MATH327: StatMech and Thermo

Wednesday, 2 April 2025

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Something to consider

Due to Pauli exclusion we have predicted unique behaviour for low-temperature ideal gases of non-relativistic fermions [e.g., the degeneracy pressure]

What physical systems could we expect to exhibit this behaviour?

Recap

Low-T non-rel. Fermion gas

$T \to 0$ Fermi energy $E_F = \mu \propto \rho_F^{2/3}$

Internal energy $\langle E \rangle_F = \frac{3}{5} \mu \langle N \rangle_F \neq 0$

Degeneracy pressure

$$P_F = \frac{2}{3} \frac{\langle E \rangle_F}{V} = \frac{\hbar^2}{5m} (3\pi^2)^{2/3} \rho_F^{5/3}$$

significant for $T \ll E_F$

Plan

Ideal Fermion gases — non-rel. examples

ultra-rel. EoS

Sommerfeld expansion highlights

Everyday metals have $\rho \sim \frac{N_A \text{ electrons}}{\text{c.c.}} \sim 10^{29}$ electrons$/m^3$

$E_F \sim 10^4 \text{ K} \sim 1 \text{ eV} \sim 10^{-19} \text{ J}$

Everyday $T \ll 10^4$ K $\to$ degenerate electron gas

Sun (on average) has similar $\rho \sim 10^{30}$ electrons/m^3 $\rightarrow E_F \sim 10^5 K$

Core $T \sim 10^7 K \gg E_F$

Fusion of hydrogen and helium heats sun

Radiation pressure balances force of gravity, reduces ρ

After H and He "fuel" exhausted, less radiation pressure

$\rightarrow$ higher ρ

$\rightarrow$ White dwarf stars

Sun's mass w/ earth radius ($\sim 100\times$ smaller)

$\rho \sim 10^6 \rho_{sun} \sim 10^{36}$ electrons/m^3 (mass ~ tonne/c.c.)

$E_F \sim (10^6)^{2/3} E_F^{sun} \sim 10^4 \times 10^5 K \sim 10^9 K \gg T \sim 10^7 K$

slowly cool to $\sim 10^3 K$ after $\sim 10^9$ years

$T \ll E_F \rightarrow$ degenerate electron gas

Electrons degeneracy pressure prevents further collapse

Binary system $\rightarrow$ white dwarf can capture matter from companion star

increasing mass and density

Chandrasekhar limit $M \sim 1.4 M_{sun} \rightarrow$ carbon & oxygen fusion

chain reaction $\rightarrow T \sim 10^9 K$ in seconds

Radiation pressure $\rightarrow$ supernova ("type Ia")

