MATH327: Statistical Physics, Spring 2024

Tutorial activity — Stirling's formula

This activity will be introduced in our 29 February tutorial, and you can continue to work on it throughout the following week. We'll review it during our next tutorial on 7 March.

We have already made use of Stirling's formula in the following form:

$$\log(N!) = N \log N - N + \mathcal{O}(\log N) \approx N \log N - N \qquad \text{for } N \gg 1,$$

which implies

$$N! \approx \exp\left[N \log N - N\right] = \left(\frac{N}{e}\right)^N.$$

This can be made more precise:

$$N! = \sqrt{2\pi N}\left(\frac{N}{e}\right)^N \left(1 + \frac{A}{N} + \frac{B}{N^2} + \frac{C}{N^3} + \cdots\right) \tag{1}$$

with calculable coefficients A, B, C, etc.[1] By performing a sequence of analyses of increasing complexity, we can build up these results.

First analysis: Derive the bounds

$$N \log N - N < \log(N!) < N \log N \tag{2}$$

for $N \gg 1$. The second bound is the easier one. There are multiple ways to obtain the first bound. One pleasant approach is to consider the series expansion for e^x. Together, these bounds establish

$$1 - \frac{1}{\log N} < \frac{\log(N!)}{N \log N} < 1 \qquad \Longrightarrow \qquad \log(N!) \sim N \log N.$$

Second analysis: Compute the first term in Eq. 1, $N! \approx \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$. This requires several steps, the first of which is to consider the **gamma function**

$$\Gamma(N+1) \equiv \int_0^\infty x^N e^{-x}\, dx.$$

Show that $\Gamma(N+1) = N!$ for integer $N \geq 0$. In other words, derive the Euler integral (of the second kind)

$$N! = \int_0^\infty x^N e^{-x}\, dx. \tag{3}$$

Again, this can be done in multiple ways, including induction with integration by parts or by taking derivatives of

$$\int_0^\infty e^{-ax}\, dx = a^{-1}$$

and then setting $a = 1$.

[1] James Stirling computed the $\sqrt{2\pi}$ while Abraham de Moivre derived the expansion in powers of $1/N$. An interesting aspect of this expansion is that it is **asymptotic** — it has a vanishing radius of convergence but can provide precise approximations if truncated at an appropriate power.

The next step in this second analysis is to approximate the gamma function as a gaussian integral. Show that the integrand $x^N e^{-x} = \exp[N \log x - x]$ of Eq. 3 is maximized at $x = N$.

For $N \gg 1$, the integrand is sharply peaked around this maximum at $x = N$. You can check this for yourself or take it as given. We can therefore focus on a small region around this peak by changing variables to $y \equiv x - N$ and considering $\left|\frac{y}{N}\right| \ll 1$. Expand the $\log x$ in the integrand, up to and including terms quadratic in $\frac{y}{N}$. You should be left with the desired result, except for the following factor, which can be approximated by a gaussian integral (note the lower bound of integration):

$$\int_{-N}^{\infty} e^{-y^2/(2N)}\, dy \approx \int_{-\infty}^{\infty} e^{-y^2/(2N)} = \sqrt{2\pi N}.$$

The error introduced by extending the integration from $(-N, \infty)$ to $(-\infty, \infty)$ is exponentially small and could be captured by computing the series of corrections suppressed by powers of $\frac{1}{N}$ in Eq. 1.

This leads us to the **third analysis**: Compute some of the leading power-suppressed corrections in Eq. 1. That is, determine the coefficients A, B, etc. Again, there are many ways to achieve this, including higher-order expansions of the $\log x$ considered above. One pleasant approach is to compare $N!$ and $(N+1)!$, now that we have derived the series prefactor $\sqrt{2\pi N}\left(\frac{N}{e}\right)^N$.