Mon 17 Apr 497 472

Logistics

HW2 due 3 May

Exam 14:30 – 16:30 Friday 26 May

Big-picture review

Prob. spaces $\rightarrow$ stat. ensembles

micro-canonical

canonical (applications to ideal gases, therm. cycles)

grand-canonical

Plan

Grand-canonical applications to quantum gases

Quantum statistics: Micro-states defined via energy level occupation numbers

Bosons $n_\ell = 0, 1, 2, \cdots$ $\rightarrow$ Bose-Einstein

$$Z_g^{(BE)} = \prod_{\ell=0}^{g} \frac{1}{1 - e^{-\beta(E_\ell - \mu)}} \qquad \mu < E_\ell$$

Fermions $n_\ell = 0, 1$ $\rightarrow$ Fermi-Dirac

$$Z_g^{(FD)} = \prod_{\ell=0}^{g} \left(1 + e^{-\beta(E_\ell - \mu)}\right)$$

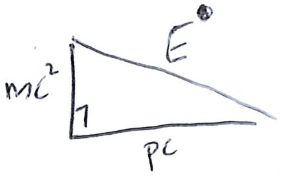
Classical limit: high temperatures w/ large negative chem. pot.

$$-\mu \gg T \gg E_\ell$$

Need energy levels and E_ℓ

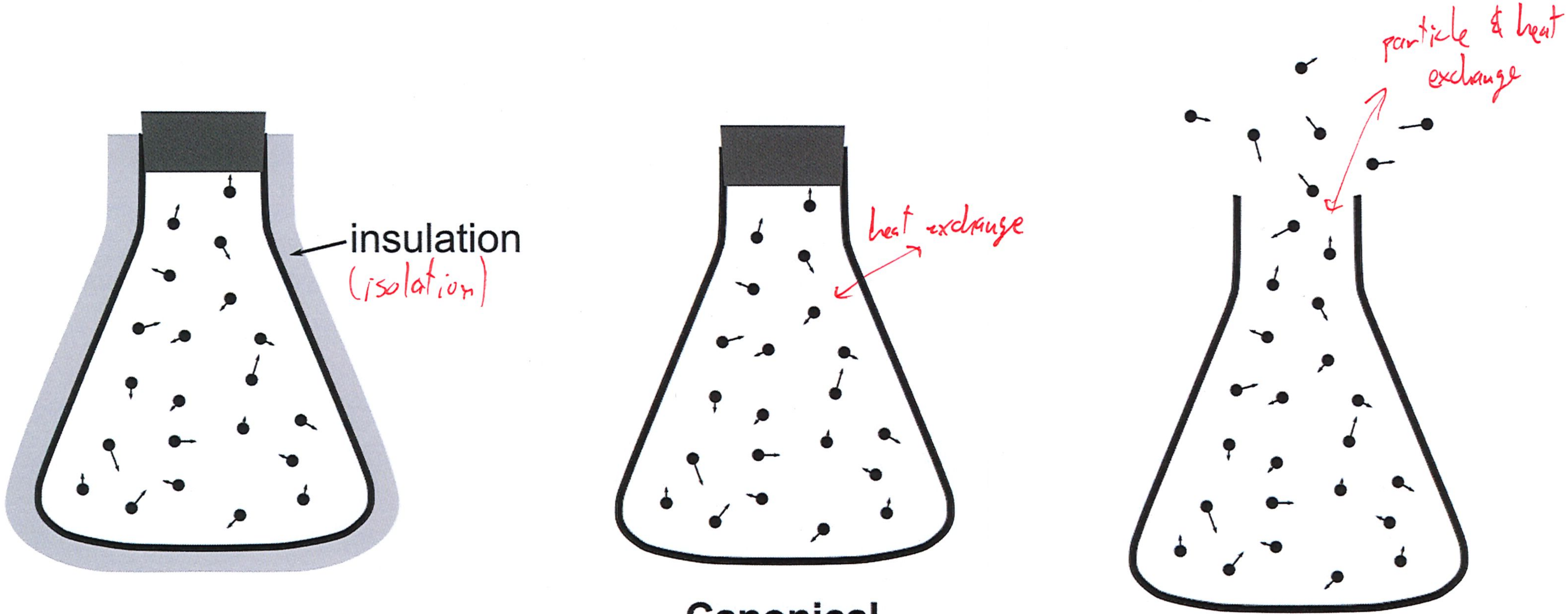
$p^2 = p_x^2 + p_y^2 + p_z^2$

In general $E^2 = (mc^2)^2 + (pc)^2$ (c: speed of light (unit conversion))



mass energy (fixed) & kinetic energy

commons.wikimedia.org/wiki/File:Statistical_Ensembles.png



Microcanonical
(const. N E)

