Thu 2 May 64 04 81

Debye model of solids

Coupled atoms → non-interacting phonons

Phonons gas → Planck spectrum (dropping all constants)

$$\langle E \rangle \propto \int_0^{\omega_{max}} \frac{\omega^3}{e^{\beta\hbar\omega}-1}\, d\omega$$

$$x = \beta\hbar\omega = \frac{\hbar\omega}{T} \qquad x_{max} \propto \frac{\omega_{max}}{T}$$

Absorb all constants into "Debye temp."

$$T_D \propto \sqrt[3]{N/V}$$

$$\langle E \rangle \propto T^4 \int_0^{T_D/T} \frac{x^3}{e^x-1}\, dx$$

High T: $\frac{T_D}{T} \ll 1$ → integrating over $0 \le x \ll 1$

$$e^x - 1 \approx x$$

$$\langle E \rangle \propto T^4 \int_0^{T_D/T} \frac{x^3}{x}\, dx \propto T^4 \left(\frac{T_D}{T}\right)^3 \propto T$$

$$c_V = \frac{\partial}{\partial T}\langle E \rangle = \text{const.}$$

same as Einstein solid

Low T: $\frac{T_D}{T} \gg 1$ $\quad \int_0^{T_D/T} \frac{x^3}{e^x-1}\, dx \approx \Gamma(4)\zeta(4) = \text{const.}$

$$\langle E \rangle \propto T^4 \rightarrow c_V \propto T^3 \checkmark$$

Electron (fermion) gas

$$\langle E\rangle_F \propto \int_0^\infty F(E)\,E^{3/2}\,dE = \int_0^\infty \frac{E^{3/2}}{e^{\beta(E-\mu)}+1}\,dE$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$u = F(E) \qquad du = \frac{dF}{dE}\,dE$$

$$dv = E^{3/2}\,dE \qquad v = \frac{2}{5}E^{5/2}$$

Boundary term $uv \propto E^{5/2}F(E)$

$\rightarrow 0$ as $E\rightarrow 0$, $0 < \frac{1}{e^{-\beta\mu}+1} < 1$

$\rightarrow 0$ as $E\rightarrow\infty$, $F(E)\sim\frac{1}{e^{\beta E}}$

$$\langle E\rangle_F \propto \int_0^\infty \frac{dF}{dE}E^{5/2}\,dE$$

$$\frac{dF}{dE} = \frac{d}{dE}\left(e^{\beta(E-\mu)}+1\right)^{-1} \qquad x=\beta(E-\mu)$$

$$= \frac{-e^x\beta}{(e^x+1)^2}$$

$$\langle E\rangle_F \propto \int_0^\infty \frac{e^x}{(e^x+1)^2}E^{5/2}\,d(\beta E) = \int_{-\beta\mu}^{\infty} \frac{e^x}{(e^x+1)^2}E^{5/2}\,dx$$

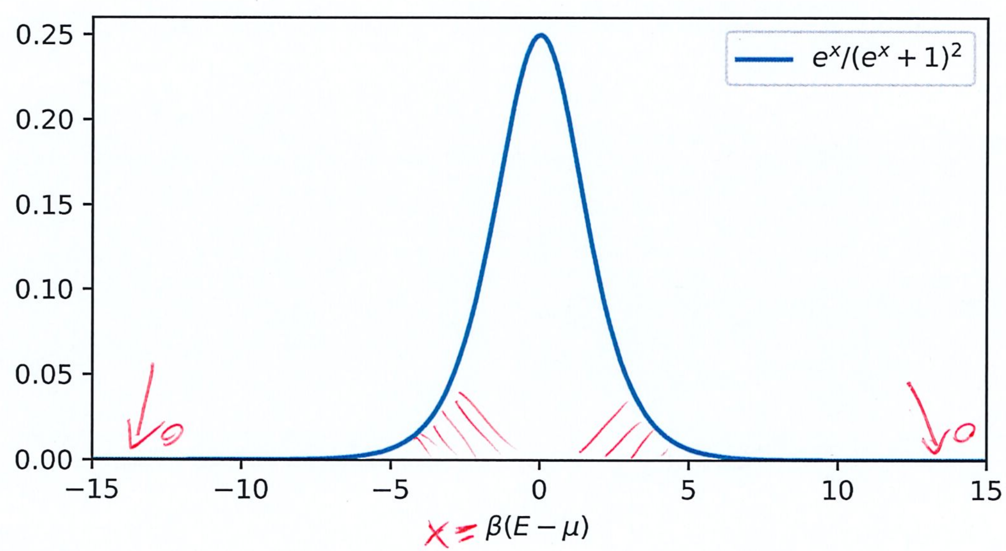
$x \ll -1$: $\frac{e^x}{(e^x+1)^2} \approx \frac{e^x}{1} \ll 1 \rightarrow$ extend range of integration

