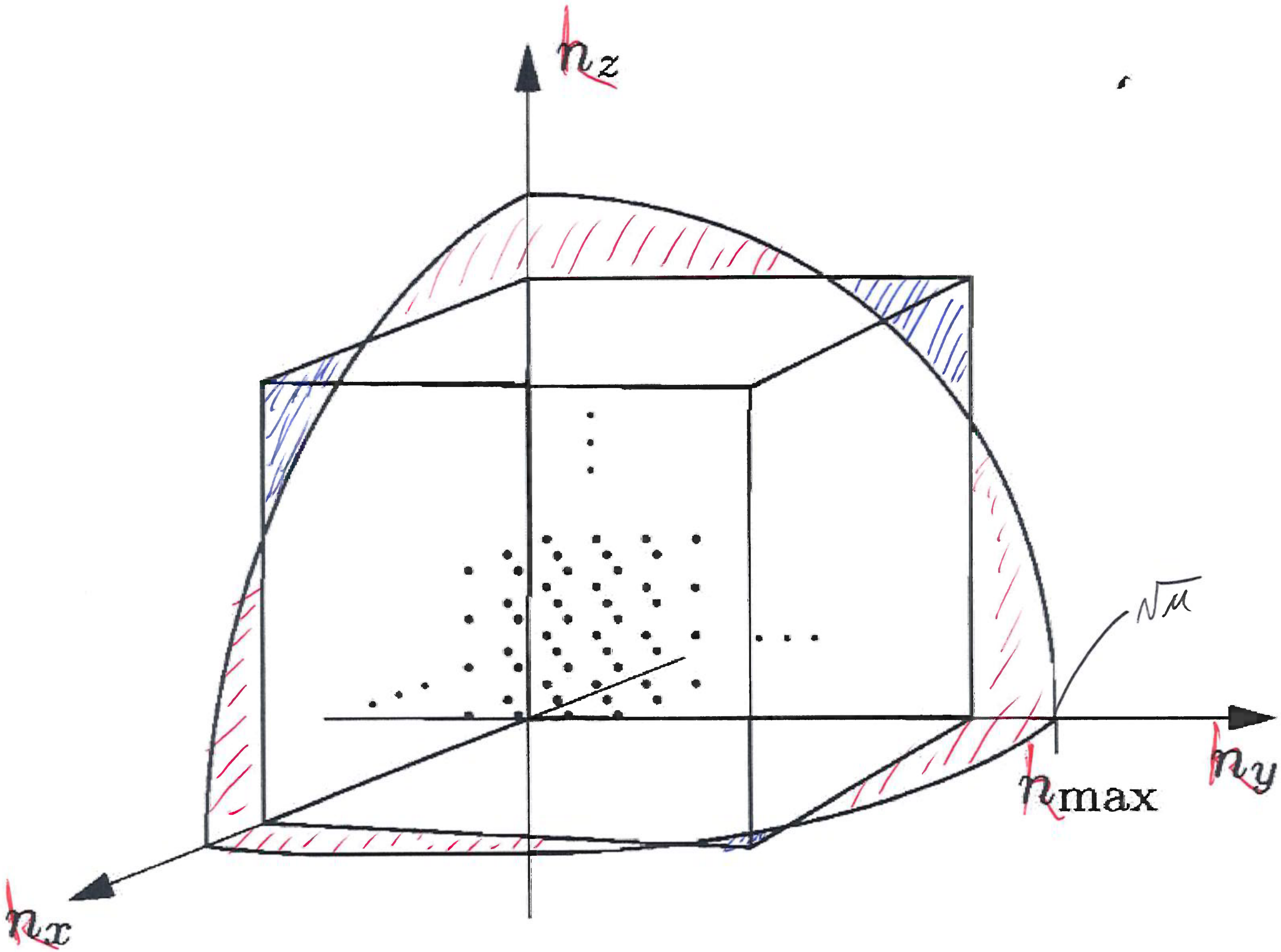

n_z
$\sqrt{\mu}$
n_{max}
n_y
n_x

Internal energy – additional E in integral

$$\boxed{\begin{aligned}\frac{\langle E\rangle_F}{V} &= \frac{\sqrt{2m^3}}{\pi^2 \hbar^3}\int_0^\infty E\,F(E)\sqrt{E}\,dE \\ &\rightarrow \frac{\sqrt{2m^3}}{\pi^2 \hbar^3}\int_0^\mu \underbrace{E^{3/2}\,dE}_{\frac{2}{5}\mu^{5/2}} \\ &= \frac{(2m\mu)^{3/2}\,\mu}{5\pi^2\hbar^3} = \frac{3}{5}\mu\frac{\langle N\rangle_F}{V} \neq 0 \text{ for } T\to 0\end{aligned}}$$

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Average $T\to 0$ energy per particle

$$\frac{\langle E\rangle_F}{\langle N\rangle_F} = \frac{3}{5}\mu$$

Same physical picture

Next tutorial will explore low $T>0$