MATH327: Statistical Physics

Tuesday, 16 April 2024

99 35 89

Something to consider

Pre-break we saw that quantum statistics matters when there would be a high probability of multiple particles occupying the same energy level.

What temperatures and chemical potentials correspond to this?

Logistics

Second HW due 26 April

Exam 14:30 – 16:30 Wed 22 May (YOLC)

Big-picture review

Prob. space → stat. ensembles

Micro-canonical

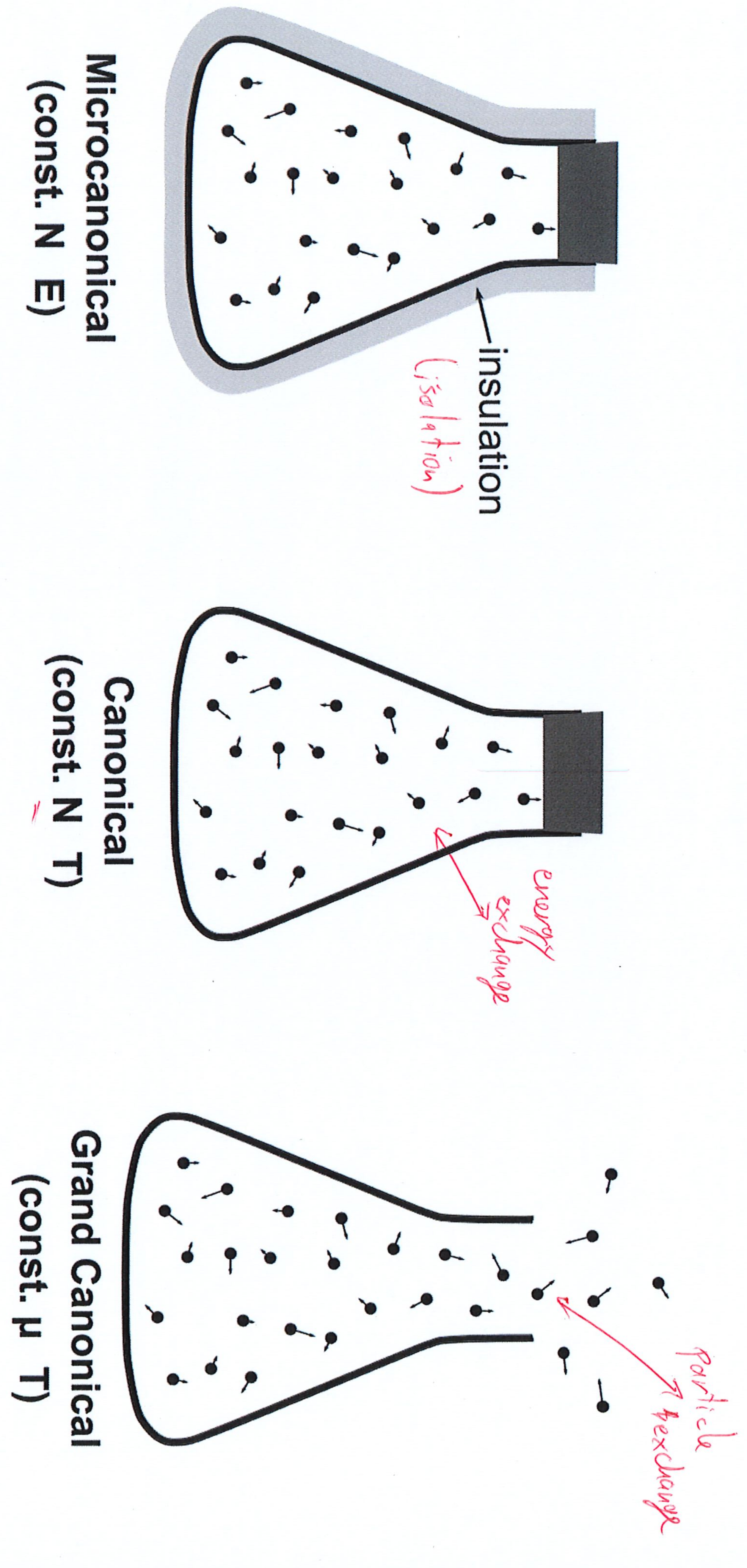
Canonical (ideal gases, therm. cycles)

Grand-canonical (quantum gases ...)

