MATH327: StatMech and Thermo

Monday, 24 February 2025

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Something to consider

Should we expect a system's internal energy expectation value $\langle E \rangle$ to depend on whether or not we can label its particles?

Recap

Canonical ensemble (T, N)

Partition Function $Z = \sum_i e^{-\beta E_i}$

Helmholtz free energy $F = -T \log Z$

$$p_i = \frac{1}{Z} e^{-\beta E_i} = e^{(F - E_i)/T}$$

Today

Observable effects from information content

Ideal gases

Dist'ble spins:

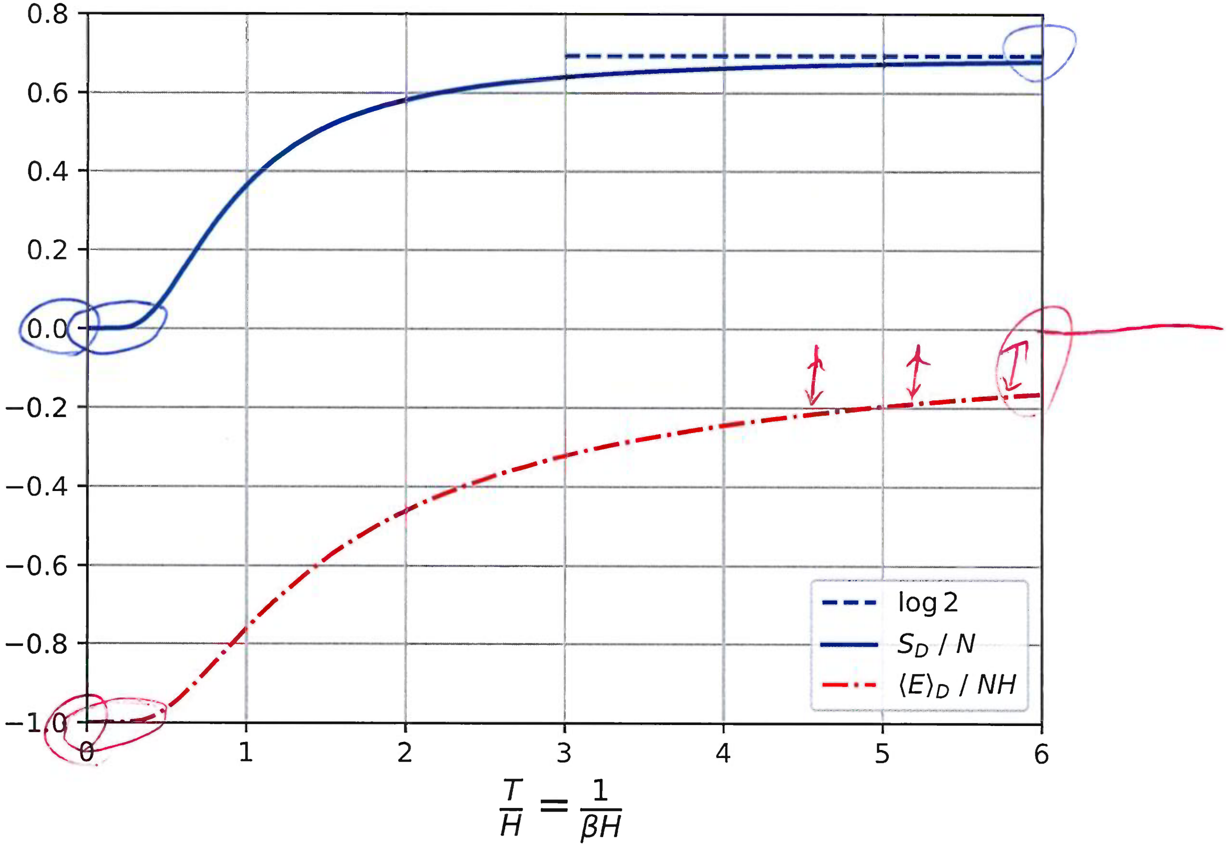
$$F_D = -\frac{N}{\beta} \log\left[2 \cosh(\beta H)\right]$$

$$\langle E \rangle_D = -NH \tanh(\beta H)$$

$$S_D = -N\beta H \tanh(\beta H) + N \log\left[2 \cosh(\beta H)\right]$$

High-temperature $\beta \to 0$

$$\frac{\langle E \rangle_D}{NH} = -\tanh(\beta H) \doteq -\beta H + \frac{(\beta H)^3}{3} + \mathcal{O}(\beta^5 H^5) \to 0 \quad \text{(linearly)}$$


