Mon 24 Apr

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Logistics:

Monday bank holidays until 15 May

HW due 3 May

Recap

Grand-canonical ideal quantum gases

Photon gas → Planck spectrum good from stars to CMB

Equation of state $P_{ph} V = \frac{\zeta(4)}{\zeta(3)} \langle N \rangle_{ph} T$

Non-rel. fermion gas $\Phi_f = -VT \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\infty \log\left(1 + e^{-\beta(E-\mu)}\right) \sqrt{E}\, dE$

Plan

Low-temperatures EoS

Interacting systems

Simplification from low temperature

Considering

$$\boxed{\begin{aligned} \frac{\langle N \rangle_F}{V} &= -\frac{\partial}{\partial \mu} \frac{\Phi}{V} = \frac{\cancel{T}\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\infty \frac{e^{-\beta(E-\mu)}\cancel{\beta}}{1+e^{-\beta(E-\mu)}} \sqrt{E}\, dE \\ &= \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\infty \frac{1}{e^{\beta(E-\mu)}+1} \sqrt{E}\, dE \\ &= \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\infty \underline{F(E)} \sqrt{E}\, dE \end{aligned}}$$

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Fermi function $F(E) = \frac{1}{e^{\beta(E-\mu)}+1}$ like dim'less $\langle n_\ell \rangle$

Assuming $\mu > 0$, threshold at $E = \mu$ where $F(E=\mu) = \frac{1}{2}$ for all T

$E > \mu$ → exponential suppression $F(E) \to 0$

$E < \mu$ → exponentially approach $F(E) \to 1$

Rearrange $F(E) = \frac{1}{\exp\left[\frac{\mu}{T}\left(\frac{E}{\mu}-1\right)\right]+1} = \frac{1}{\left(\exp\left(\frac{E}{\mu}-1\right)\right)^{\mu/T}+1}$

Smaller $\frac{T}{\mu}$ $\rightarrow$ faster approach to limits

Low-temperature simplification: Approximate $F(E)$ as step func.

$$F(E) \approx \begin{cases} 1 & \text{for } E<\mu \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\begin{aligned} \frac{\langle N\rangle_F}{V} &= \frac{\sqrt{2m^3}}{\pi^2\hbar^3}\int_0^\infty F(E)\sqrt{E}\,dE \approx \frac{\sqrt{2m^3}}{\pi^2\hbar^3}\int_0^\mu \sqrt{E}\,dE \\ &= \frac{\sqrt{2m^3}}{\pi^2\hbar^3}\left(\frac{2}{3}E^{3/2}\right)_0^\mu = \frac{(2m\mu)^{3/2}}{3\pi^2\hbar^3} \end{aligned}}$$

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Leading order of expansion in powers of $\frac{T}{\mu} \ll 1$

Sommerfeld

Physical picture: All energy levels with $E_\ell < \mu$ occupied $n_\ell = 1$

$E_\ell \propto k^2 \rightarrow$ # fill (part of) sphere with radius $\sqrt{\mu}$ $\rightarrow \propto \mu^{3/2}$

For $T\to 0$, max. energy is <u>Fermi energy</u>

$$E_F = \mu = \frac{\hbar^2}{2m}\left(3\pi^2\frac{\langle N\rangle_F}{V}\right)^{2/3}$$

Fermion gas <u>internal energy</u>

Just like photon gas, only need extra factor in E in integral

$$\boxed{\begin{aligned} \frac{\langle E\rangle_F}{\cancel{V}} &= \frac{\sqrt{2m^3}}{\pi^2\hbar^3}\int_0^\infty E\,F(E)\sqrt{E}\,dE \rightarrow \frac{\sqrt{2m^3}}{\pi^2\hbar^3}\int_0^\mu E^{3/2}\,dE \\ &= \frac{(2m\mu)^{3/2}}{5\pi^2\hbar^3}\mu = \frac{3}{5}\mu\frac{\langle N\rangle_F}{\cancel{V}} \neq 0 \end{aligned}}$$