Canonical
(const. N T)

Grand Canonical
(const. μ T)

chemical potential

$m=0$ er $p \gg mc$ $\rightarrow$ ultra-relativistic $E=pc$

Non-relativistic limit $p \ll mc$

$$E = mc^2\sqrt{1+\frac{(pc)^2}{(mc^2)^2}} = mc^2\left(1+\frac{p^2}{2m^2c^2}+\mathcal{O}\left(\frac{p^4}{m^4c^4}\right)\right)$$

$$= \underbrace{mc^2}_{\text{irrelevant}} + \frac{p^2}{2m} + \mathcal{O}\left(\frac{p^4}{m^3c^2}\right)$$

page 103

$$E = \frac{p^2}{2m} = \frac{\hbar^2\pi^2}{2mL^2}\left(k_x^2+k_y^2+k_z^2\right) \quad \text{in } L^3 \text{ volume}$$

now with $k_{x,y,z} = 1,2,3,\ldots$ (Heisenberg)

$\varepsilon = \frac{\hbar^2\pi^2}{2mL^2}$

Quantum ground state $\vec{k}=(1,1,1) \rightarrow E=3\varepsilon$

excited state $\vec{k}=(2,1,1) \rightarrow E=6\varepsilon$

$\vec{k}=(2,2,1) \rightarrow E=9\varepsilon$ — each $3\times$ degenerate

$\vec{k}=(3,1,1) \rightarrow E=11\varepsilon$

$\vec{k}=(2,2,2) \rightarrow E=12\varepsilon$

page 103

Consider gas of photons (bosonic) with $m=0$ $\quad E_{ph} = pc = \hbar c\frac{\pi}{L}k$

Photons are quanta of electromagnetic waves (light)