0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
0
1
2
3
4
5
6
$\frac{T}{H} = \frac{1}{\beta H}$
$\log 2$
S_D / N
$\langle E \rangle_D$ / NH

$$\frac{S_D}{N} = -(\beta H)^2 + \mathcal{O}(\beta^4 H^4) + \log\left[2\cosh(\beta H)\right]$$

$$\log 2 + \log\left(1 + \tfrac{1}{2}(\beta H)^2 + \mathcal{O}(\beta^4 H^4)\right)$$

$$= \log 2 + \tfrac{1}{2}(\beta H)^2 + \mathcal{O}(\beta^4 H^4)$$

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$$\frac{S_D}{N} = \log 2 - \tfrac{1}{2}(\beta H)^2 + \mathcal{O}(\beta^4 H^4)$$

As $T \to \infty$ $\quad S_D \to N \log 2 = \log 2^N = \log M$ ~ micro-canonical

All $p_i = \frac{1}{Z} \underbrace{e^{-E_i/T}}_{1} \approx \frac{1}{\sum_i \underbrace{e^{-E_i/T}}_{1}} \to \frac{1}{M}$

Indist'able spins move (slowly) in gas (one-dim'l)

$H \Uparrow \quad \uparrow \; \downarrow \; \uparrow \; \downarrow \; \downarrow \; \ldots$

$N=2$ spins $\to$ 3 micro-states $\quad \uparrow\uparrow \quad \downarrow\downarrow \quad \{\uparrow\downarrow$ and $\downarrow\uparrow\}$

$\underline{\text{not}}$ 2^N

Only knowable info is total $\{n_+, n_-\} \to E_{n_-}$

One micro-state for each energy level!

Example: $N=4$ micro-states are $E = -4H, -2H, 0, 2H, 4H$

(all up) (all down)

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In general only $M_I = N+1$ micro-states

Label w_k for $E_k = -NH + \underbrace{2H}_{\Delta E}k = -H(N-2k) \quad k = 0, 1, \ldots N$

Start with partition function

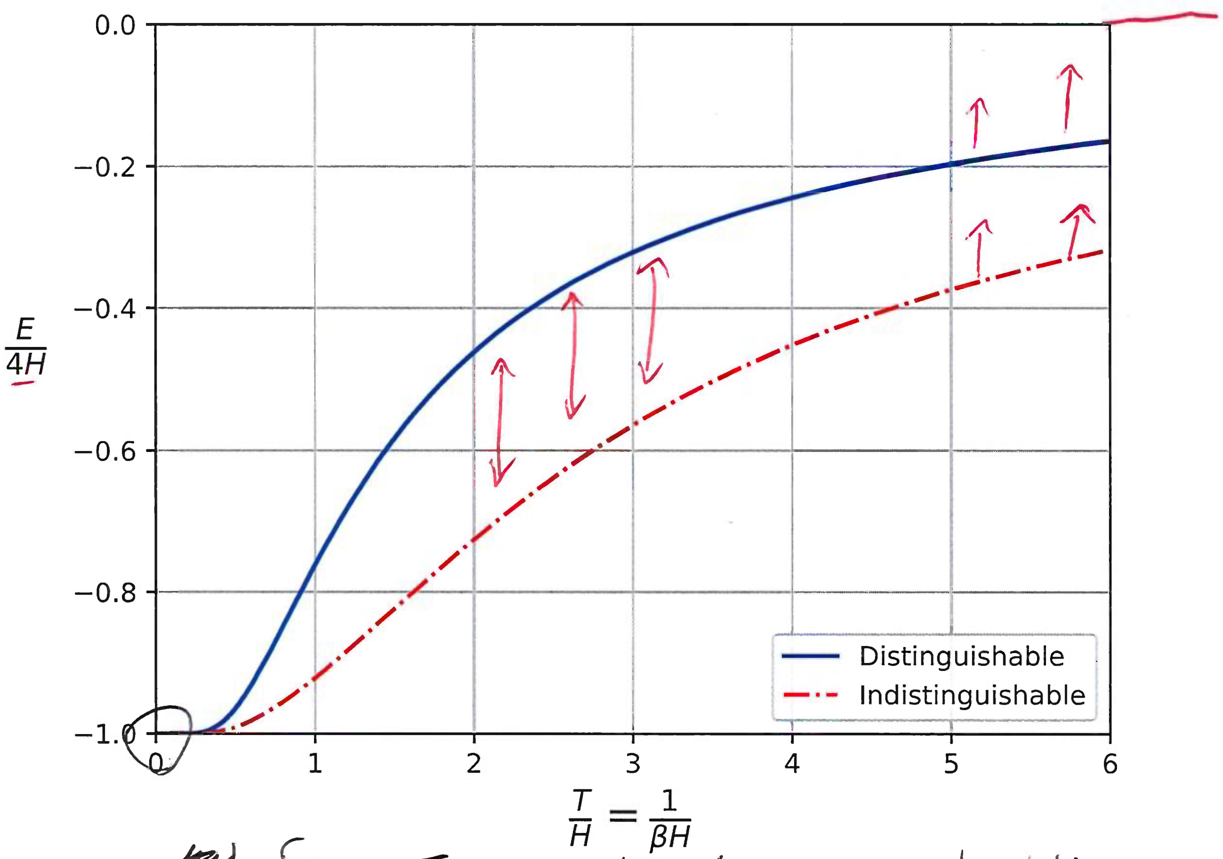
$$Z_I = \sum_{k=0}^{N} e^{-\beta E_k} = \sum_k e^{\beta H(N-2k)} = e^{N\beta H} \sum_{k=0}^{N} \left(e^{-2\beta H}\right)^k$$

