MATH327: StatMech and Thermo

Monday, 31 March 2025

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Something to consider

Moving from gases of photons to gases of fermions,
what effects do you expect Pauli exclusion will have?

For what temperatures and chemical potentials
do you expect these effects to be significant?

Recap

Ideal photon gas – Planck spectrum, real ~~sa~~ systems, EoS
$\mu = 0$

Non-rel. fermion gas, $\mu > 0$ at low T

$$\frac{\langle N \rangle_F}{V} = \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\infty F(E) \sqrt{E}\, dE$$

$$F(E) = \frac{1}{e^{\beta(E-\mu)} + 1} \xrightarrow[T \to 0]{} \begin{cases} 1 & 0 \leq E < \mu \\ 0 & \text{otherwise} \end{cases}$$

Plan

$T \to 0$ fermion gas

Fermi energy

Degeneracy pressure

Examples

$$\boxed{\frac{\langle N \rangle_F}{V} \to \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\mu \sqrt{E}\, dE = \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \left(\frac{2}{3} \mu^{3/2}\right) = \frac{(2m\mu)^{3/2}}{3\pi^2 \hbar^3}}$$

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Leading order of (Sommerfeld) expansion in powers of $\frac{T}{\mu} \ll 1$

Physical picture: All energy levels with $E_\ell < \mu$ occupied, $n_\ell = 1$

$E_\ell \propto k^2 \rightarrow \langle N \rangle_F$ fills octant of sphere with $\underbrace{\sqrt{\mu}}_{\propto \mu^{3/2}}$

For $T \to 0$ max energy is Fermi energy

$$E_F = \mu = \frac{\hbar^2}{2m}\Big(3\pi^2 \underbrace{\frac{\langle N \rangle_F}{V}}_{\rho_F}\Big)^{2/3} = \frac{\hbar^2}{2m}(3\pi^2)^{2/3} \rho_F^{2/3}$$

Fermion gas internal energy $\rightarrow$ just extra E in integral

$$\frac{\langle E \rangle_F}{V} = \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\infty E\, F(E) \sqrt{E}\, dE \rightarrow \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\mu E^{3/2}\, dE$$

$$= \frac{(2m\mu)^{3/2} \mu}{5 \pi^2 \hbar^3} = \frac{3}{5} \mu \frac{\langle N \rangle_F}{V} \neq 0 \text{ for } T \to 0$$

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Average $T \to 0$ energy per particle $\frac{\langle E \rangle_F}{\langle N \rangle_F} = \frac{3}{5}\mu$