Speed of light $c = \frac{\lambda\omega}{2\pi}$ from wavelength λ

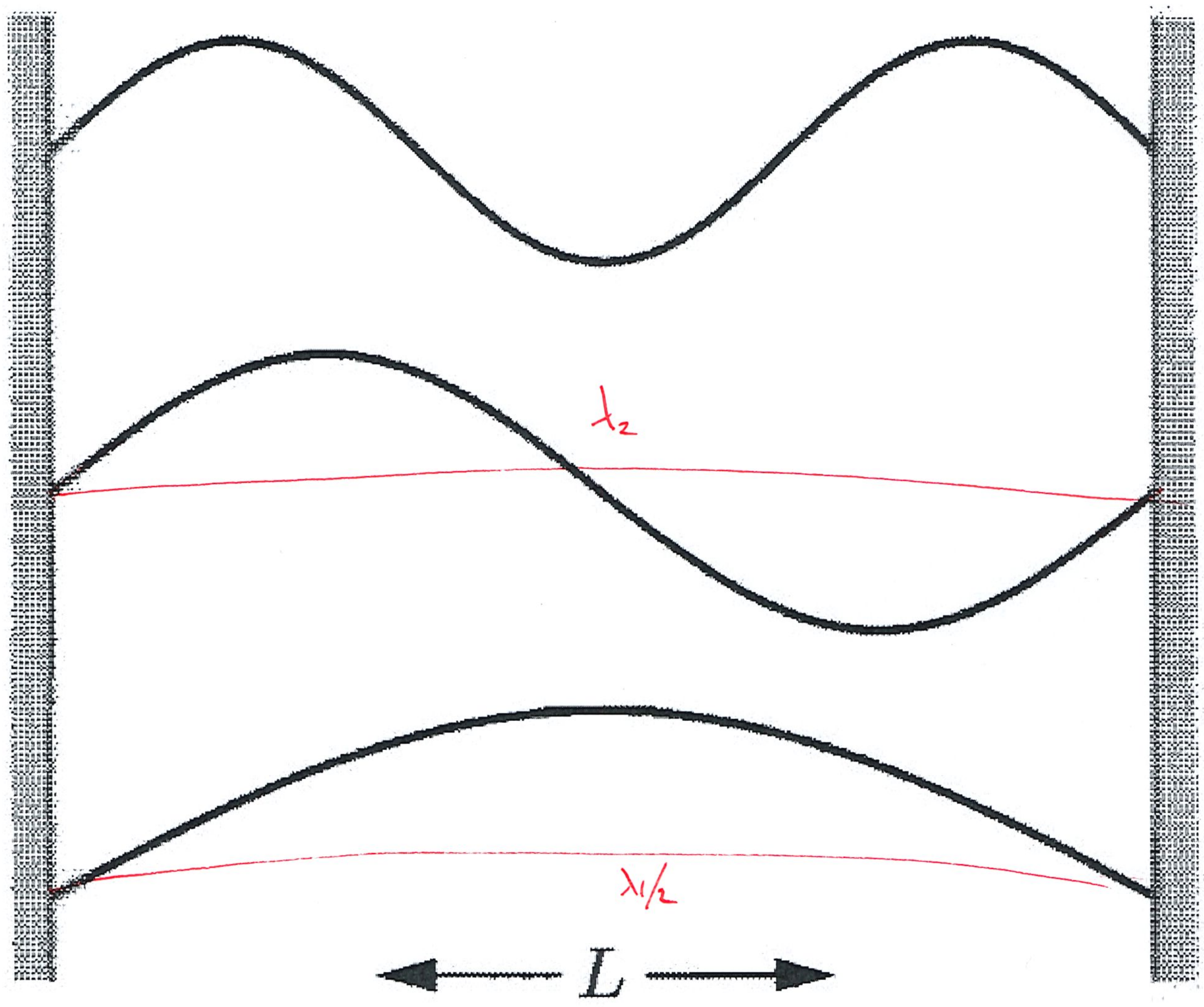
angular frequency $\omega = 2\pi f$

Volume $V=L^3$ quantizes frequency like momenta

$$L = k_x\left(\frac{\lambda}{2}\right) \rightarrow \lambda = \frac{2L}{k_x} \qquad k_x = 1,2,\ldots$$

$$\omega = \frac{2\pi c}{\lambda} = c\frac{\pi}{L}k_x = \left(\frac{c}{\hbar}\right)p_x$$

