MATH327: Statistical Physics

Tuesday, 7 May 2024

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Something to consider

We have used non-interacting phonons as an approximation to analyze complicated coupled atoms in solids.

Can we do something similar for the simpler interactions among spins in the Ising model?

Recap

Ising model $E = -\sum_{(jk)} s_j s_k - H \sum_n s_n$

Magnetization $\langle m \rangle = \frac{1}{N\beta} \frac{\partial}{\partial H} \log Z$ is order parameter

distinguishish high-T disordered phase $\langle |m| \rangle \to 0$

vs. low-T ordered phase $\langle |m| \rangle \to 1$

$N \to \infty$ discontinuity $\longleftrightarrow$ phase transition

Next: Simple approximation

Note $\langle m \rangle = \frac{1}{N} \sum_n \langle s_n \rangle$ is (mean) average value of spin ∟config.-independent

Expand $s_j s_k = \left[(s_j - \langle m \rangle) + \langle m \rangle \right] \times \left[(s_k - \langle m \rangle) + \langle m \rangle \right]$

$= (s_j - \langle m \rangle)(s_k - \langle m \rangle) + (s_j + s_k)\langle m \rangle - \langle m \rangle^2$

negligible

Suppose on average only small fluctuations around mean spin

$$\rightarrow E \approx -\sum_{(j,k)} \left[(s_j+s_k)\langle m\rangle - \langle m\rangle^2\right] - H\sum_n s_n$$

each spin appear 2d times in sum over $d\cdot N$ links

$$E_{MF} = d\cdot N\langle m\rangle^2 - \underbrace{(2d\langle m\rangle + H)}_{H_{eff}}\sum_n s_n$$ is <u>mean-field approx.</u>

Effective magnetic field averaging over 2d n.n. of s_n

Upon flipping $s_j \rightarrow -s_j$

$$\boxed{\begin{aligned} E_{MF} &= d\cdot N\langle m\rangle^2 - H_{eff}\Big(s_j + \sum_{k\neq j} s_k\Big) \\ &\rightarrow d\cdot N\langle m\rangle^2 - H_{eff}\Big(-s_j + \sum_{k\neq j} s_k\Big) \end{aligned}\ \Bigg\}\ \Delta E_j = 2H_{eff}\, s_j}$$

Independent of s_k for $k\neq j$ → non-interacting!

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Remnant of interactions in H_{eff}

Canonical part. func. factorizes!

$$Z_{MF} = \sum_{\{s_n\}} \exp\Big[-\beta d\cdot N\langle m\rangle^2 + \beta H_{eff}\sum_n s_n\Big]$$

$$= \exp\left(-\beta d\cdot N\langle m\rangle^2\right)\Big(\sum_{s_1=\pm1} e^{\beta H_{eff} s_1}\Big)\cdots\Big(\sum_{s_N=\pm1} e^{\beta H_{eff} s_N}\Big)$$

$$= C\left(2\cosh(\beta H_{eff})\right)^N$$

$$= C\Big(2\cosh\big[\beta(\underbrace{2d\langle m\rangle + H}_{\propto \frac{\partial}{\partial H}\log Z})\big]\Big)^N$$

Demand $$\langle m\rangle = \frac{1}{N\beta}\frac{\partial}{\partial H}\log Z_{MF}$$

$$= \frac{N\beta}{N\beta}\,\frac{\sinh(\beta(2d\langle m\rangle + H))}{\cosh(\beta(2d\langle m\rangle + H))} = \tanh(\beta(2d\langle m\rangle + H))$$

→ <u>self-consistency condition</u> for mean-field approx

Plot $\langle m \rangle$ and $\tanh(\beta(2d\langle m\rangle + H))$ vs. $\langle m \rangle$

Find intersection(s)

Example: $d=2$, $H=0$, $T=4 \rightarrow \beta = \frac{1}{4}$

$\langle m \rangle = 0 \rightarrow$ disordered phase

Turn on $H \neq 0 \rightarrow$ shifts tanh (2d, $T=4$, $H=\pm 2$ $\rightarrow m_0 \approx 0.88$)