$x \gg 1$: $\frac{e^x}{(e^x+1)^2} \approx \frac{e^x}{e^{2x}} = \frac{1}{e^x} \ll 1$

$\rightarrow$ Integrand peaked around $x=0 \leftrightarrow E=\mu$

Need to express $E^{5/2}$ in terms of $x=\beta(E-\mu)=\frac{E-\mu}{T}$

$$\frac{e^x}{(e^x+1)(e^x+1)} = \frac{1}{(e^x+1)(1+e^{-x})}$$



Taylor expanded around $E = \mu$

$$E^{5/2} = \mu^{5/2} + \frac{5}{2}(E-\mu)\mu^{3/2} + \frac{15}{8}(E-\mu)^2 \mu^{1/2} + \ldots$$

$$= \mu^{5/2} + \frac{5}{2} xT \mu^{3/2} + \frac{15}{8}(xT)^2 \mu^{1/2} + \ldots$$

$$\langle E \rangle_F \approx \underbrace{A \int_{-\infty}^{\infty} \frac{e^x}{(e^x+1)^2} dx}_{T\text{-indep.}} + BT \underbrace{\int_{-\infty}^{\infty} \frac{e^x x}{(e^x+1)^2} dx}_{0 \text{ since odd}} + CT^2 \underbrace{\int_{-\infty}^{\infty} \frac{e^x x^2 dx}{(e^x+1)^2}}_{\text{const.}}$$

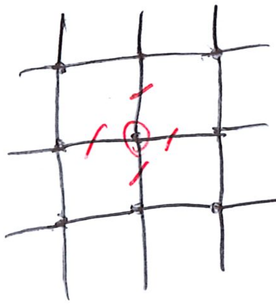
$$C_V = \frac{\partial}{\partial T} \langle E \rangle_F \propto T \checkmark$$

Lattices – generalize spins in a row (dist'able)

Each spin interacts only with its nearest neighbours (n.n.)

$$E = -J \sum_{\langle jk \rangle} s_j s_k \quad - H \sum_{s_n} s_n \qquad s_n = \pm 1 \qquad \text{Ising model}$$

1d lattice:  $c = 2$ n.n. per site
coordination #

2d square  $c = 4 \rightarrow 2d$ in d dimensions

Simplify lattice w/ periodic boundary conditions

$\rightarrow$ d' dim'l torus (zero gaussian curvature, equally spaced sites)