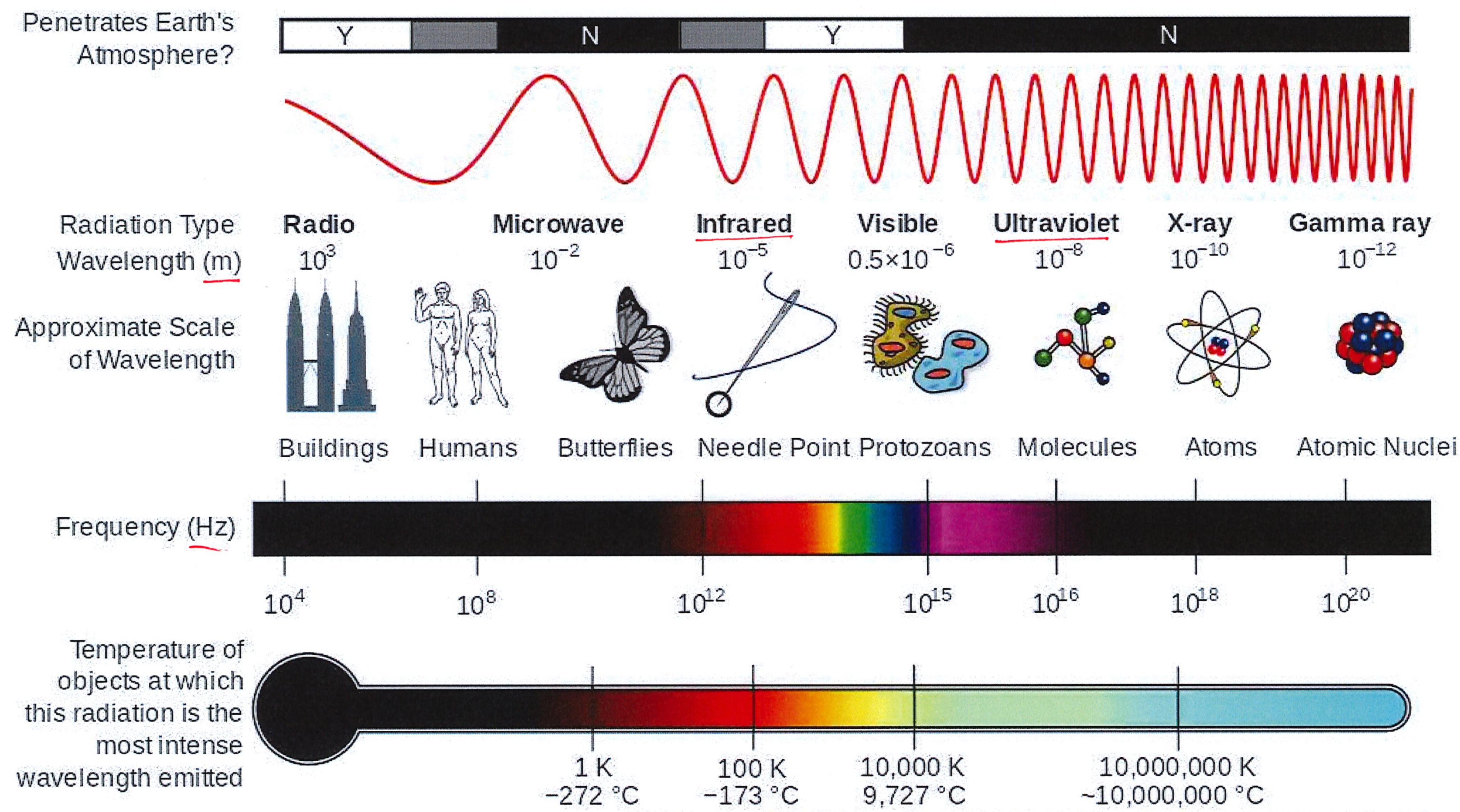
$$E_{ph} = \hbar\omega$$



$$\lambda_3 = \frac{2L}{3}$$

$$\lambda_2 = \frac{\cancel{2}L}{\cancel{2}}$$

$$\lambda_1 = 2L$$



commons.wikimedia.org/wiki/File:EM_Spectrum_Properties_edit.svg

Photon gas grand-canonical potential

$$\Phi_{ph} = T \sum_{l=0}^{\infty} \log\left(1 - e^{-\beta(E_{ph}-\mu)}\right) = \underbrace{2T}_{\text{two polarizations per } \vec{k}} \sum_{\vec{k}} \log\left(1 - e^{-\beta(\hbar\omega - \mu)}\right)$$

Simplification: Photons easy to create & absorb

Little energy to photon #

$$\mu = \frac{\partial E}{\partial N}\bigg|_S \approx 0 \qquad E_l = \hbar\omega > 0 \rightarrow \mu < E_l \ ✓$$

Simplification: Integrate over closely spaced momenta

$$\Phi_{ph} \approx 2T \int \log\left(1 - e^{-\beta\hbar\omega}\right) d\hat{k}_x \, d\hat{k}_y \, d\hat{k}_z$$

$\omega \propto k \rightarrow$ spherical coordinates

positive $\rightarrow$ single octant

$$\int_0^\infty d\hat{k}_x \int_0^\infty d\hat{k}_y \int_0^\infty d\hat{k}_z = \int_0^\infty \hat{k}^2 d\hat{k} \underbrace{\int_0^{\pi/2} \sin\theta \, d\theta}_{1} \underbrace{\int_0^{\pi/2} d\phi}_{\pi/2} = \frac{\pi}{2} \int_0^\infty \hat{k}^2 d\hat{k}$$

plug in $k = \omega\left(\frac{L}{c\pi}\right)$

$$\Phi_{ph} \approx \pi T \int_0^\infty \hat{k}^2 \log\left(1 - e^{-\beta\hbar\omega}\right) d\hat{k}$$

$$= \frac{VT}{c^3\pi^2} \int_0^\infty \omega^2 \log\left(1 - e^{-\beta\hbar\omega}\right) d\omega$$

Internal energy

$$\langle E \rangle = -T^2 \frac{\partial}{\partial T}\left(\frac{\Phi}{T}\right) + \underbrace{\mu}_{0} \langle N \rangle = \frac{\partial}{\partial \beta}(\beta\Phi)$$

$$\boxed{\begin{aligned} \langle E \rangle_{ph} &= \frac{V}{c^3\pi^2} \int_0^\infty \omega^2 \frac{\partial}{\partial\beta} \log\left(1 - e^{-\beta\hbar\omega}\right) d\omega \\ &= \frac{V}{c^3\pi^2} \int_0^\infty \frac{\omega^2 \left(-e^{-\beta\hbar\omega}\right)\left(-\hbar\omega\right)}{1 - e^{-\beta\hbar\omega}} d\omega = \frac{V\hbar}{c^3\pi^2} \int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega} - 1} d\omega \end{aligned}}$$

page 106

Energy density $\frac{\langle E \rangle_{ph}}{V} = \int_0^\infty \underbrace{P(\omega)}_{\text{spectral density}} d\omega$

For photons $P(\omega) = \left(\frac{\hbar}{c^3\pi^2}\right)\frac{\omega^3}{e^{\beta\hbar\omega}-1}$ is Planck spectrum

