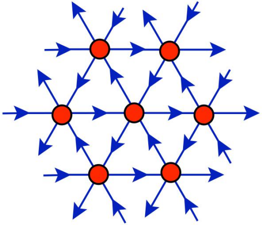
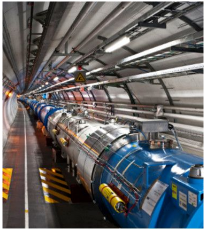
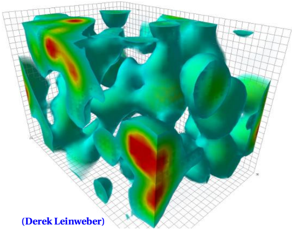
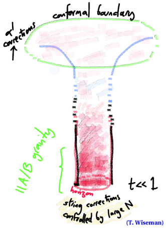
# Lattice supersymmetric field theories — Part 1

David Schaich (University of Liverpool)








Nonperturbative and Numerical Approaches
to Quantum Gravity, String Theory and Holography

International Centre for Theoretical Sciences, Bangalore, 22 August 2022

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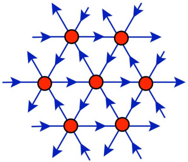
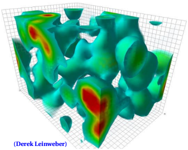
# Overview and plan



**Overcoming challenges opens many opportunities for lattice studies of supersymmetric QFTs**

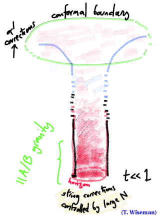
Motivation, background, formulation

- Supersymmetry breaking in discrete space-time
- Supersymmetry preservation in discrete space-time




Applications with significant recent progress

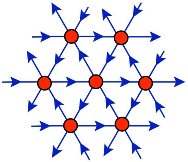
- Maximal $\mathcal{N} = 4$ super-Yang–Mills
- Lower dimensions $d < 4$
- Minimal $\mathcal{N} = 1$ super-Yang–Mills




Remaining challenges: Super-QCD; Sign problems

# Overview and plan

Overcoming challenges opens many opportunities
for lattice studies of supersymmetric QFTs



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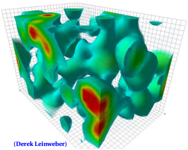


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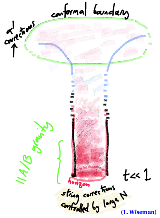


Conceptual focus with interaction encouraged

"It's better to uncover a little than to cover a lot" (V. Weisskopf)

# Further resources

**Lattice studies of supersymmetric gauge theories**

David Schaich*

*Department of Mathematical Sciences,*
*University of Liverpool, Liverpool L69 7ZL, United Kingdom*

(Dated: 17 August 2022)

Updated version of arXiv:2208.03580 at icts.res.in/program/numstrings2022/talks

arXiv:0903.4881 by Catterall, Kaplan and Ünsal remains most detailed review

Expect connections with lectures by:

Anna Hasenfratz — Introduction to Lattice Field Theory

Georg Bergner — Matrix Models, Gauge-Gravity Duality, and Simulations…

# Further resources

Expect connections with lectures by:

Anna Hasenfratz — Introduction to Lattice Field Theory

Georg Bergner — Matrix Models, Gauge-Gravity Duality, and Simulations…

Will try to avoid pre-empting research talks coming up later in this program

**Many people have contributed over many years**

Alessandro D'Adda, Georg Bergner, Simon Catterall, Andy Cohen, Chris Culver, Poul Damgaard, Tom DeGrand, Joel Giedt, Masanori Hanada, Anosh Joseph, Raghav Jha, Daisuke Kadoh, Issaku Kanamori, Noboru Kawamoto, David B. Kaplan, So Matsuura, Angel Sherletov, Fumihiko Sugino, Mithat Ünsal, Urs Wenger, Andreas Wipf, …

# Motivations (I)

Lattice field theory promises first-principles predictions
for strongly coupled supersymmetric QFTs

Formally $\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \ \mathcal{O}(\Phi) \ e^{-S[\Phi]}$

Regularize by formulating theory in finite, discrete, euclidean space-time
↖Gauge invariant, non-perturbative, *d*-dimensional

