MATH327: StatMech and Thermo, Spring 2025

Tutorial exercises — Entropy bounds

These exercises will be introduced in our 27 February tutorial, where there should be plenty of time to analyze the $N_1 = N_2 = 10$ case. You can keep working on the larger-N cases afterwards; we'll review them during our next tutorial on 6 March.

We met the second law of thermodynamics by considering what happens when two subsystems are brought into thermal contact — allowed to exchange energy but not particles. Conservation of energy means that if subsystem Ω_1 has energy e_1, the other subsystem Ω_2 must have energy $E - e_1$, where E is the total energy of the overall micro-canonical system Ω. We found (in Eq. 21 on page 32 of the lecture notes) that the total number of micro-states of the overall system is

$$M = \sum_{e_1} M^{(1)}_{e_1} M^{(2)}_{E-e_1}$$

where $M^{(S)}_e$ is the number of micro-states of subsystem $S \in \{1, 2\}$ with energy e.

Because M is a sum of strictly positive terms, we can easily set bounds on it. Say the sum over e_1 has $N_{\text{terms}} \geq 1$ terms, and define $\max$ be the largest of those terms. Then $\max \leq M$, with equality only when $N_{\text{terms}} = 1$. Similarly, $M \leq N_{\text{terms}} \cdot \max$, with equality when every term in the sum is the same. All together, we have

$$\max \leq M \leq N_{\text{terms}} \cdot \max\,.$$

This can be more powerful than it may initially appear, thanks to the large numbers involved in statistical systems. For illustration, suppose $\max \sim e^N$ and $N_{\text{terms}} \sim N$ for a system with N degrees of freedom. (Such exponential behaviour is reasonable — we have already seen $M = 2^N = e^{N \log 2}$ for N spins with $H = 0$, while $H > 0$ introduces factors of $N!$ that Stirling's formula can recast in terms of $N^N = e^{N \log N}$.) Then

$$e^N \lesssim M \lesssim N e^N.$$

For a micro-canonical ensemble in thermodynamic equilibrium, the resulting bounds on the entropy $S = \log M$ are

$$N \lesssim S \lesssim N + \log N.$$

With $N \sim 10^{23}$, we have $\log N \sim 50$ and $10^{23} \lesssim S \lesssim 10^{23} + 50$, a very tight range in relative terms, with the upper bound only $\sim 10^{-20}\%$ larger than the lower bound.

To see how this works in practice, let each of Ω_1 and Ω_2 be a spin system with $N_1 = N_2 = 10$ spins and $H = 1$. Fix $E = -10$ for the combined system and numerically compute the bounds on its entropy,

$$\log(\max) \leq S \leq \log(N_{\text{terms}} \cdot \max)\,.$$

What fraction of the true entropy S is accounted for by $\log(\max)$? How do these answers change for $N_1 = N_2 = 20, 30, 40, \cdots$, still with fixed $E = -10$?

By considering the sort of spin configurations that produce $\max$, you can gain insight into the emergence of an 'arrow of time'!