Equivalently write as $P(\lambda)$ with $\lambda = \frac{2\pi c}{\omega}$ $\qquad \omega = \frac{2\pi c}{\lambda}$ $\qquad d\omega = \frac{-2\pi c}{\lambda^2}d\lambda$

$$\frac{\langle E\rangle_{ph}}{V} = \frac{\hbar}{c^3\pi^2}\int_{\infty}^{0} \frac{(2\pi c/\lambda)^3}{e^{2\pi\beta\hbar c/\lambda}-1}\left(-\frac{2\pi c}{\lambda^2}\right)d\lambda = 16\pi^2\hbar c\int_0^{\infty}\frac{d\lambda}{\lambda^5\left(e^{2\pi\beta\hbar c/\lambda}-1\right)}$$

page 106 $$P(\lambda) = \left(\frac{16\pi^2\hbar c}{\lambda^5}\right)\frac{1}{e^{2\pi\beta\hbar c/\lambda}-1}$$

UV: $\lambda \to 0$

Exponential factor dominates $\frac{1}{\lambda^5}$ term

IR: λ large compared to $\hbar c$

$$e^{2\pi\beta\hbar c/\lambda}-1 \approx \frac{2\pi\beta\hbar c}{\lambda} \to P(\lambda) = \left(\frac{16\pi^2\cancel{\hbar c}}{\lambda^5}\right)\left(\frac{\lambda}{2\pi\beta\cancel{\hbar c}}\right)$$

$$= \frac{8\pi T}{\lambda^4}$$

also for small β (high temperatures)