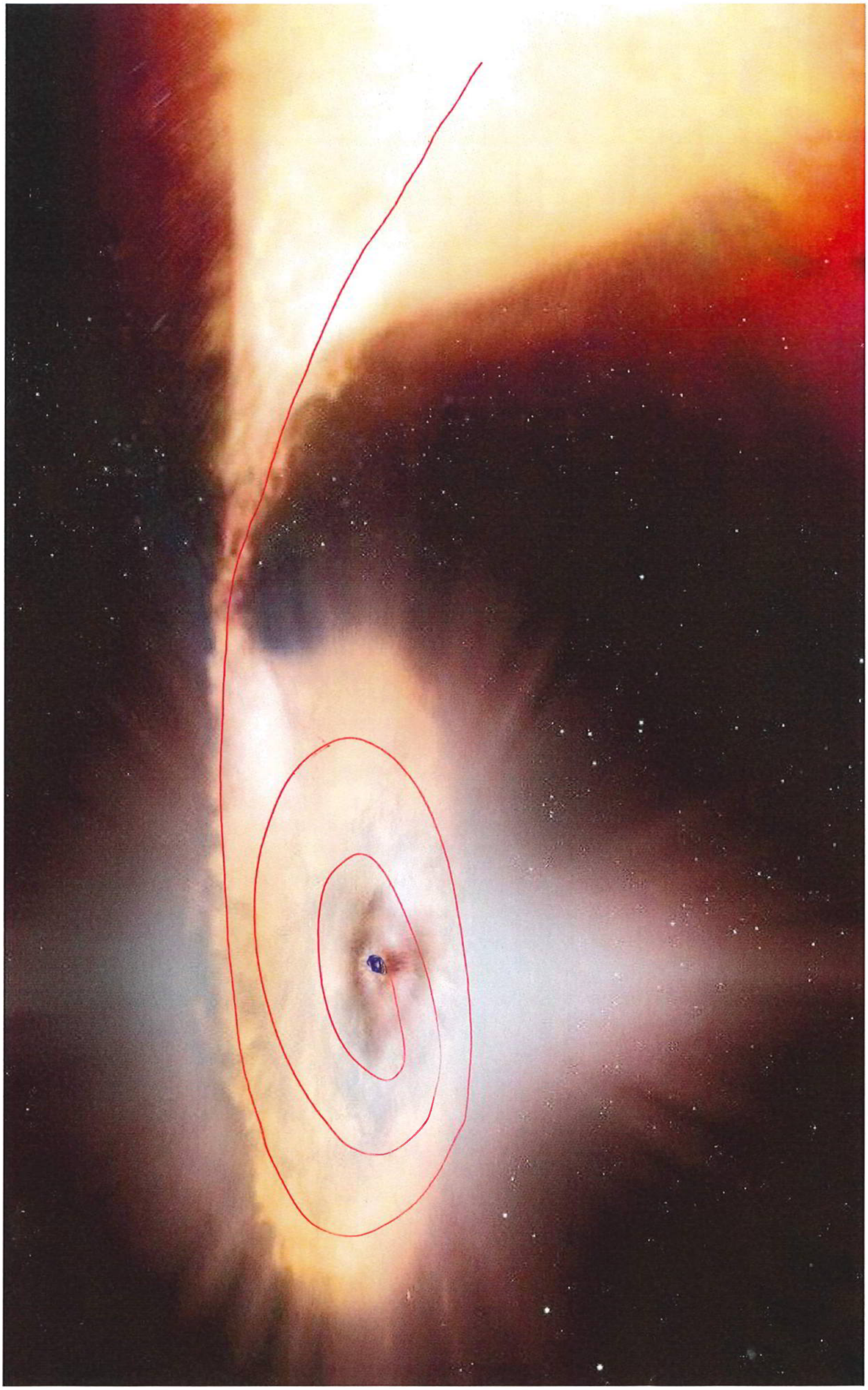
~5 billion time brighter than sun

Regularity $\rightarrow$ "standard candle" measuring distance vs. time

$\rightarrow$ accelerating expansion of Universe (Nobel 2011)

$T \sim 10^9 K \sim 0.3$ MeV vs. $m_e c^2 = 0.511$ MeV

Both mc^2 and pc needed for Chandrasekhar



Ultra-relativistic fermion gas, briefly

Example: Neutrinos mass $\lesssim 10^{-37}$ kg $\sim \frac{m_e}{10^6}$

$$E_\nu = c p_\nu = \hbar\omega \qquad \omega = \frac{2\pi c}{\lambda} = c\frac{\pi}{L}k \qquad k_{x,y,z} = 1,2,3,\dots$$

Like photon gas with different signs (2 spins ~ 2 pol.)

$$\Phi_\nu = -\frac{VT}{\pi^2 c^3}\int_0^\infty \omega^2 \log\left(1 + e^{-\beta\hbar\omega}\right) d\omega$$

$\mu \approx 0$

$$\langle N\rangle_\nu = \frac{-\partial}{\partial\mu}\Phi_\nu\Big|_{\mu=0} = \frac{VT}{\pi^2 c^3}\int_0^\infty \omega^2 \frac{\partial}{\partial\mu}\log\left(1 + e^{-\beta\hbar\omega}e^{\beta\mu}\right) d\omega\Big|_{\mu=0}$$

$$= \frac{V\cancel{T}}{\pi^2 c^3}\int_0^\infty \frac{\omega^2 e^{-\beta\hbar\omega}\cancel{\beta}}{1+e^{-\beta\hbar\omega}}d\omega = \frac{V}{\pi^2 c^3}\left(\frac{T}{\hbar}\right)^3 \underbrace{\int_0^\infty \frac{x^2}{e^x+1}dx}_{\left(1-\frac{1}{2^2}\right)\Gamma(3)\zeta(3)}$$

