MATH327: Statistical Physics

Friday, 26 April 2024

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Something to consider

When we move on from the photon gas to gases of fermions, what effects do you expect Pauli exclusion will have?

For what temperatures and chemical potentials do you expect these effects to be significant?

Recap

Photon gas, Planck spectrum, sun, CMB

Internal energy $\langle E\rangle_{ph} = \frac{\Gamma(4)\zeta(4)}{\pi \hbar^3 c^3} VT^4 = \frac{\pi^{2}}{15\hbar^3 c^3} VT^4$

Compares w/classical, canonical, non-rel

$$\langle E\rangle = \frac{3}{2} NT$$

$\rightarrow$ compute $\langle N\rangle_{ph} = \frac{-\partial}{\partial\mu}\Phi_{ph}\Big|_{\mu=0}$

$$\langle N\rangle_{ph} = \frac{-VT}{c^3\pi^2}\int_0^\infty w^2 \frac{\partial}{\partial\mu}\log\left(1 - e^{-\beta\hbar w}e^{\beta\mu}\right) dw\Big|_{\mu=0}$$

$$= \frac{+V\cancel{T}}{c^3\pi^2}\int_0^\infty \frac{w^2\left(+e^{-\beta\hbar w}e^{\beta\mu}\right)\cancel{\beta}}{1-e^{-\beta\hbar w}e^{\beta\mu}} dw\Big|_{\mu=0}$$

$$= \frac{V}{c^3\pi^2}\int_0^\infty \frac{w^2}{e^{\beta\hbar w}-1} dw = \frac{V}{c^3\pi^2}\left(\frac{T}{\hbar}\right)^3\int_0^\infty \frac{x^2}{e^x-1} dx$$

$\hookrightarrow \Gamma(3)\zeta(3) = 2\zeta(3)$

$x = \beta\hbar w$

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$$\boxed{\langle N\rangle_{ph} = \frac{2\zeta(3)}{\pi^2\hbar^3c^3} VT^3 \propto \frac{\langle E\rangle_{ph}}{T}}$$ — as before $\langle E\rangle_{ph} \propto \langle N\rangle_{ph} T$

Constant factor $$\left(\frac{\pi^2}{15\cancel{\hbar^3c^3}}\right)\left(\frac{\pi^2\cancel{\hbar^3c^3}}{2\zeta(3)}\right) = \frac{6\pi^4/90}{2\zeta(3)}$$
$$= \frac{\Gamma(4)\zeta(4)}{\Gamma(3)\zeta(3)} \simeq 2.7$$

Radiation pressure $$P_{ph} = -\frac{\partial}{\partial V}\langle E\rangle_{ph}\Big|_{S_{ph}}$$

Need constant entropy $$S_{ph} = \frac{\langle E\rangle_{ph} - \Phi_{ph}}{T}$$

$$\frac{\Phi_{ph}}{T} = \frac{V}{c^3\pi^2}\int_0^\infty \omega^2 \log\left(1-e^{-\beta\hbar\omega}\right)d\omega = \frac{VT^3}{\pi^2\hbar^3c^3}\int_0^\infty x^2\log\left(1-e^{-x}\right)dx$$
$\to -2\zeta(4) = -\frac{\pi^4}{45}$

$$S_{ph} = VT^3\left(\frac{\pi^2}{\hbar^3c^3}\right)\left(\frac{1}{15}+\frac{1}{45}\right) = \underbrace{VT^3}_{\text{constant when } T = bV^{-1/3}}\left(\frac{4\pi^2}{45\hbar^3c^3}\right)$$

$$\boxed{\begin{aligned} P_{ph} &= \frac{-\pi^2}{15\hbar^3c^3}\frac{\partial}{\partial V}\left(b^4V^{-1/3}\right) \\ &= \frac{1}{3V}\left(\frac{\pi^2}{15\hbar^3c^3}b^4V^{-1/3}\right) = \frac{\langle E\rangle_{ph}}{3V}\end{aligned}}$$