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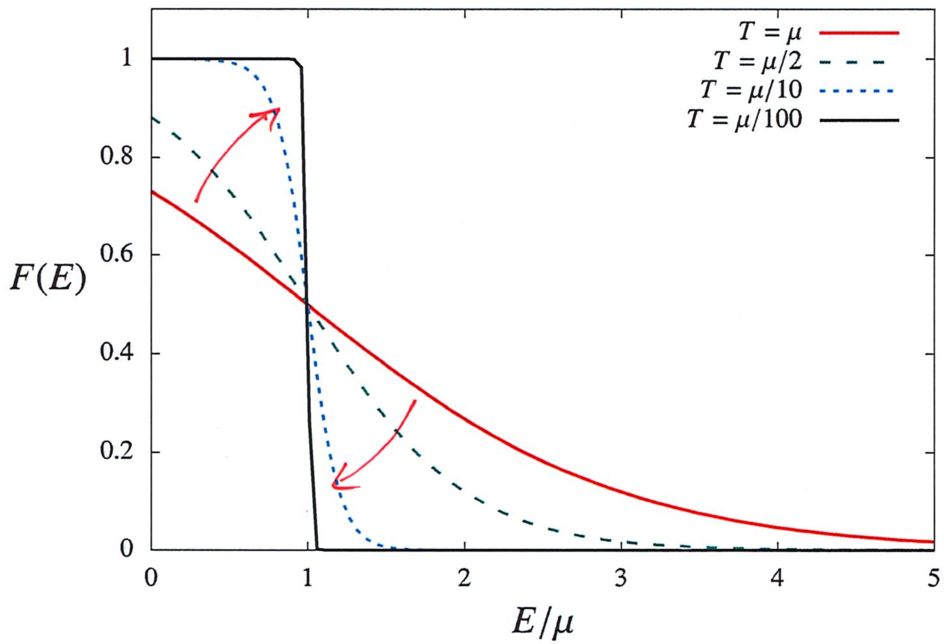
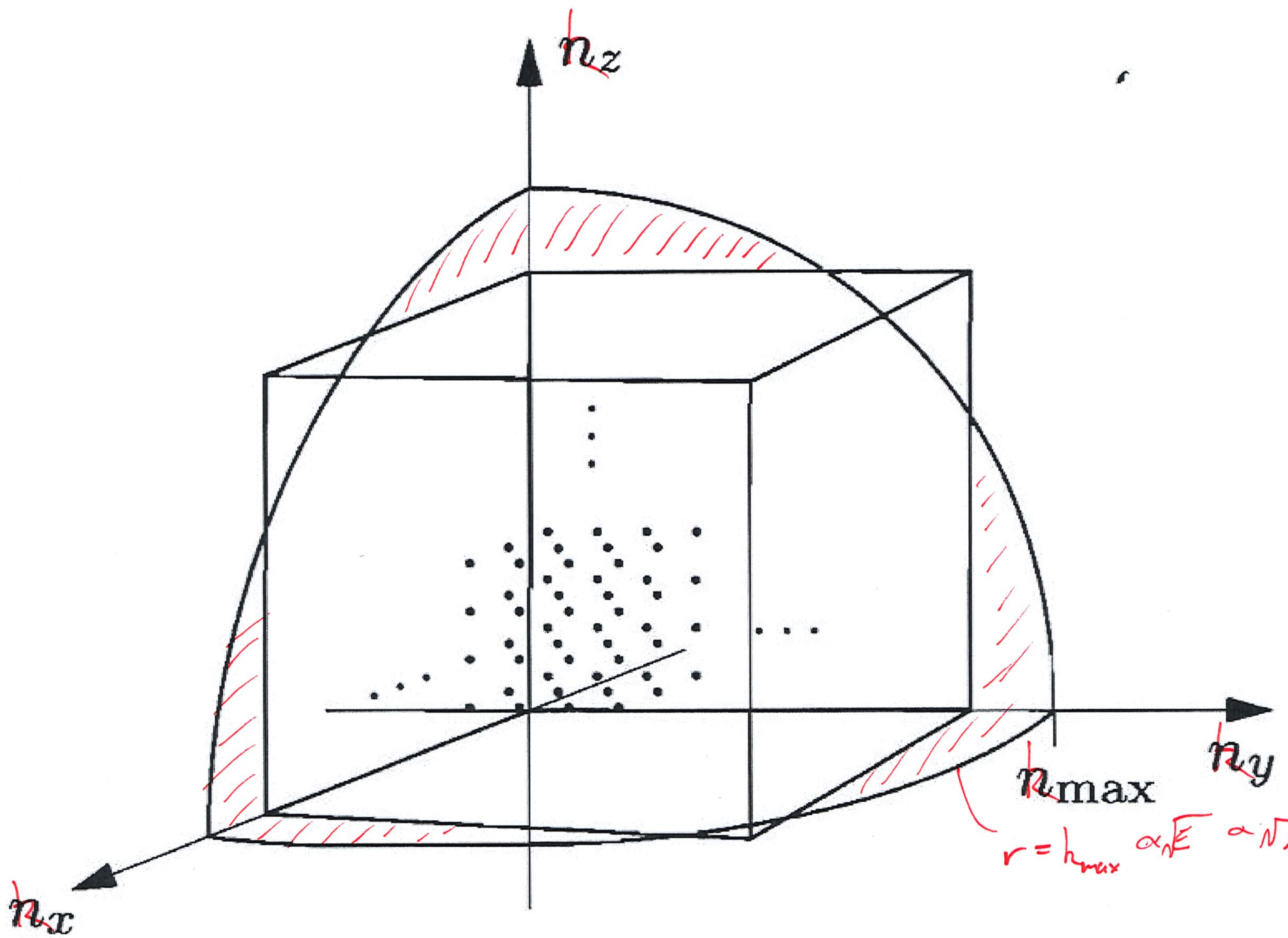
Average $T\to 0$ energy per particle $\frac{\langle E\rangle_F}{\langle N\rangle_F} = \frac{3}{5}E_F$