classical Rayleigh-Jeans spectrum

Classical spectrum diverges $\sim\frac{1}{\lambda^4}$ as $\lambda \to 0$

"ultraviolet catastrophe" $\to$ quantum

Max of Planck $P(\lambda)$ at shorter λ for higher temperatures

Peak in λ of visible light for $T \approx 5000\,K$

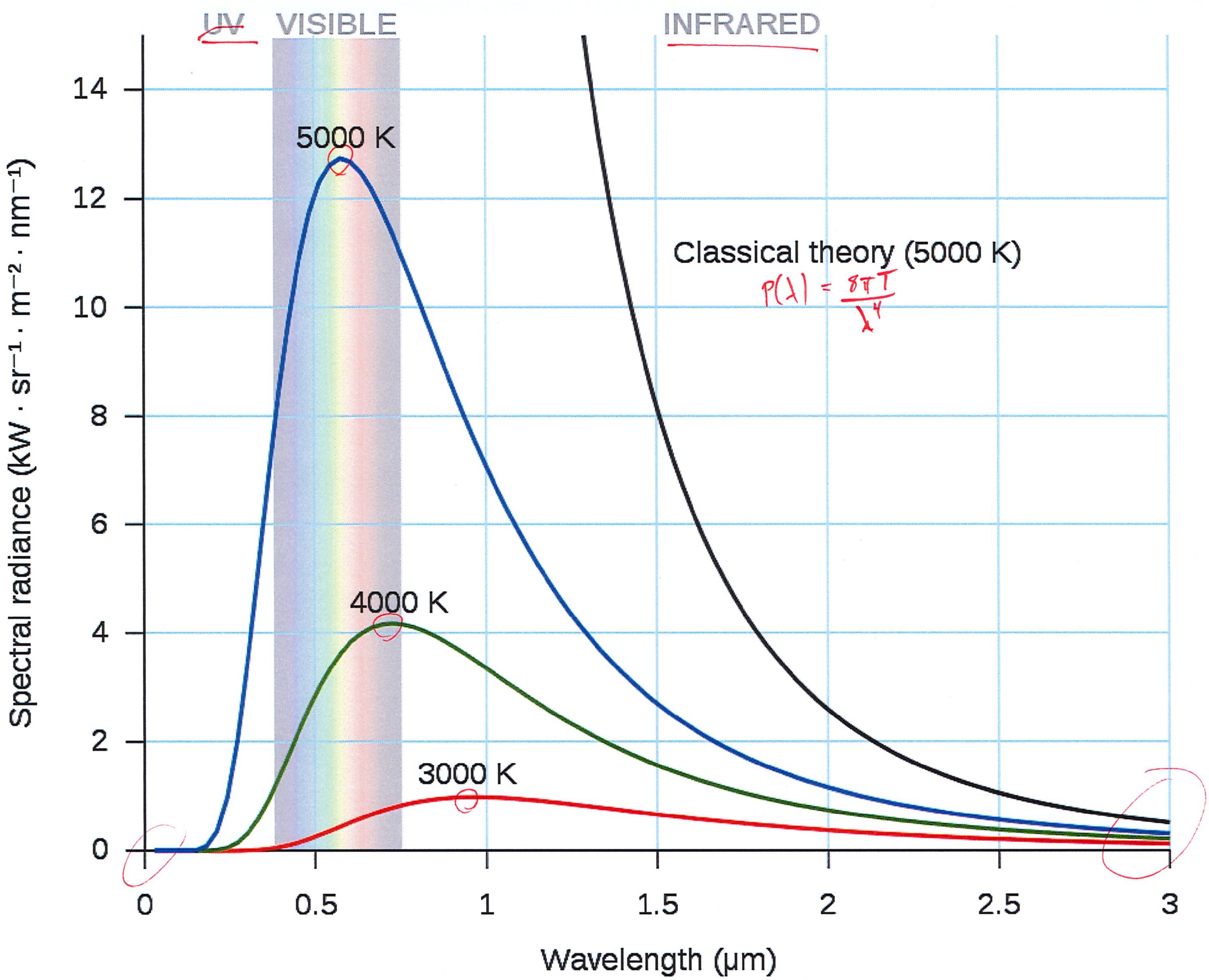
because sunlight has $T \simeq 5778\,K$

Determine effective surface temperature by fitting $P(\lambda)$

Red stars $\to T \lesssim 3500\,K$

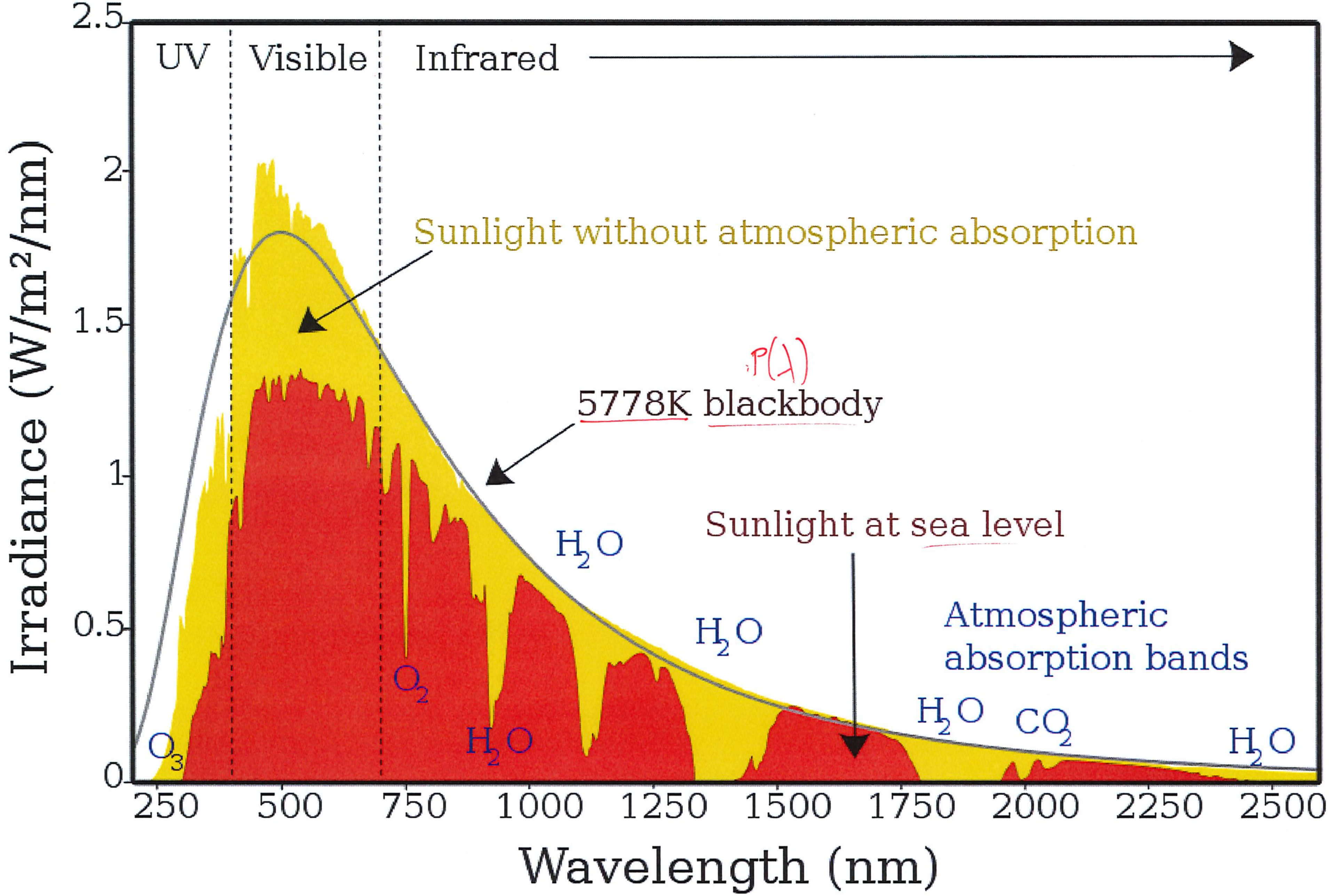