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$$P_{ph}V = \frac{1}{3}\langle E\rangle_{ph} = \left(\frac{\cancel{\Gamma(3)}}{\cancel{\Gamma(4)}}\right)\left(\frac{\cancel{\Gamma(4)}\zeta(4)}{\cancel{\Gamma(3)}\zeta(3)}\right)\langle N\rangle_{ph}T = \frac{\pi^4}{90\zeta(3)}\langle N\rangle_{ph}T$$ 0.9004

Photon gas equation of state similar to ideal gas law

Ideal gases of fermionic particles

$n_\ell = 0,1 \longrightarrow \Phi_F = -T\sum_\ell \log\left(1+e^{-\beta(E_\ell-\mu)}\right)$

Back to non-rel. $$E = \frac{\hbar^2\pi^2}{2mL^2}\underbrace{\left(k_x^2+k_y^2+k_z^2\right)}_{k^2}$$ $k_{x,y,z} = 1,2,3\ldots$

$$\Phi_F = -2T \sum_{\vec{k}} \log\left[1 + \exp\left(\frac{-\hbar^2 \pi^2 \hat{k}^2}{2mL^2 T} + \frac{\mu}{T}\right)\right]$$

two "spin" states per $\vec{k}$

Same simplification: Integrate over $\hat{k}_{x,y,z} > 0$ in spherical coords

$$\Phi_F \simeq -\pi T \int_0^\infty \hat{k}^2 \log\left[1 + \exp\left(\frac{-\hbar^2 \pi^2 \hat{k}^2}{2mL^2 T} + \frac{\mu}{T}\right)\right] d\hat{k}$$

Change variables to energy $\hat{k} = \frac{L\sqrt{m}}{\pi \hbar}\sqrt{2E}$

$$d\hat{k} = \frac{L\sqrt{m}}{\pi\hbar}\frac{dE}{\sqrt{2E}}$$

$$\Phi_F \simeq -\pi T \left(\frac{L\sqrt{m}}{\pi\hbar}\right)^3 \int_0^\infty (2E) \log\left(1 + e^{-\beta(E-\mu)}\right) \frac{dE}{\sqrt{2E}}$$

$$= -VT\frac{\sqrt{2m^3}}{\pi^2\hbar^3}\int_0^\infty \log\left(1 + e^{-\beta(E-\mu)}\right)\sqrt{E}\, dE$$

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Next simplification: Consider low temperatures

Start w/average particle density

$$\frac{\langle N \rangle_F}{V} = \frac{-\partial}{\partial \mu}\frac{\Phi_F}{V} = \cancel{T}\frac{\sqrt{2m^3}}{\pi^2\hbar^3}\int_0^\infty \frac{e^{-\beta(E-\mu)}\cancel{\beta}}{1+e^{-\beta(E-\mu)}}\sqrt{E}\, dE$$

$$= \frac{\sqrt{2m^3}}{\pi^2\hbar^3}\int_0^\infty \frac{1}{e^{\beta(E-\mu)}+1}\sqrt{E}\, dE$$

$$= \frac{\sqrt{2m^3}}{\pi^2\hbar^3}\int_0^\infty F(E)\sqrt{E}\, dE$$

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$F(E) = \frac{1}{e^{+\beta(E-\mu)}+1} \sim \langle n_\ell \rangle_{FD}$ is Fermi function

Assume $\mu > 0$, threshold $F(E=\mu) = \frac{1}{2}$ for all T

$E > \mu \rightarrow$ exponentially suppressed $F(E) \rightarrow 0$

$E < \mu \rightarrow$ exponentially approach $F(E) \rightarrow 1$

Lower T (larger β) $\rightarrow$ faster approach to limits

Low~~er~~-T simpli~~f~~fication

Approximate $F(E)$ as a step function ~~$F(E) = \begin{cases} 1 & \text{for } 0 \le E < \\ 0 \end{cases}$~~

$$F(E) = \begin{cases} 1 & 0 \le E < \mu \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\langle N \rangle_F}{V} = \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\infty F(E) \sqrt{E} \, dE \approx \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\mu \sqrt{E} \, dE$$

$$= \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \left(\frac{2}{3} E^{3/2} \right)_0^\mu = \frac{(2m\mu)^{3/2}}{3\pi^2 \hbar^3}$$