$\frac{T}{\mu} = 1$
2
10
100



n_z
n_y
n_x
n_{max}
$r = n_{max} \propto \sqrt{E} \propto \sqrt{u}$

Pauli exclusion → Fermions fill energy levels up to E_F

~~M~~ Justify $\mu > 0$ for low-temperature Fermion gas

$$\mu = \frac{\partial E}{\partial N}\Big|_{S,V}$$

Single micro-state (energy levels filled up to E_F)

$$S = -\sum_i p_i \log p_i = 0$$ constant even ~~M~~ if N changes

Add particles, $\Delta N > 0$ → fill energy level above $E_F = \mu$

$$\Delta E = E_F \, \Delta N > 0 \rightarrow \mu > 0$$ ✓

Higher temperatures → $\mu < 0$ for classical limit

→ Sommerfeld expansion pencast

Pressure $$P_F = \frac{-\partial}{\partial V} \langle E \rangle_F \Big|_{N, S_F} = \frac{-\partial}{\partial V}\left(\frac{3}{5}\mu \langle N \rangle_F\right)\Big|_{N, S_F}$$

Again $T \to 0$ gives single micro-state → constant $S_F = 0$

Insert $$E_F = \mu = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{\langle N \rangle_F}{V}\right)^{2/3}$$

$$P_F = -\frac{3}{5}\left(\frac{\hbar^2}{2m}\right)(3\pi^2)^{2/3} \langle N \rangle_F^{5/3} \frac{\partial}{\partial V} V^{-2/3} = \frac{2}{3V} \langle E \rangle_F$$