Generalizations

Honeycomb $c=3 \to d+1$

Triangular $c=6 \to 2(d+1)$

Kagome (2d) $c=4$ but not square

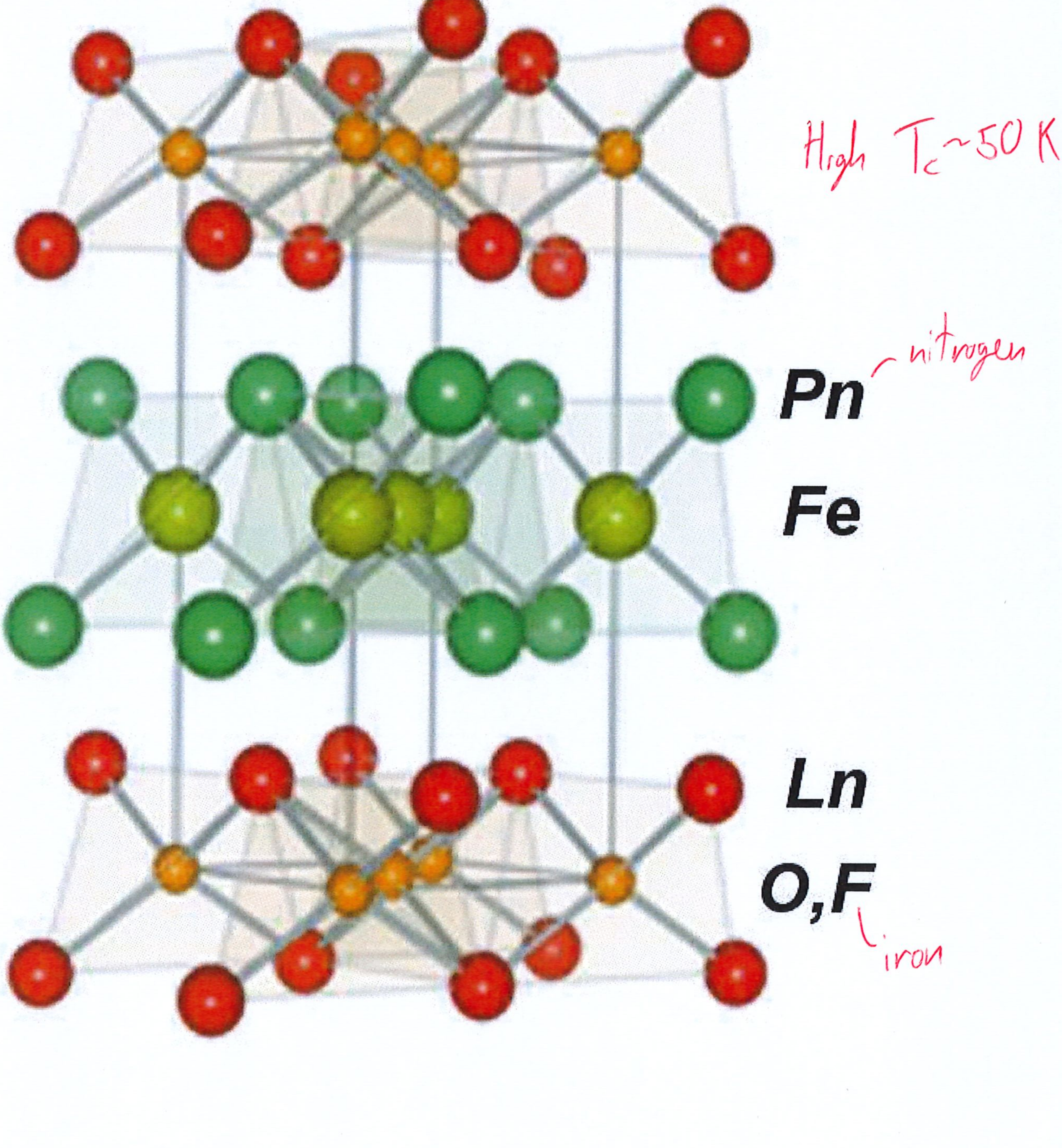
What are Ising model ground state for $J>0$ vs. $J<0$

Fully connected lattice

How many n.n. pairs as function of N

Solve Ising model on fully connected lattice

closed-form expression for part. func. Z


High Tc ~50 K
Pn
nitrogen
Fe
Ln
O,F
iron