Blue stars $\to T \gtrsim 10{,}000\,K$

Intergalactic space $\to T \simeq 2.725\,K$



UV
VISIBLE
INFRARED
5000 K
4000 K
3000 K
Classical theory (5000 K)
$P(\lambda) = \frac{8\pi T}{\lambda^4}$
Spectral radiance (kW · sr⁻¹ · m⁻² · nm⁻¹)
0
2
4
6
8
10
12
14
0
0.5
1
1.5
2
2.5
3
Wavelength (μm)

Spectrum of Solar Radiation (Earth)



Cosmic ~~B~~ Microwave Background left over from Big Bang ~14 Gyr

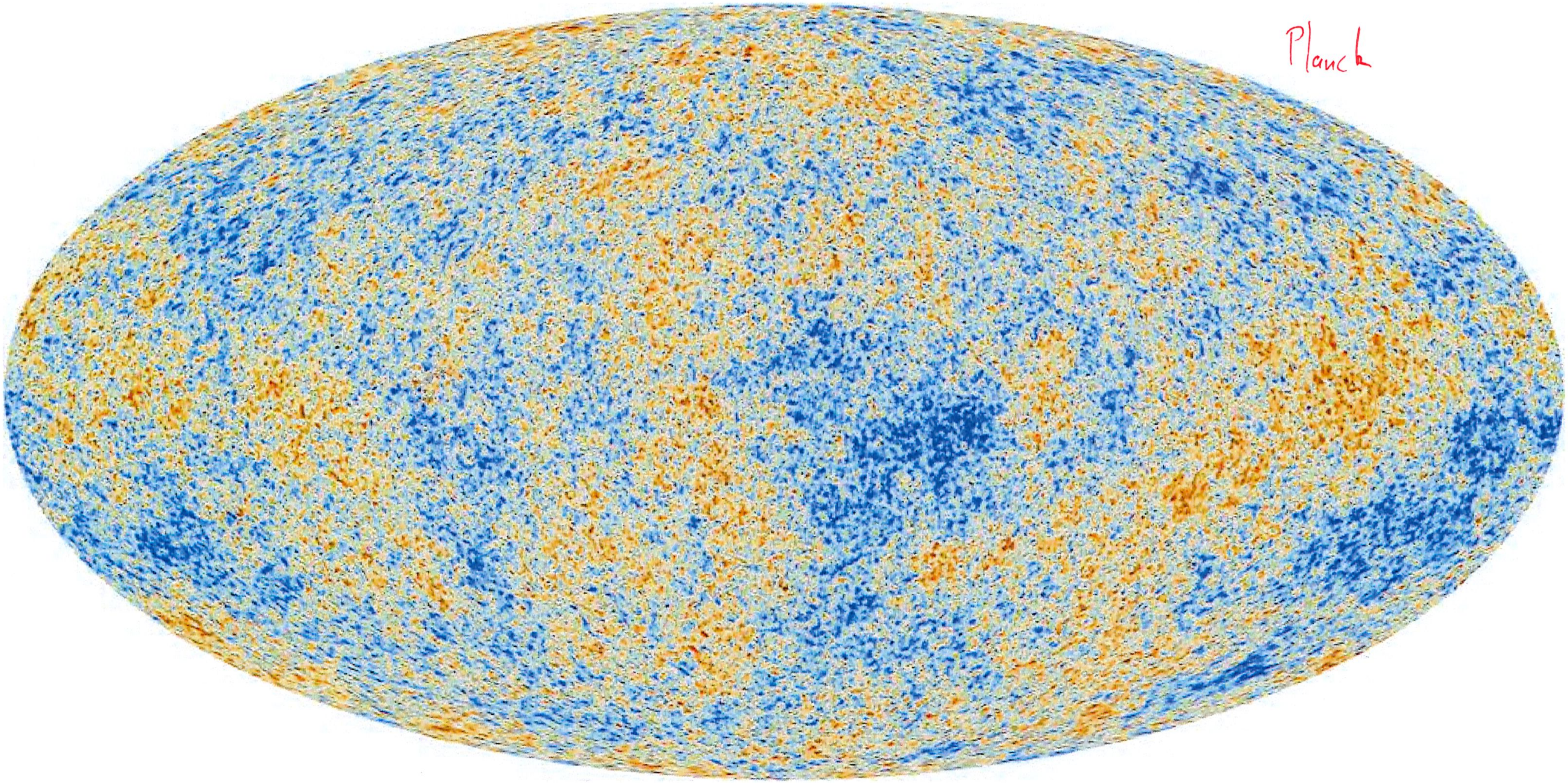
T at each point in sky from photons after subtracting galaxies