Intersection at $\langle m \rangle = \pm m_0 \neq 0$

$\rightarrow$ ordered phase, aligned w/ mag. field

Reduce temperature $\rightarrow$ larger β in tanh

$\rightarrow$ faster change in tanh vs. $\langle m \rangle$

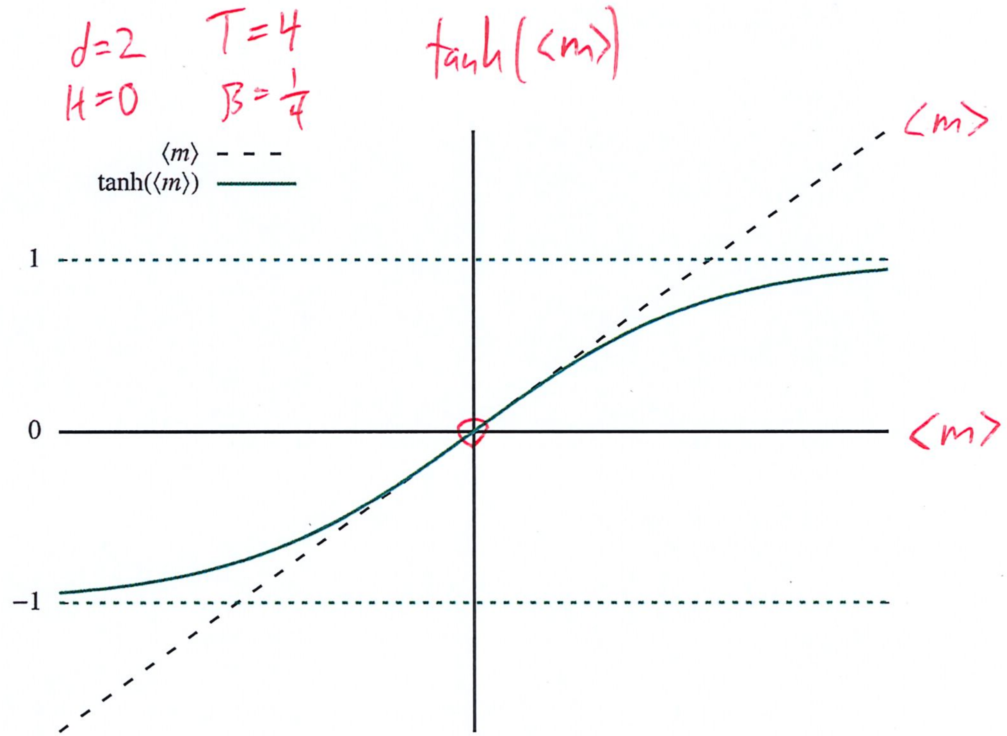
$\rightarrow \langle m \rangle \approx \pm 1$ (max. magnitude)

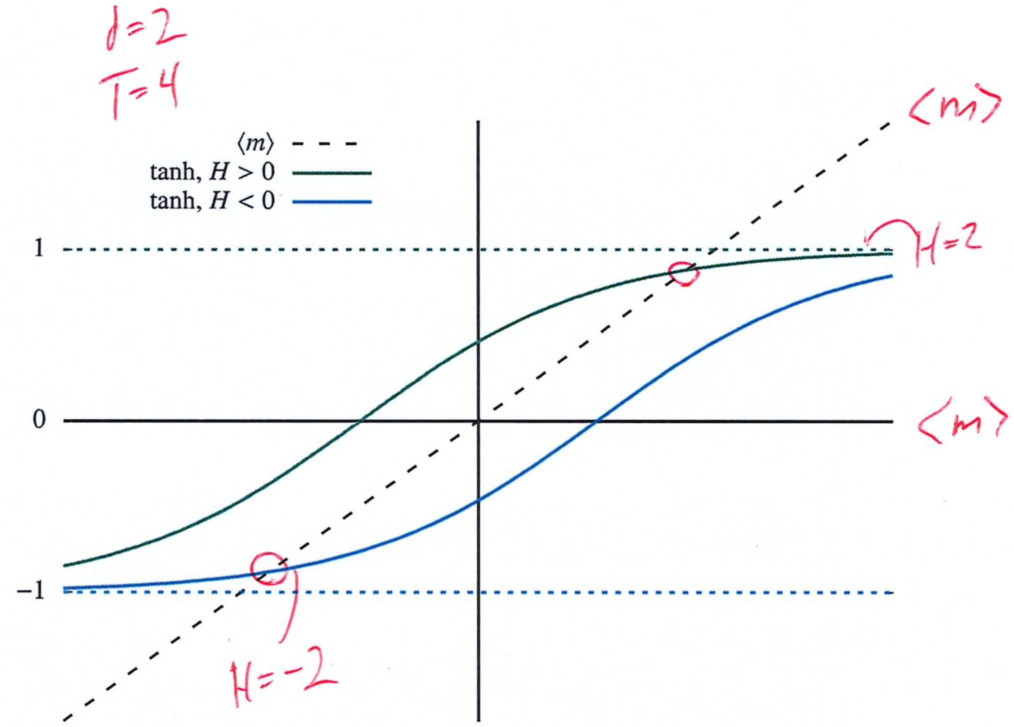
Expect low-T ordered phase even with $H=0$