$$\sum_{k=0}^{N} x^k = \sum_{k=0}^{\infty} x^k - \sum_{k=N+1}^{\infty} x^k = \frac{1}{1-x} - x^{N+1} \sum_{l=0}^{\infty} x^l = \frac{1-x^{N+1}}{1-x}$$

$$x = e^{-2\beta H} < 1 \qquad Z_I = e^{N\beta H} \left(\frac{1 - e^{-2(N+1)\beta H}}{1 - e^{-2\beta H}} \right)$$

$$F_I = -\frac{\log Z_I}{\beta} = -NH - \frac{1}{\beta} \log\left(1 - e^{-2(N+1)\beta H}\right) + \frac{1}{\beta} \log\left(1 - e^{-2\beta H}\right)$$

N=4



Same $\langle E \rangle \to 0$
as $T \to \infty$

Same $T \to 0$ limit (single ground state)
$\langle E \rangle_I$ approaches faster

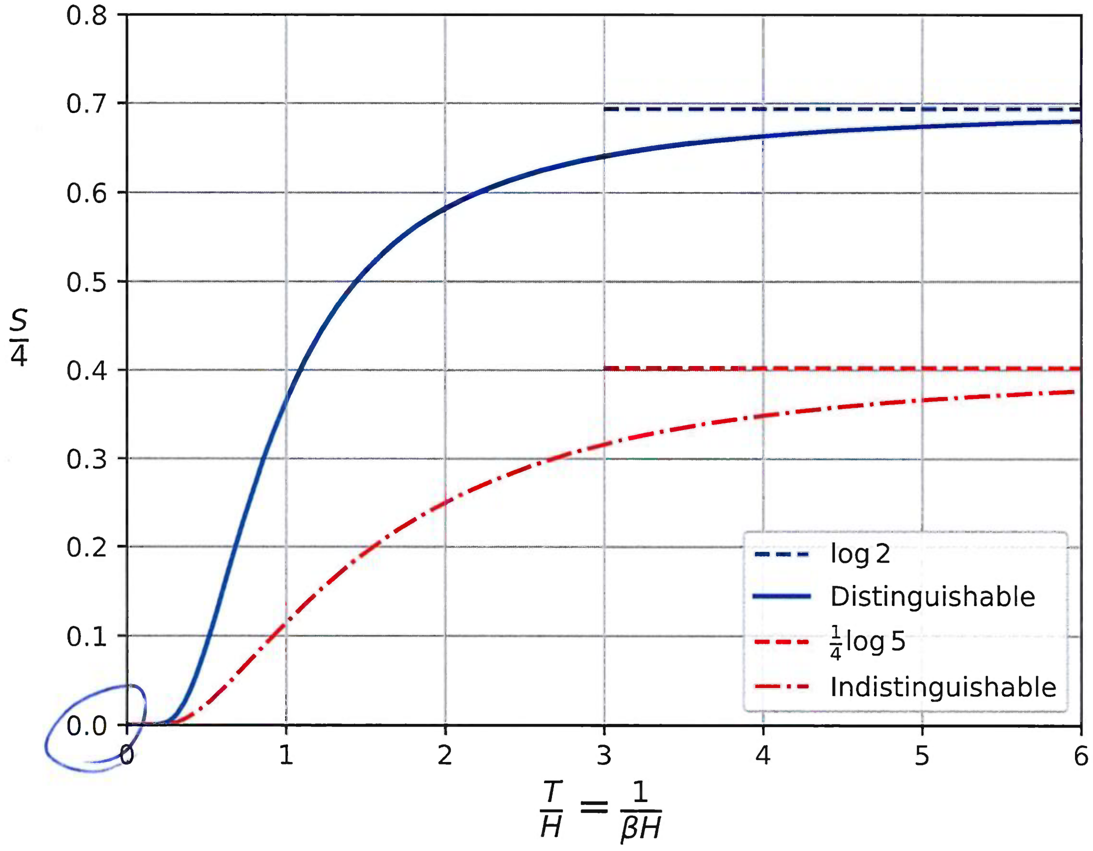
Physically measurable effects from intrinsic info content
"Information is physical"

Both $S \to \log M$

$N=4$

$\log 2 \leftarrow M_D = 2^N$

$\frac{1}{N}\log(N+1) \leftarrow M_I = N+1$



Same $S \to 0$ as $T \to 0$