MATH327: Statistical Physics, Spring 2024

Second homework assignment

Instructions

Complete all four questions below and submit your solutions by file upload on Canvas. Clear and neat presentations of your workings and the logic behind them will contribute to your mark. Use of resources beyond the module materials must be explicitly referenced in your submissions. This assignment is **due Friday, 26 April**. Anonymous marking is turned on and I will aim to return feedback shortly after 10 May.

The Department recently published more detailed academic integrity guidance. To summarize it:
By submitting solutions to this assessment you affirm that you have read and understood the Academic Integrity Policy detailed in Appendix L of the Code of Practice on Assessment, and that you have successfully passed the Academic Integrity Tutorial and Quiz in the course of your studies. You also affirm that the work you are submitting is your own and you have not commissioned production of the work from a third party or used artificial intelligence (AI) software in an unacceptable manner to generate the work. (Generative AI software applications include, but are not limited to, ChatGPT, Bing Chat, DALL.E and Bard.) You also affirm that you have not copied material from another person or source, nor committed plagiarism, nor fabricated, falsified or embellished data when completing the attached piece of work. You also affirm that you have not colluded with any other student in the preparation or production of this work. Marks achieved on this assessment remain provisional until they are ratified by the Board of Examiners in June 2024. Please note that your submission will be analysed by Turnitin for plagiarism and an Artificial Intelligence detection tool.

Question 1: Energy and entropy

Using the canonical ensemble with inverse temperature $\beta = 1/T$, consider a system of $N \gg 1$ indistinguishable non-interacting spins in an external magnetic field with strength $H > 0$.

(a) What are the internal energy $\langle E \rangle_I$ and the entropy S_I as functions of β, N and H?

[4 marks]

(b) Use your results to confirm the following low- and high-temperature limits:

$$\lim_{T \to 0} \langle E \rangle_I = -NH \qquad \lim_{T \to 0} S_I = 0$$
$$\lim_{T \to \infty} \langle E \rangle_I = 0 \qquad \lim_{T \to \infty} S_I = \log(N+1)$$

[6 marks]

(c) For low but non-zero temperatures, expand both $\langle E \rangle_I$ and S_I in terms of $\varepsilon \equiv e^{-2\beta H} \ll 1$. Show that the largest temperature-dependent term in the energy expansion is proportional to ε and find the constant of proportionality. Show that the largest temperature-dependent term in the entropy expansion is proportional to $\beta\varepsilon$ and find the constant of proportionality.

[8 marks]

(d) For high but finite temperatures, expand both $\langle E \rangle_I$ and S_I in terms of $x \equiv 2\beta H \ll 1$. Show that the largest temperature-dependent term in the energy expansion is proportional to x and find the constant of proportionality. Show that the largest temperature-dependent term in the entropy expansion is proportional to x^2 and find the constant of proportionality.

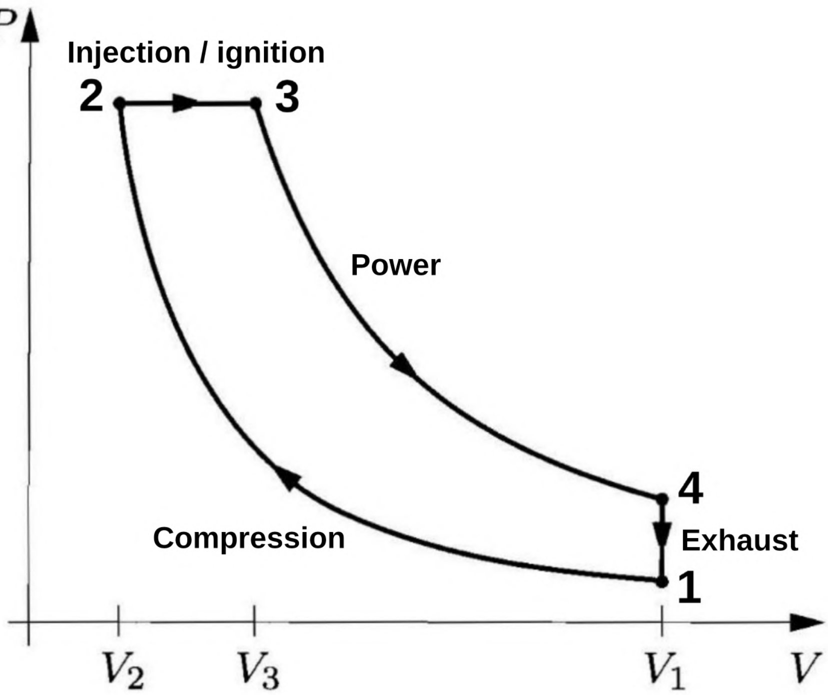
[8 marks]