$$x = \beta\hbar\omega = \frac{\hbar}{T}\omega$$

$$\frac{3}{2}\zeta(3)$$

$$\langle N\rangle_\nu = \frac{3\zeta(3)}{2\pi^2\hbar^3 c^3}VT^3$$

For internal energy, add factor of $E = \hbar\omega = xT$

$$\langle E\rangle_\nu = \frac{VT^4}{\pi^2\hbar^3 c^3}\underbrace{\int_0^\infty \frac{x^3}{e^x+1}dx}_{\left(1-\frac{1}{2^3}\right)\Gamma(4)\zeta(4)} = \left(\frac{7}{8}\right)6\cdot\frac{\pi^4}{90}\left(\frac{1}{\pi^2\hbar^3c^3}\right)VT^4$$

$$\langle E\rangle_\nu = \frac{(7/8)\,\cancel{\Gamma(4)}\,\zeta(4)}{(3/4)\,\cancel{\Gamma(3)}\,\zeta(3)}\langle N\rangle_\nu T = \frac{7}{2}\frac{\zeta(4)}{\zeta(3)}\langle N\rangle_\nu T$$

$$= \frac{7\pi^4}{\underbrace{180\,\zeta(3)}_{3.151}}\langle N\rangle_\nu T$$

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EoS needs pressure $\rightarrow$ constant $S_\nu = \frac{1}{T}\left(\langle E\rangle_\nu - \Phi_\nu\right) \propto VT^3$

$$\Phi_\nu = \frac{-VT}{\pi^2c^3}\left(\frac{T}{\hbar}\right)^3 \underbrace{\int_0^\infty x^2\log\left(1+e^{-x}\right)dx}_{\frac{7}{4}\zeta(4) = \frac{7\pi^4}{360}} \propto VT^4$$

Constant entropy: $T = b V^{-1/3}$

Pressure
$$\boxed{P_\nu = -\frac{\partial}{\partial V} \langle E \rangle_\nu \Big|_{S_\nu} = -\frac{7}{8}\left(\frac{\Gamma(4)\zeta(4)}{\pi^2 \hbar^3 c^3}\right)\frac{\partial}{\partial V}\left(b^4 V^{-1/3}\right) = \frac{1}{3V}\langle E \rangle_\nu}$$

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Equation of state
$$P_\nu V = \frac{1}{3}\langle E \rangle_\nu = \frac{7}{6}\frac{\zeta(4)}{\zeta(3)}\langle N \rangle_\nu T$$
$$= \frac{7\pi^4}{540\, \underbrace{\zeta(3)}_{1.05}}\langle N \rangle_\nu T$$

Compare to photon gas, just extra factor of $\frac{7}{6}$

Extend non-rel. fermion gas to $T > 0$

Go beyond step-func. approx. for $F(E)$

→ Low-T heat capacity

→ Chemical potential $\mu(T)$ approaches classical limit

Better notation

$$\langle N \rangle_F = \int_0^\infty g(E)\, F(E)\, dE \qquad \langle E \rangle_F = \int_0^\infty E\, g(E)\, F(E)\, dE$$

Density of states $g(E) = g_0 \sqrt{E} = V\frac{\sqrt{2m^3}}{\pi^2 \hbar^3}\sqrt{E}$

