Thu 8 May

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Plan

Ising model ground states vs. lattice structure

Fully connected solution highlights

$H=0 \rightarrow E = -J \sum_{(jk)} s_j s_k$

Ground states for $J>0$ (includes $J=1$)

All spins align, degeneracy 2

Ordered (Ferromagnetic) phase, order param. $|m|=1$

All links interchangeable — lattice structure doesn't matter

$J<0$: Aligned spins increase energy

Ground state maximizes pairs of opposing n.n. spins

All spins alternating if possible

Anti-Ferromagnetic phase, degeneracy 2

$m = \frac{n_+ - n_-}{n_+ + n_-} = 0$ same as disordered phase but clear order/patterns

Order param: $m_s = \frac{1}{N}\sum_n (-1)^n s_n$ "staggered magnetization"

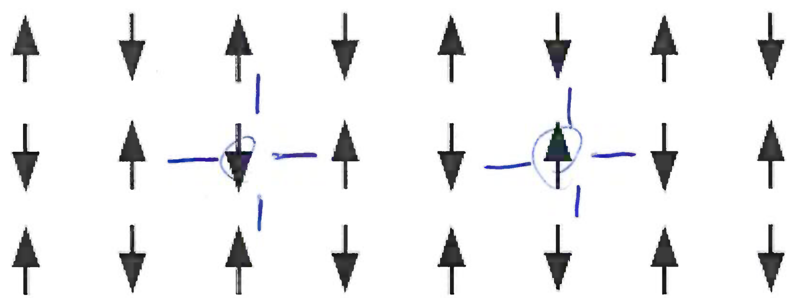
even/odd lattice labels

Ground state $|m_s|=1$ vs. $m_s=0$ in disordered phase

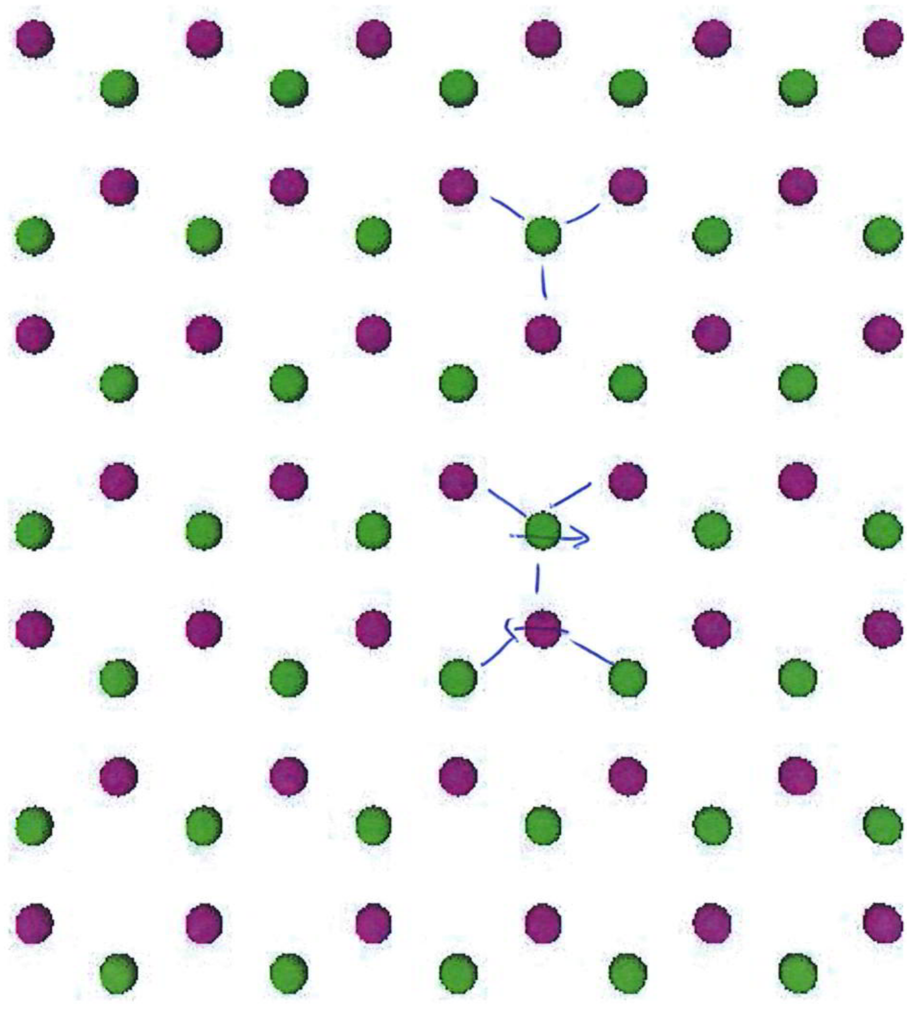
Relate to $F=-T\log Z$ by adding "staggered mag. field"