Blue / red small fluctuations around average $T_{CMB} \simeq 2.725\ K$

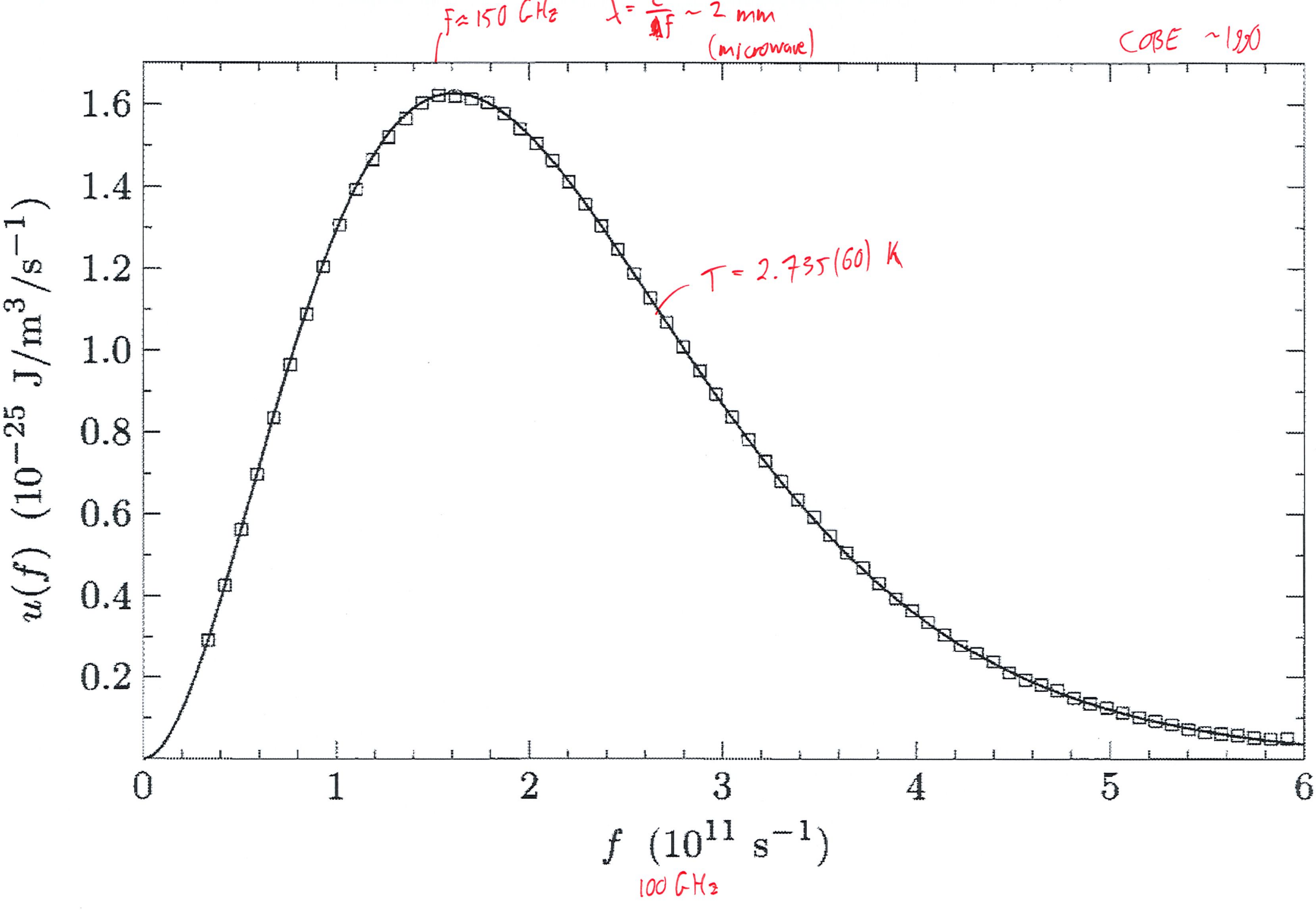
$\Delta T \simeq 0.0002\ K$

Pattern of fluctuations → dark matter

Amazingly accurate descriptions given assumption of non-interacting ideal gas



Planck



f ≈ 150 GHz
λ = c/f ~ 2 mm
(microwave)
COBE ~1990
T = 2.735(60) K
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
u(f) (10^−25 J/m^3/s^−1)
0
1
2
3
4
5
6
f (10^11 s^−1)
100 GHz