Three relations:
$$P_F = \frac{2}{3}\frac{\langle E \rangle_F}{V} = \frac{2}{5}\mu \frac{\langle N \rangle_F}{V} = \frac{\hbar^2}{5m}(3\pi^2)^{2/3} \rho_F^{5/3}$$

density $$\rho_F = \frac{\langle N \rangle_F}{V}$$

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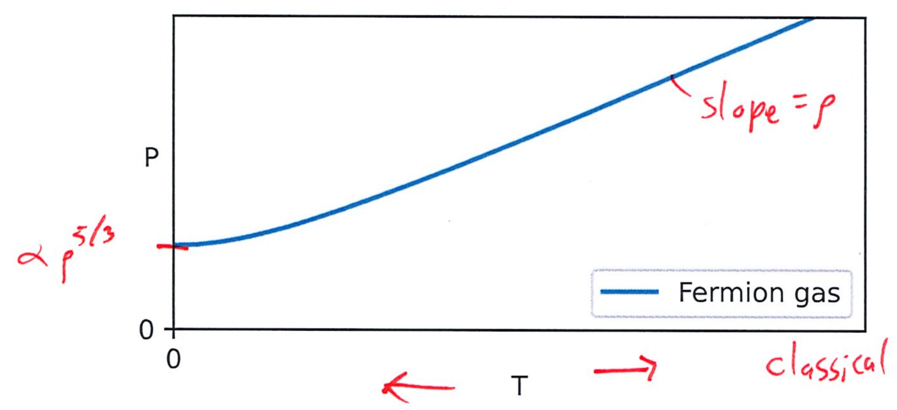
Like energy, non-zero pressure for $T \to 0$, unlike ideal gas law

"degeneracy" pressure from Pauli exclusion
not force between Fermions

High-temperature classical limit → $P = \frac{N}{V} T = \rho T$



$P = \rho T$

Supernovas

Degeneracy pressure matters when $T \ll E_F$ even if $T > 0$

$E_F \propto \rho^{2/3}$

Larger densities $\rightarrow$ larger E_F

Everyday metals have $\rho \sim \frac{N_A \text{ electrons}}{cc} \sim \frac{10^{23} \text{ electrons}}{cc}\left(\frac{100\text{ cm}}{m}\right)^3$

$\sim 10^{29}$ electrons/m^3

$\rightarrow E_F \sim 10^4 K \sim 1 eV \sim 10^{-19} J$

Everyday temperatures $T \ll E_F \rightarrow$ degenerate electron gas

Sun has similar $\rho \sim 10^{30}$ electrons/m^3 (on average), $E_F \sim 10^5 K$

but much higher temperature up to $\sim 10^7 K \gg E_F$ in core

Radiation pressure from fusion hydrogen & helium

heats up sun & balances gravitation force $\rightarrow$ reduce density

After H & He 'fuel' exhausted, less radiation pressure

$\rightarrow$ higher density

White dwarf stars have sun's mass w/earth's radium

(~100 times smaller than sun's)