of (single-particle) energy levels per unit energy

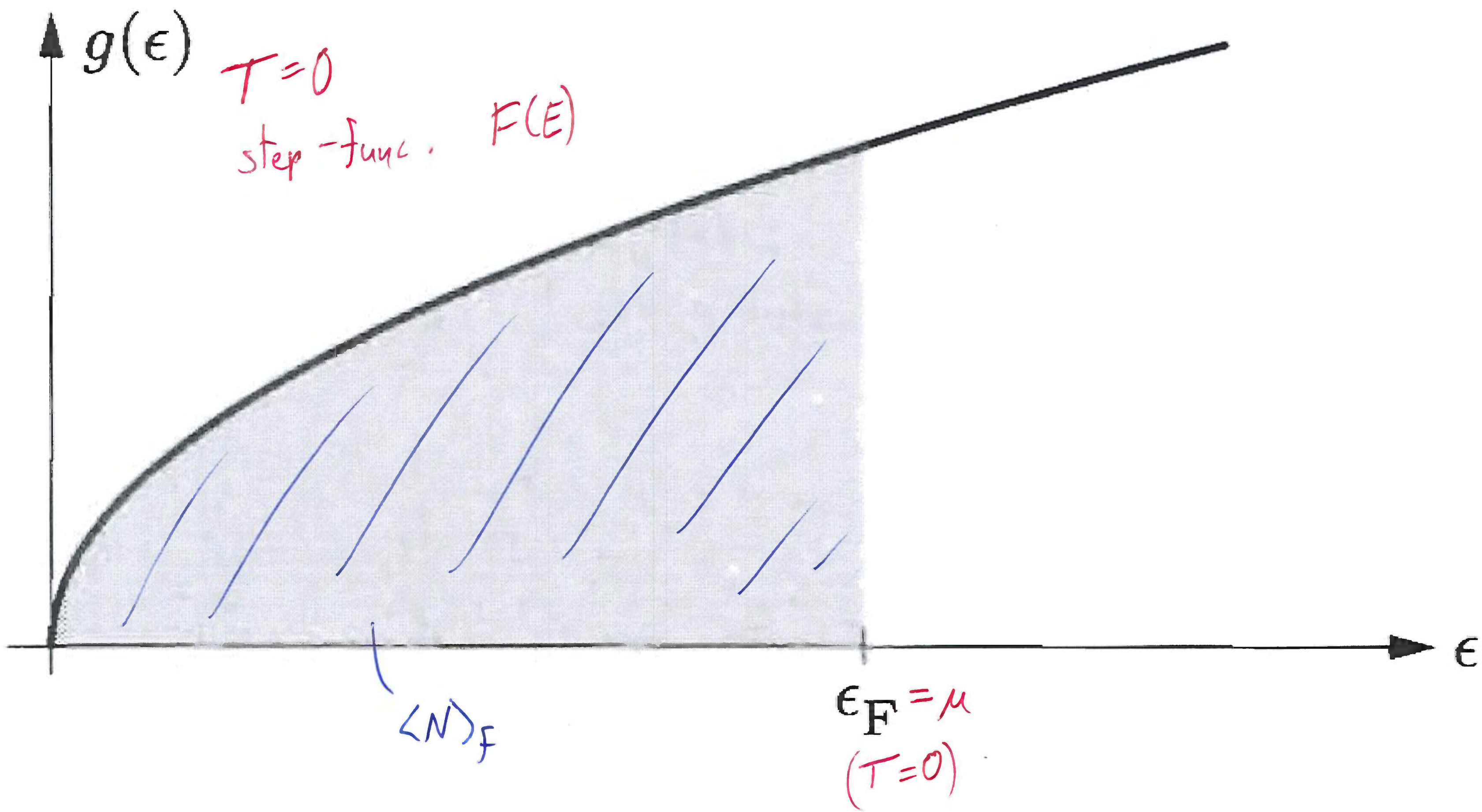
$F(E) = \frac{1}{e^{\beta(E-\mu)}+1}$ is occupation prob.