$$E = -J\sum_{(jk)} s_j s_k - H_s \sum_n (-1)^n s_n$$

$$\langle m_s\rangle \propto \frac{\partial}{\partial H_s}\log Z \quad \text{then set } H_s=0$$



commons.wikimedia.org/wiki/File:Antiferromagnetic_ordering.svg



$J<0$ triangular lattice can't have fully alternating spins
"Geometrical Frustration" because non-bipartite
(can also be frustrated by $E = -\sum_{j \neq k} J_{jk} s_j s_k$
with positive & negative J_{jk})

Single unit cell → 6 degenerate configs
with $E = -J(-2+1) = J < 0$

With $N \gg 1$, no order or order param
Many degenerate micro-states
→ $S > 0$ even at absolute zero T

For spin glasses w/frustration from random signs for J_{jk}
Finding w_i with $E_i \leq E_0$ is NP-complete
both NP and NP-hard

Motivates quantum computing

Fully connected Ising model
Exactly solvable, but simplified by approximations
Sum over links → sums over sites

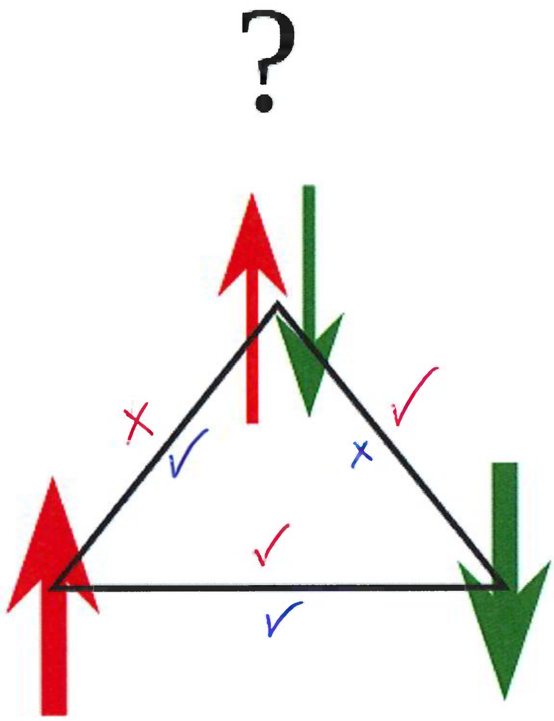
$$\frac{1}{2}\sum_{j \neq k} s_j s_k = \frac{1}{2}\left(\sum_j s_j\right)\left(\sum_k s_k\right) - \frac{1}{2}\sum_n s_n^2 = \frac{N^2 m^2}{2} - \frac{N}{2}$$

($\sum_n s_n^2 = N$)

$$E = \frac{-J}{2N}\sum_{j \neq k} s_j s_k - H\sum_n s_n = \frac{-JNm^2}{2} + \frac{J}{2} - HNm$$

$$Z = \sum_{n_+=0}^{N} \binom{N}{n_+} \exp\left[\frac{\beta J}{2}(Nm^2 - 1) + \beta HNm\right]$$

$$\rightarrow \int_{-1}^{1} \binom{N}{n_+} \exp\left[\frac{\beta J}{2}(Nm^2 - 1) + \beta HNm\right] dm$$



