MATH327: StatMech and Thermo, Spring 2025
Extra practice — Potts model

The Potts model is a simple generalization of the Ising model. It is defined by the energy

$$E(\sigma) = -\sum_{(ij)} \delta_{\sigma_i,\sigma_j},$$

where the spin variable $\sigma_n \in \{1, 2, \cdots, q\}$ at lattice site n has $q \geq 2$ possible values. ($q = 2$ corresponds to the Ising model.) Recall the Kronecker delta $\delta_{\sigma_i,\sigma_j} = 1$ for $\sigma_i = \sigma_j$ and vanishes for $\sigma_i \neq \sigma_j$. As always, the canonical partition function at temperature $T = 1/\beta$ is

$$Z(\beta, N) = \sum_{\{\sigma_n\}} \exp\left[-\beta E(\sigma)\right].$$

Consider the Potts model on the fully connected lattice with N sites, where each site is a nearest neighbour of every other site. Is the mean-field approximation reliable for the fully connected lattice with $N \gg 1$? Explain why or why not with clear reasoning.

Let $\{x_1, x_2, \cdots, x_q\}$ be the fraction of spins with value $\{1, 2, \cdots, q\}$. Then $x = \max\{x_1, x_2, \cdots, x_q\}$ is an order parameter for the Potts model, with $\frac{1}{q} \leq x \leq 1$. For $q \geq 3$, the mean-field approximation gives the self-consistency condition

$$x = \frac{e^{A\beta x} - 1}{e^{A\beta x} + q - 1}$$

for this order parameter, where A is a real positive constant. Find the temperature T_c at which $x_c = \frac{q-2}{q-1}$ satisfies this self-consistency condition.

As a warm-up, you can carry out the calculation for some fixed $q \geq 3$.