$\rho \sim 10^6 \rho_{sun} \sim 10^{36}$ electrons/m^3 ~ tonne/cc

$E_F \sim (10^6)^{2/3} E_F^{(sun)} \sim 10^9 K \gg T \sim 10^7 K$

slowly cools to ~$10^3 K$ after ~14 Gyr

$T \ll E_F \rightarrow$ low-temperature electron gas

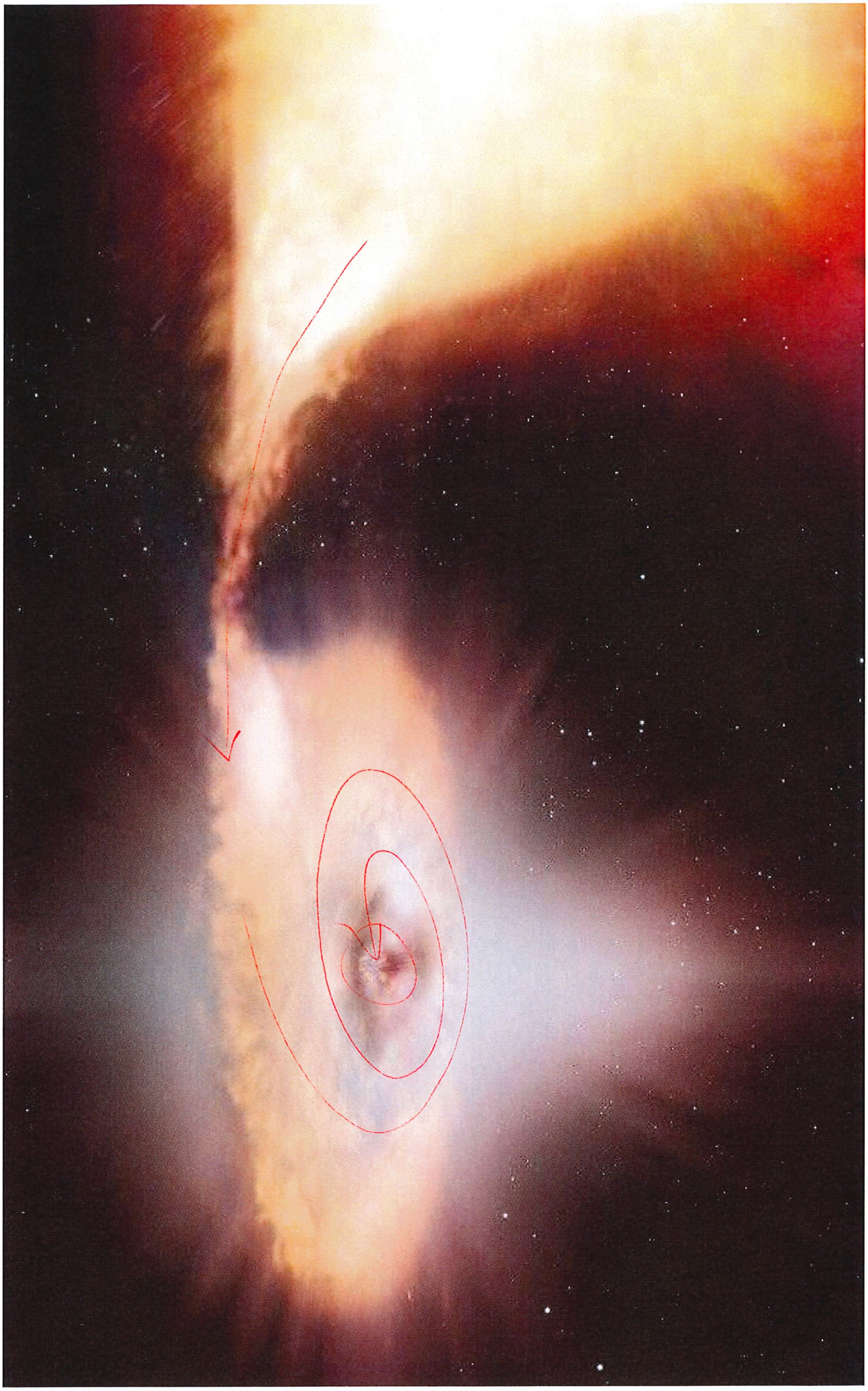
Degeneracy pressure prevent further collapse to black holes

In binary system, white dwarf can capture matter from companion star

Larger density $\rightarrow$ eventually reaches $M \sim 1.4 M_{sun}$ (Chandrasekhar limit)

$\rightarrow$ carbon & oxygen fusion chain reaction

$\rightarrow T \sim 10^9 K$ within seconds $\rightarrow$ supernova (Type-Ia) ~5 billion sun



Type-Ia supernovas have ~~m~~ large amount regularity → "standard candle"
Provide distance vs. time → accelerating expansion of Universe
(Nobel Prize 2011)

Relativistic fermion ideal gas
and corrections to low-temperature limit
→ asynch. pencasts

Instead turn to interacting (non-ideal) systems
Have seen good predictions from ideal systems (Planck; c_v)
Insufficient to describe phenomena like phase transitions
Interactions much harder to analyze
generally no close-~~form~~ form predictions
or even accurate approximations