Same physical picture

Tutorial: Integrate $\int_0^\infty F(E)\, E^{3/2}\, dE$ by parts

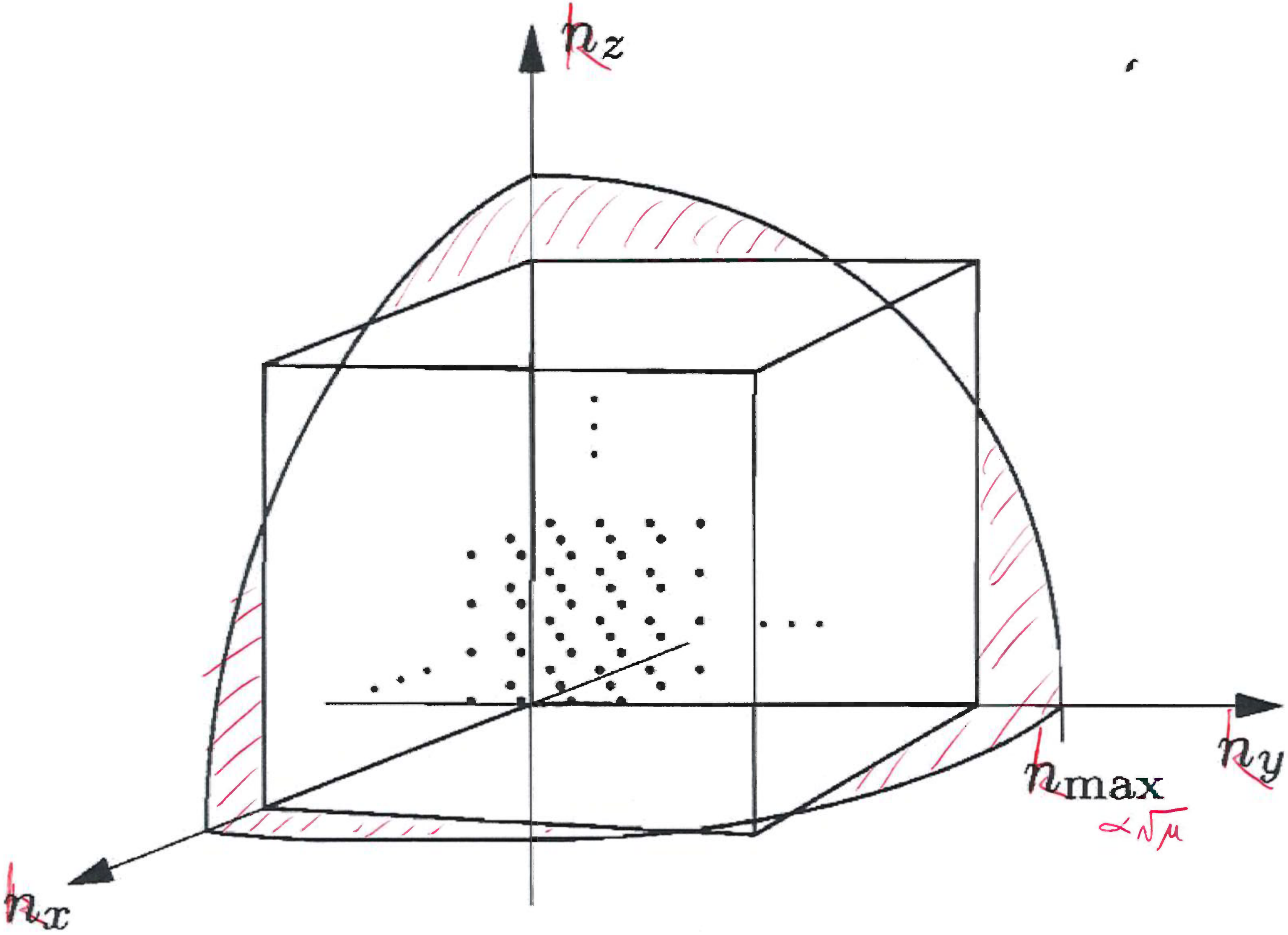
Argue boundary term vanishes

For low T, remaining integrand sharply peaked around $E = \mu$

Taylor expand $\rightarrow C_V \propto T$

Confirm $\mu > 0$ for low-T Fermion gas

$$\mu = \frac{\partial E}{\partial N}\Big|_{S,V}$$


n_z
n_y
n_{max}
$\propto \sqrt{\mu}$
n_x

For $T \to 0$, single micro-states, $S = -\sum_i p_i \log p_0 = -\log 1 = 0$

S unchanged for $\Delta N > 0 \to$ add particles filling next energy levels ~~Above~~ above $E_F = \mu$

$\therefore \Delta E \approx E_F (\Delta N) > 0 \longrightarrow \mu > 0$

For higher T, can use Sommerfeld expansion to confirm $\mu < 0$

$\rightarrow$ consistent classical limit

<u>Pressure</u> $P_F = \frac{-\partial}{\partial V} \langle E \rangle_F \Big|_{S_F} = \frac{-3}{5} \frac{\partial}{\partial V} \left(\mu \langle N \rangle_F\right)\Big|_{S_F}$

Again $S_F = 0$ (const.) for single $T \to 0$ micro-state

$$\boxed{\begin{aligned} P_F &= \frac{-3}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} \langle N \rangle_F^{5/3} \frac{\partial}{\partial V} V^{-2/3} = \frac{2}{3} \frac{\langle E \rangle_F}{V} \\ &= \frac{2}{5} \mu \frac{\langle N \rangle_F}{V} = \frac{\hbar^2}{5m} (3\pi^2)^{2/3} \rho_F^{5/3} \end{aligned}}$$

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New feature $P_F \neq 0$ as $T \to 0$

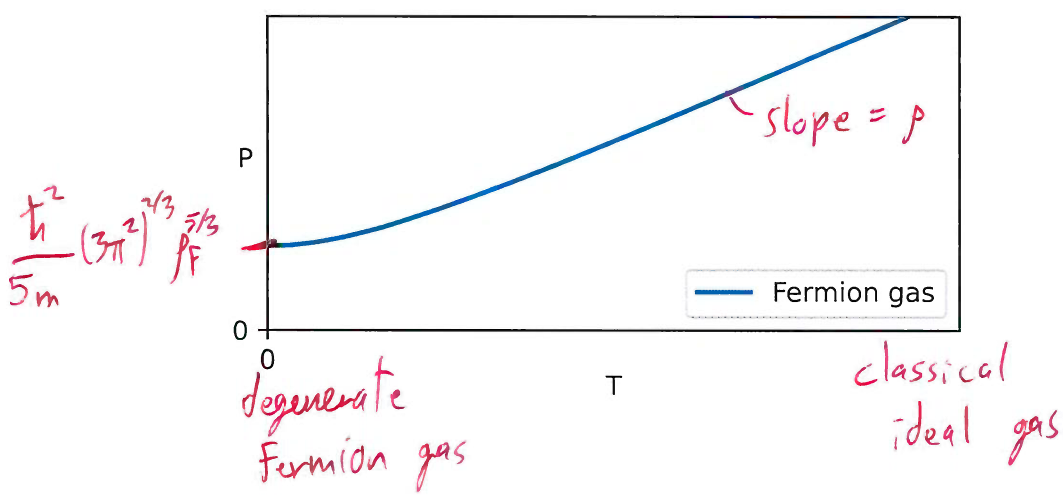
"Degeneracy pressure" from Pauli exclusion

(unrelated to degenerate energy levels)

High-T classical limit: $P = \frac{\langle N \rangle}{V} T = \rho T$

Degeneracy pressure matters for $T \ll E_F$ $\quad (E_F \propto \rho_F^{2/3})$

low T or high density (or both)



$\frac{\hbar^2}{5m}(3\pi^2)^{2/3}\rho_F^{5/3}$
P
0
0
T
Fermion gas
slope = ρ
degenerate Fermion gas
classical ideal gas