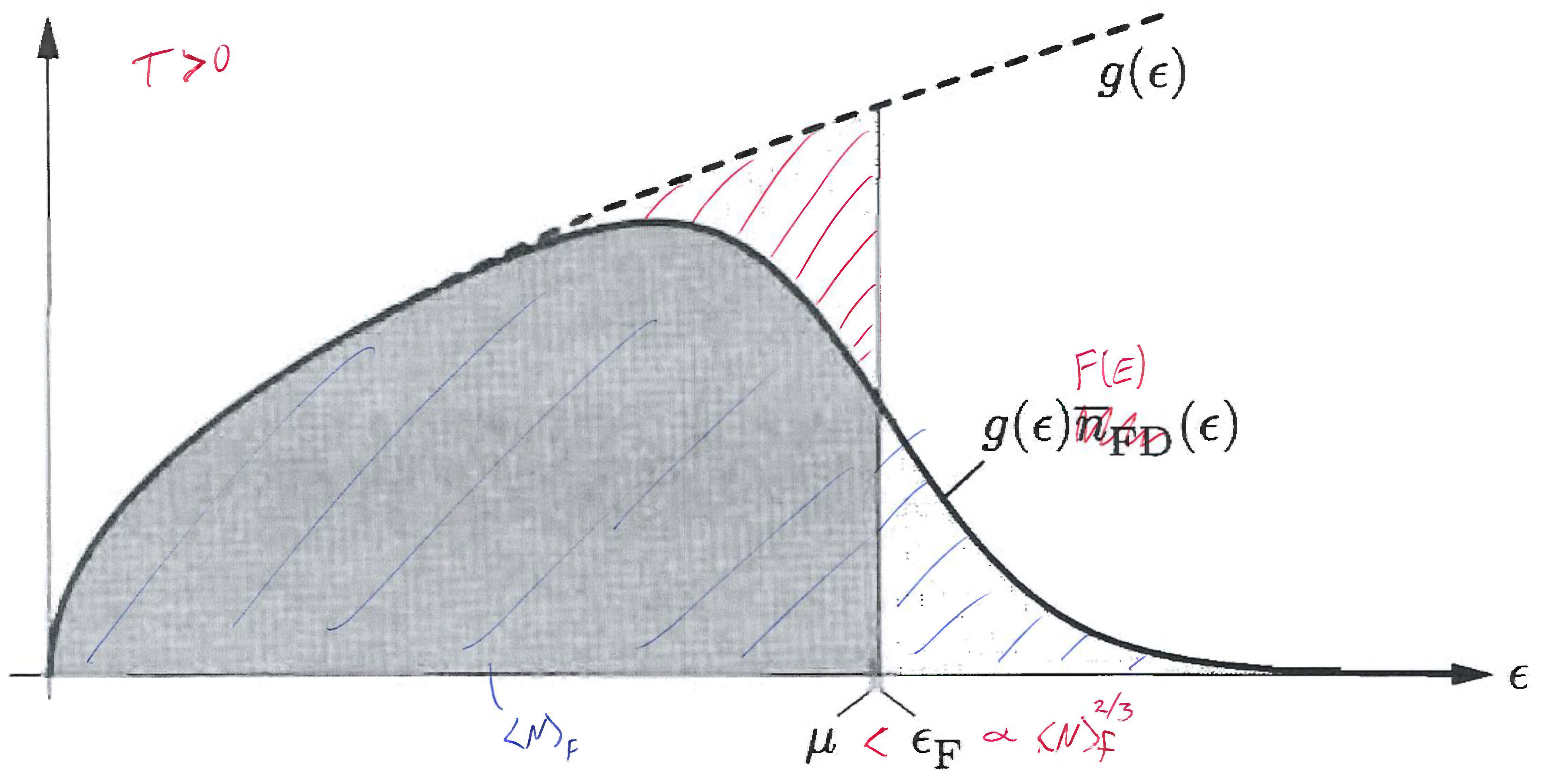
$T > 0 \to$ exp. suppressed occupation with $E > E_F$

$\therefore$ unoccupied energy levels with $E < E_F$

Recall $F(E=\mu) = \frac{1}{2}$

Will see $E_F > \mu$ when $T > 0$


$g(\epsilon)$
$T=0$
step-func. $F(E)$
$\langle N \rangle_F$
$\epsilon_F = \mu$
$(T=0)$
ϵ



$T > 0$
$g(\epsilon)$
$F(\epsilon)$
$g(\epsilon)\overline{n}_{FD}(\epsilon)$
ϵ
$\langle N \rangle_F$
$\mu < \epsilon_F \propto \langle N \rangle_F^{2/3}$

$$\frac{\langle N\rangle_F}{g_0} = \int_0^\infty E^{1/2} F(E)\, dE = \frac{2}{3} E^{3/2} F(E)\Big|_0^\infty - \frac{2}{3}\int_0^\infty E^{3/2}\left(\frac{dF}{dE}\right) dE$$

$u = F(E)$

$dv = E^{1/2} dE \rightarrow v = \frac{2}{3} E^{3/2}$

$$-\frac{d}{dE}\left(e^{\beta(E-\mu)}+1\right)^{-1} = \frac{\beta e^{\beta(E-\mu)}}{\left(e^{\beta(E-\mu)}+1\right)^2} = \frac{\beta e^x}{(e^x+1)^2}$$

$x = \beta(E-\mu)$

$$\frac{\langle N\rangle_F}{g_0} = \frac{2}{3}\int_0^\infty \frac{e^x}{(e^x+1)^2} E^{3/2}\, d(\beta E) = \frac{2}{3}\int_{-\beta\mu}^\infty \frac{e^x}{(e^x+1)^2} E^{3/2}\, dx$$

depends on x

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Sharply peaked around $x=0$
$E=\mu$

