MATH327: StatMech and Thermo, Spring 2025

Extra practice — 2d fermions

Consider a gas of fermions with mass m confined to a two-dimensional surface of area A, with temperature $T = 1/\beta$ and chemical potential μ. Its grand-canonical potential is

$$\Phi(T,\mu) = -\frac{mAT}{2\pi\hbar^2}\int_0^\infty \log\left[1 + e^{-(E-\mu)/T}\right] dE\,.$$

(a) Show that the particle density is

$$\frac{\langle N\rangle}{A} = \alpha\int_0^\infty F(E)\,dE, \qquad\qquad F(E) = \frac{1}{e^{(E-\mu)/T}+1},$$

where the constant $\alpha = \frac{m}{2\pi\hbar^2}$ and $F(E)$ is the usual Fermi function.

(b) Evaluate the particle density in the low-temperature limit where $F(E)$ becomes a step function. Use the result to find the Fermi energy $E_F = \mu$.

(c) Starting from the internal energy density

$$\frac{\langle E\rangle}{A} = \frac{m}{2\pi\hbar^2}\int_0^\infty E\;F(E)\,dE,$$

show $\langle E\rangle = \gamma\,\langle N\rangle^2$ in the low-temperature limit, and find the constant γ.

(d) The pressure is now the force per unit length on the boundary of the area A. In the low-temperature limit it is given by

$$P = -\left.\frac{\partial\langle E\rangle}{\partial A}\right|_N.$$

Calculate P to show $PA = \kappa\,\langle N\rangle\,E_F$ in the low-temperature limit, and determine the constant κ.