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Leading order of expansion in powers of $\frac{T}{\mu} \ll 1$

Sommerfeld

Physical picture: All energy levels with $E_\ell < \mu$ occupied, $n_\ell = 1$

$E_\ell \propto k^2 \rightarrow \langle N \rangle_F$ fills octant of sphere with radius $\sqrt{\mu}$

$\propto \mu^{3/2}$

For $T \rightarrow 0$, max energy is Fermi energy

$$E_F = \mu = \frac{\hbar^2}{2m} \left(3 \frac{\langle N \rangle_F}{V} \pi^2 \right)^{2/3}$$

Fermion gas internal energy

Just extra E in integral

$$\frac{\langle E \rangle_F}{V} = \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\infty E \, F(E) \sqrt{E} \, dE \approx \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\mu E^{3/2} \, dE$$

$$= \frac{(2m\mu)^{3/2}}{5\pi^2 \hbar^3} \mu = \frac{3}{5} \mu \frac{\langle N \rangle_F}{V} \neq 0 \text{ as } T \rightarrow 0$$

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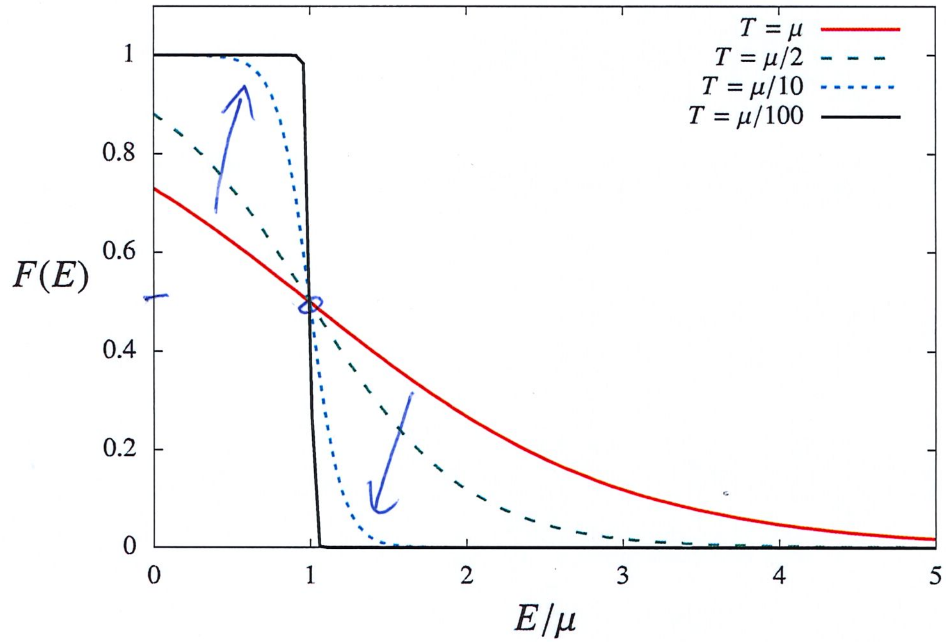
$$F(E) = \frac{1}{\exp\left[\frac{\mu}{T}\left(\frac{E}{\mu}-1\right)\right]+1} = \frac{1}{\left[\exp\left(\frac{E}{\mu}-1\right)\right]^{\mu/T}+1}$$