For $x \ll -1$ $\quad \frac{e^x}{(e^x+1)^2} \simeq \frac{e^x}{1} \ll 1$

$x \gg 1$ $\quad \frac{e^x}{(e^x+1)^2} \simeq \frac{e^x}{e^{2x}} = \frac{1}{e^x} \ll 1$

Peak allows approximations:

1) Extend $\int_{-\beta\mu}^\infty \rightarrow \int_{-\infty}^\infty$ since $\mu > 0$ for low T (large β)

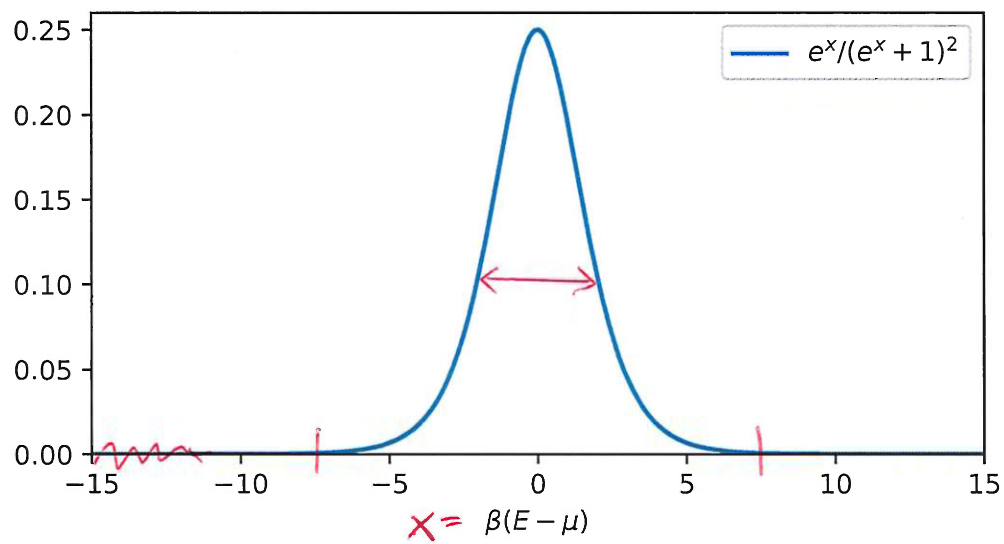
2) Expand $E^{3/2} \approx \mu^{3/2} + (E-\mu)\frac{\partial}{\partial E}E^{3/2}\Big|_{E=\mu}$

$$+ \frac{1}{2}(E-\mu)^2 \frac{\partial^2}{\partial E^2} E^{3/2}\Big|_{E=\mu}$$

$$= \mu^{3/2} + \frac{3}{2}(E-\mu)\mu^{1/2} + \frac{3}{8}(E-\mu)^2 \mu^{-1/2}$$

Low-T Sommerfeld expansion

$$= \mu^{3/2} + \frac{3}{2} xT \mu^{1/2} + \frac{3}{8}(xT)^2 \mu^{-1/2}$$


0.25
0.20
0.15
0.10
0.05
0.00
−15
−10
−5
0
5
10
15
$e^x/(e^x+1)^2$
x = $\beta(E-\mu)$

$$\frac{\langle N \rangle_f}{g_0} \approx \frac{2}{3}\mu^{3/2} \underbrace{I_0}_{1} + T\mu^{1/2} \underbrace{I_1}_{0} + \frac{1}{4}T^2 \mu^{-1/2} I_2$$

$$\boxed{\begin{aligned} I_0 &= \int_{-\infty}^{\infty} \frac{e^x}{(e^x+1)^2}\,dx = \int_{-\infty}^{\infty} \frac{-dF}{dE}\,dE = -F(E)\Big|_{-\infty}^{\infty} = -0+1 = 1 \\ I_1 &= \int_{-\infty}^{\infty} \frac{x e^x}{(e^x+1)^2}\,dx = \int_{-\infty}^{\infty} \frac{x}{(e^x+1)(1+e^{-x})}\,dx = 0 \\ I_2 &= \int_{-\infty}^{\infty} \frac{x^2 e^x}{(e^x+1)^2}\,dx \rightarrow \frac{\pi^2}{3} \end{aligned}}$$

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$$\langle N \rangle_f \approx g_0\left[\frac{2}{3}\mu^{3/2} + \frac{\pi^2}{12\mu^{1/2}} T^2\right]$$