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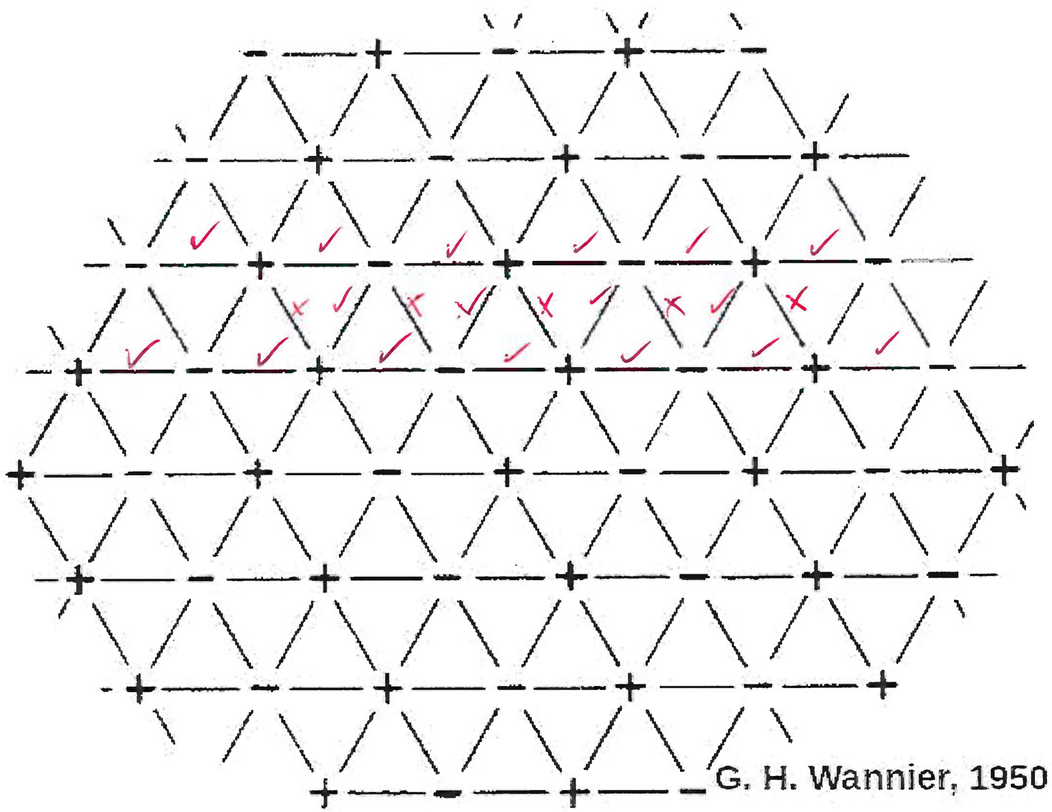
G. H. Wannier, 1950

$$\exp\left(\log\binom{N}{n_+}\right) = \exp\left[\log\left(\frac{N!}{n_+!\,n_-!}\right)\right] \quad \text{use Stirling}$$

$$m = \frac{n_+ - n_-}{N} = \frac{2n_+}{N} - 1 = 1 - \frac{2n_-}{N}$$

$$n_\pm = \frac{N}{2}(1 \pm m)$$

$$Z = e^{-\beta J/2} \int_{-1}^{1} \left[e^{f(m)}\right]^N dm$$

$$f(m) = \frac{1}{2}\log\left(\frac{4}{1-m^2}\right) + \frac{m}{2}\log\left(\frac{1-m}{1+m}\right) + \frac{\beta J m^2}{2} + \beta H m$$

$$-m\,\operatorname{arctanh}(m)$$

Like Sommerfeld, expand around max. of $f(m)$

$$\frac{\partial f}{\partial m} = \beta(Jm + H) - \operatorname{arctanh}(m) = 0$$

$$m = \tanh(\beta(Jm + H))$$

Mean-field $\rightarrow$ $T_c = J$

$b = 1/2$