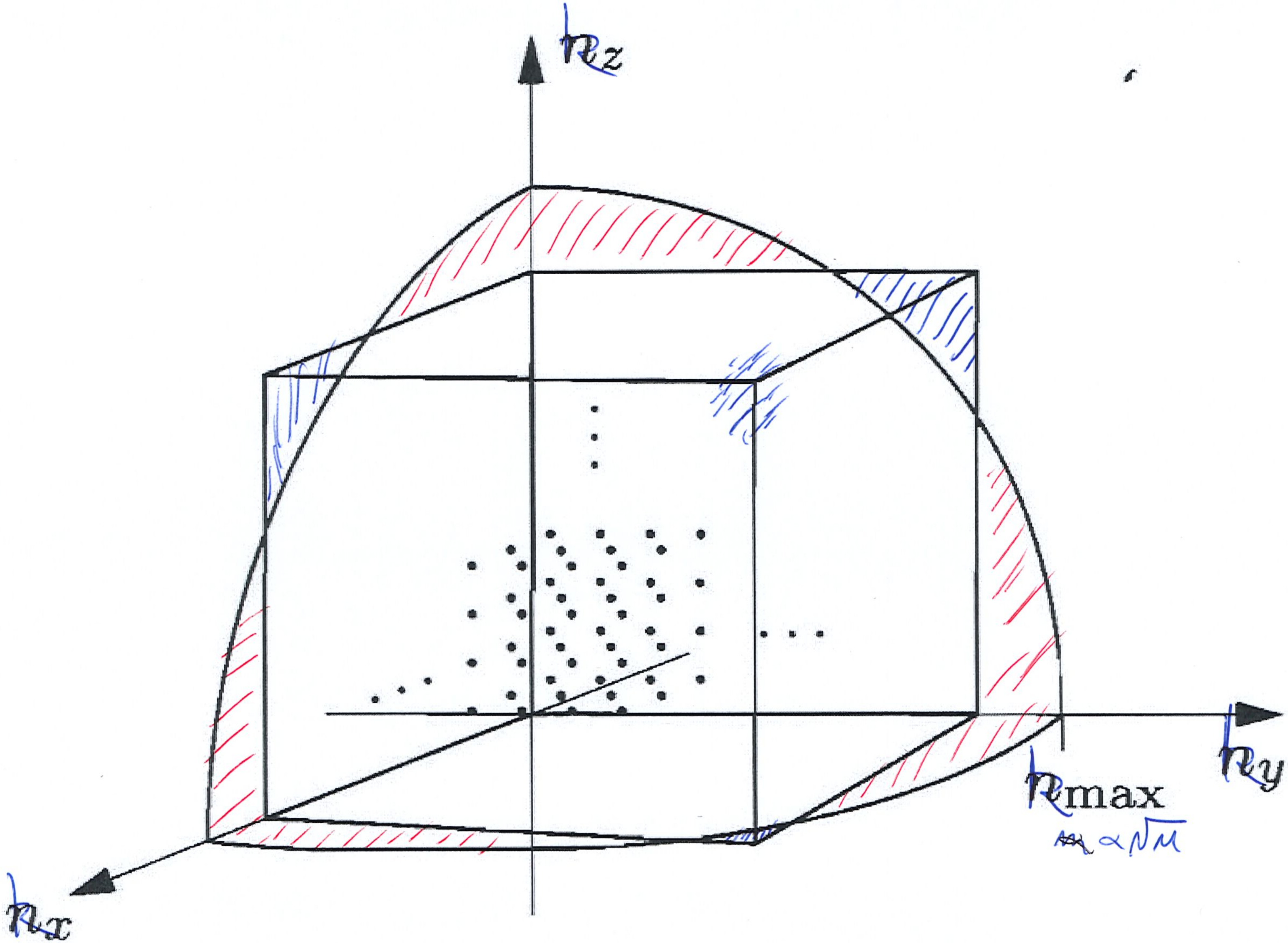
μ/T

1

1/2

1/10

1/100



n_z
n_y
n_x
n_{max}
$\propto \sqrt{N}$

Average $T \to 0$ energy per particle $\frac{\langle E \rangle_F}{\langle N \rangle_F} = \frac{3}{5} E_F$