Compare $T=2, 4, 8$ $\quad\langle m \rangle = 0$ always possible

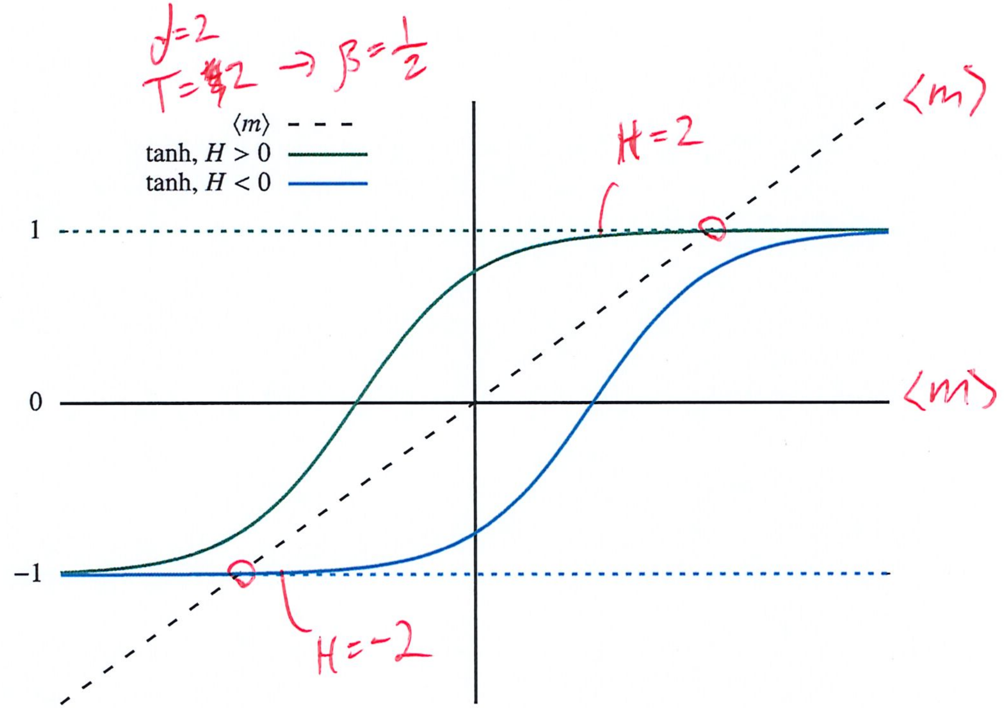
For lower T, additional $\langle |m| \rangle = m_0 > 0$, $m_0 \rightarrow 1$ as $T \rightarrow 0$

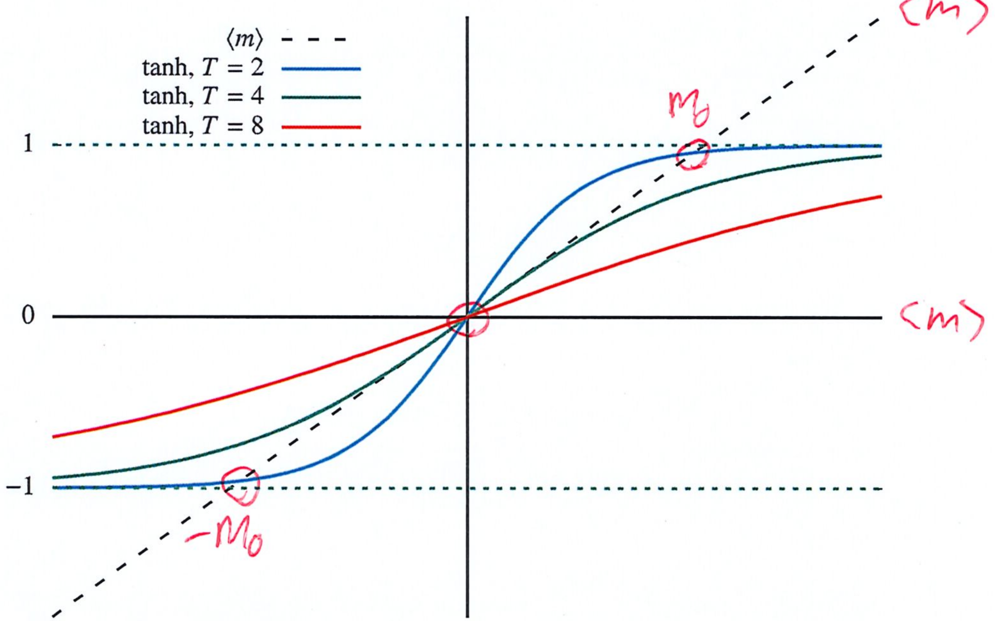
Which of these solutions to the self-consistency condition is the true solution realized by the system


d=2 T=4
H=0 β=1/4
tanh(⟨m⟩)
⟨m⟩
tanh(⟨m⟩)
1
0
−1
⟨m⟩
⟨m⟩



$d=2$
$T=4$
$\langle m \rangle$
tanh, $H > 0$
tanh, $H < 0$
1
0
−1
$\langle m \rangle$
$H=2$
$\langle m \rangle$
$H=-2$


J=2
T=2 → β=1/2
⟨m⟩
tanh, H > 0
tanh, H < 0
1
0
−1
H=2
H=−2
⟨m⟩
⟨m⟩


⟨m⟩
tanh, T = 2
tanh, T = 4
tanh, T = 8
1
0
−1
m0
−m0
⟨m⟩
⟨m⟩