The following famous results may be useful:

$$\frac{1}{1 - e^{-x}} = \frac{1}{x} + \frac{1}{2} + \frac{x}{12} - \frac{x^3}{720} + \frac{x^5}{30\,240} + \mathcal{O}(x^6)$$
$$\log\left[1 - e^{-x}\right] = \log(x) - \frac{x}{2} + \frac{x^2}{24} - \frac{x^4}{2880} + \frac{x^6}{181\,440} + \mathcal{O}(x^7)\,.$$

Question 2: Diesel cycle

Consider the Diesel cycle defined by the PV diagram shown below, in which the 'compression' stage $1 \to 2$ and the 'power' stage $3 \to 4$ are both adiabatic, while the pressure is constant during the 'injection/ignition' stage $2 \to 3$, and the volume is constant during the 'exhaust' stage $4 \to 1$. The compression ratio is $r \equiv V_1/V_2 > 1$ and the cutoff ratio is $C \equiv V_3/V_2 > 1$, with $C < r$.



(a) By computing W_{out}, W_{in} and Q_{in}, show that the Diesel cycle's efficiency is

$$\eta_D = 1 - \frac{f(C)}{r^{2/3}}$$

and determine the function $f(C)$ that depends only on the cutoff ratio.

[18 marks]

(b) Show $f(C) > 1$ for $C > 1$. This will confirm a claim made in a tutorial, that the Diesel cycle is less efficient than the Otto cycle with the same compression ratio r.

[6 marks]

Question 3: Particle number fluctuations

Consider the fugacity expansion of the grand-canonical partition function (Eq. 82 in the lecture notes):

$$Z_g(T, \mu) = \sum_{N=0}^{\infty} \xi^N \, Z_N(T).$$

Here $\xi = e^{\beta\mu} = e^{\mu/T}$ is the fugacity and $Z_N(T)$ is the N-particle canonical partition function, which is independent of ξ. Recall that $\Phi(T, \mu) = -T \log Z_g(T, \mu)$ is the corresponding grand-canonical potential.

(a) Derive a relation between the average particle number $\langle N \rangle$ and the derivative $\dfrac{\partial}{\partial \log \xi} \Phi$.

[8 marks]

(b) Derive a relation between $\left\langle (N - \langle N \rangle)^2 \right\rangle$ and $\left(\dfrac{\partial}{\partial \log \xi} \right)^2 \Phi$.

[8 marks]

(c) Specialize to the case of Maxwell–Boltzmann statistics, for which the fugacity expansion simplifies to $Z_g^{\text{MB}}(T, \mu) = \exp[\xi Z_1(T)]$, where Z_1 is the single-particle partition function. Use the relations you have derived to determine $\langle N \rangle$ and $\left\langle (N - \langle N \rangle)^2 \right\rangle$ for this case, and show

$$\frac{\sqrt{\left\langle (N - \langle N \rangle)^2 \right\rangle}}{\langle N \rangle} = \frac{1}{\sqrt{\langle N \rangle}}.$$

It may be useful to note $\dfrac{\partial}{\partial \log \xi} = \xi \dfrac{\partial}{\partial \xi}$.

[8 marks]

Question 4: Bosons and fermions

Using the grand-canonical ensemble with temperature $T = 1/\beta$ and chemical potential μ, consider a quantum system that has only three energy levels, with energies $E_0 = 0$, $E_1 = \varepsilon$ and $E_2 = 2\varepsilon$.

(a) If the particles in this system are fermions, how many micro-states are there?

[2 marks]

(b) How many micro-states would there be if the particles were bosons?

[2 marks]

(c) For the fermionic case, write down every term in the grand-canonical partition function.

[6 marks]

(d) For the fermionic case, what is the average internal energy $\langle E \rangle$?

[8 marks]

(e) For the fermionic case, what are the average occupation numbers $\langle n_0 \rangle$, $\langle n_1 \rangle$ and $\langle n_2 \rangle$ for the three energy levels?

[8 marks]

It may be useful to consider a system with only two energy levels as a warm-up exercise.