Pauli exclusion $\to$ fermions fill energy levels up to E_F at $T=0$

Tutorial: For low $T>0$, integrate $\int_0^\infty F(E)\, E^{3/2}\, dE$ by parts

Argue boundary term vanishes
Remaining integrand sharply peaked around $E \approx \mu$
Taylor expand $\to C_V \propto T$

Justify $\mu > 0$ for low-temperature fermion gas

$$\mu = \frac{\partial E}{\partial N}\bigg|_{S,V}$$

For $T \to 0$, $S = -\sum_i p_i \log p_i = 0$ constant even for $\Delta N \neq 0$

single micro-state, $p_1 = 1$

Add particles $\Delta N > 0 \to$ fill next energy level above $E_F = \mu$

$\Delta E \approx (\Delta N) E_F > 0 \to \mu > 0$ ✓

Sommerfeld expansion for higher temperatures $\to \mu < 0$ consistent classical limit

<u>Pressure</u> $P_F = \frac{-\partial}{\partial V} \langle E \rangle_F \Big|_{S_F} = \frac{-\partial}{\partial V}\left(\frac{3}{5}\mu \langle N \rangle_F\right)\Big|_{S_F}$

Again $T \to 0$ gives constant $S_F = 0$ from single micro-state

Insert $E_F = \mu = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{\langle N \rangle_F}{V}\right)^{2/3}$

$$P_F = \frac{-3}{5}\left(\frac{\hbar^2}{2m}\right)(3\pi^2)^{2/3} \langle N \rangle_F^{5/3} \frac{\partial}{\partial V} V^{-2/3} = \frac{2}{3V}\langle E \rangle_F$$

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Three expressions:

$$P_F = \frac{2}{3}\frac{\langle E\rangle_F}{V} = \frac{2}{5}\mu\frac{\langle N\rangle_F}{V} = \frac{\hbar^2}{5m}(3\pi^2)^{2/3}\rho_F^{5/3}.$$

ρ_F density $\frac{\langle N\rangle_F}{V}$

$P_F \neq 0$ as $T \to 0$

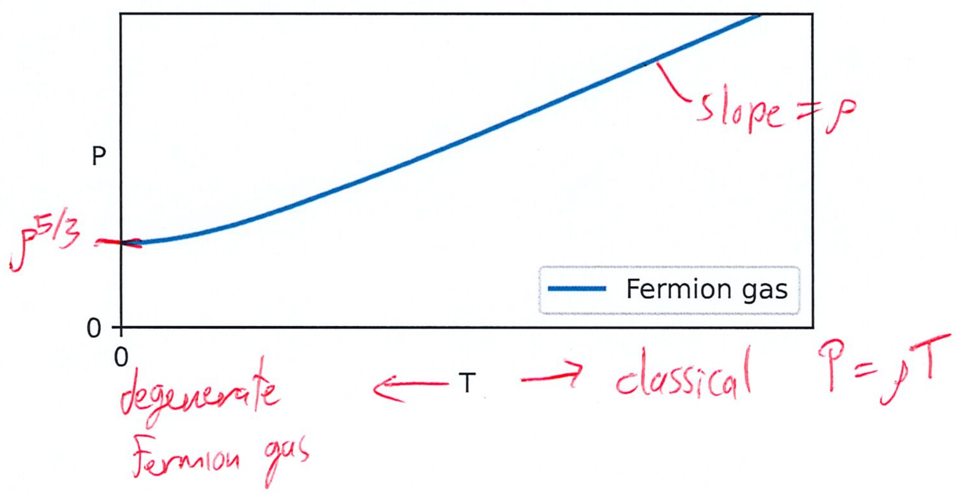
"Degeneracy pressure" from Pauli exclusion

(unrelated to degenerate energy levels)

High-T classical limit: $P = \frac{N}{V}T = \rho T$

Degeneracy pressure matters when $T \ll E_F$ even if $T > 0$

$E_F \propto \rho^{2/3}$


P
0
0
T
Fermion gas
$\rho^{5/3}$
slope $= \rho$
degenerate
Fermion gas
classical
$P = \rho T$