Quantum statistics review

Define micro-states by occupation numbers n_ℓ of discrete energy level ε_ℓ

(not based on single-particle part. func. w/over-counting correction only works if all particles in diff. energy levels) commons.wikimedia.org/wiki/File:Statistical_Ensembles.png



In 3d, two possibilities for occupation numbers

Bosons $n_\ell = 0, 1, 2, \ldots$ } different quantum statistics

Fermions $n_\ell = 0, 1$

Grand-canonical part. func. for bosons

Sum over $n_\ell = 0, 1, 2, \ldots$ for every energy level ε_ℓ

Simplify: System with single ε_0 (energy E_0)

$\boxed{\text{Micro-state energy} \quad E_i = N_i E_0 = n_0 E_0}$ page 95

$$Z_g = \sum_{n_0=0}^{\infty} e^{-\beta(n_0 E_0 - \mu n_0)} = \sum_{n_0} \left[e^{-\beta(E_0-\mu)}\right]^{n_0}$$

$$= \frac{1}{1 - e^{-\beta(E_0-\mu)}}$$

Geometric series only converges for $e^{-\beta(E_0-\mu)} < 1$

$$\beta > 0 \rightarrow E_0 - \mu > 0 \rightarrow E_0 > \mu \checkmark$$

$$E_0 \geq 0 \qquad \mu = -T \left.\frac{\partial S}{\partial N}\right|_E < 0$$

Generalize to ε_ℓ with $\ell = 0, 1, 2, \ldots L$

$\{n_\ell\}$ defines micro-state w_i

Non-interacting particles: $E_i = \sum_\ell n_\ell E_\ell$

$$N_i = \sum_\ell n_\ell$$

$$Z_g = \sum_{n_0} \sum_{n_1} \cdots \sum_{n_L} \exp\left[-\beta \sum_\ell (E_\ell - \mu) n_\ell\right]$$

$$= \left(\sum_{n_0} e^{-\beta(E_0-\mu)n_0}\right)\left(\sum_{n_1} e^{-\beta(E_1-\mu)n_1}\right) \cdots \left(\sum_{n_L} e^{-\beta(E_L-\mu)n_L}\right)$$

(factorization)

$$= \prod_{\ell=0}^{L} \frac{1}{1 - e^{-\beta(E_\ell-\mu)}} \qquad E_\ell > \mu$$

Bose-Einstein g.c. part. func.

For fermions only difference in $n_\ell \in \{0, 1\}$

Single energy level $\varepsilon_0 \to Z_g = \sum_{n_0=0}^{1} e^{-\beta(E_0-\mu)n_0} = 1 + e^{-\beta(E_0-\mu)}$

General $\varepsilon_\ell \to Z_g = \sum_{n_0}\sum_{n_1}\cdots\sum_{n_L} \exp\left[-\beta\sum_{\ell=0}^{L}(E_\ell-\mu)n_\ell\right]$

$$= \left(\sum_{n_0} e^{-\beta(E_0-\mu)n_0}\right)\cdots\left(\sum_{n_L} e^{-\beta(E_L-\mu)n_L}\right)$$

$$= \prod_\ell \left(1 + e^{-\beta(E_\ell-\mu)}\right)$$

Fermi-Dirac g.c. part. func.

Collect results for $\Phi = -T \log Z_g$

Maxwell-Boltzmann $\Phi_{MB}(T,\mu) = -T \log\left[\prod_\ell \exp\left(e^{-\beta(E_\ell-\mu)}\right)\right]$

$$= -T\sum_\ell e^{-\beta(E_\ell-\mu)}$$

$$\Phi_{BE} = T\sum_\ell \log\left(1 - e^{-\beta(E_\ell-\mu)}\right)$$

$$\Phi_{FD} = -T\sum_\ell \log\left(1 + e^{-\beta(E_\ell-\mu)}\right)$$

Classical MB statistics should emerge from quantum when $N \ll$ # of accessible energy levels

Recall canonical spin system $\to$ low T exponentially suppresses $p_i \propto e^{-E_i/T}$ for $E_i > E_0$

$\to$ Expect high T for classical emergence ...