# Recap: Lattice field theory in a nutshell

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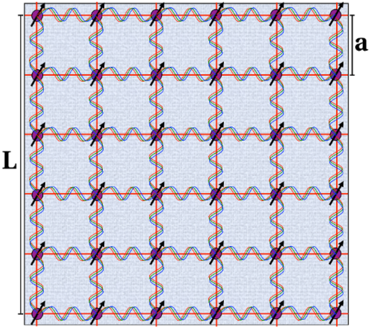


P. Vranas LLNL

Spacing between lattice sites ("*a*")
$\longrightarrow$ UV cutoff scale $1/a$

Remove cutoff: $a \to 0$ $(L/a \to \infty)$

Discrete $\longrightarrow$ continuous symmetries ✓

# Recap: Lattice field theory in a nutshell

High-performance computing
$\longrightarrow$ evaluate up to
$\sim$billion-dimensional integrals

(Dirac op. as $\sim 10^9 \times 10^9$ matrix)

## Importance sampling Monte Carlo

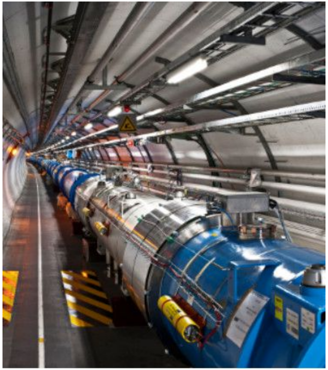
Algorithms sample field configurations with probability $\frac{1}{\mathcal{Z}} e^{-S[\Phi]}$

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \;\; \mathcal{O}(\Phi) \;\; e^{-S[\Phi]} \longrightarrow \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(\Phi_i) \text{ with stat. uncertainty } \propto \frac{1}{\sqrt{N}}$$

# Motivations (II)

Lattice field theory promises first-principles predictions
for strongly coupled supersymmetric QFTs

**BSM**



New physics beyond the standard model

Nature demands supersymmetry breaking

Expect non-perturbative, **dynamical** breaking

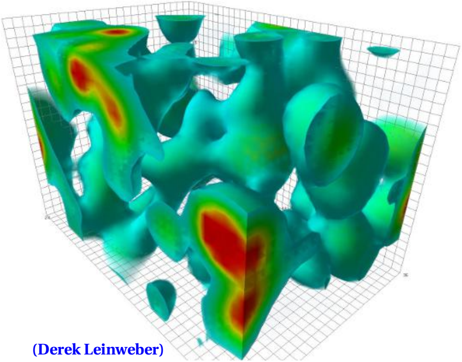
Also expect susy breaking based on
chiral gauge theories — not covered here!

# Motivations (II)

Seiberg duality and conformal window in super-QCD;
Montonen–Olive S-duality in $\mathcal{N} = 4$ SYM; scattering amplitudes etc.

**BSM**

**QFT**



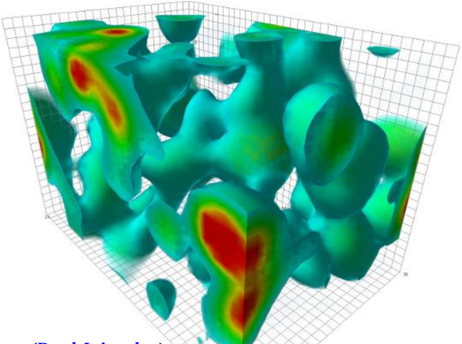
(Derek Leinweber)

# Motivations (II)

Can numerically test holographic conjectures,
or rely on holography to non-perturbatively define string theory

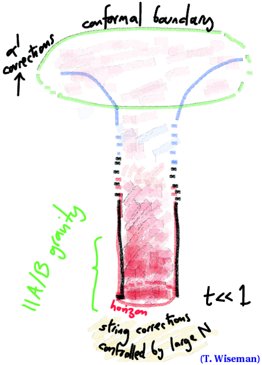
**BSM**

**QFT**



(Derek Leinweber)

**Holography**




(T. Wiseman)

# Motivations (II)

Can numerically test holographic conjectures,
or rely on holography to non-perturbatively define string theory

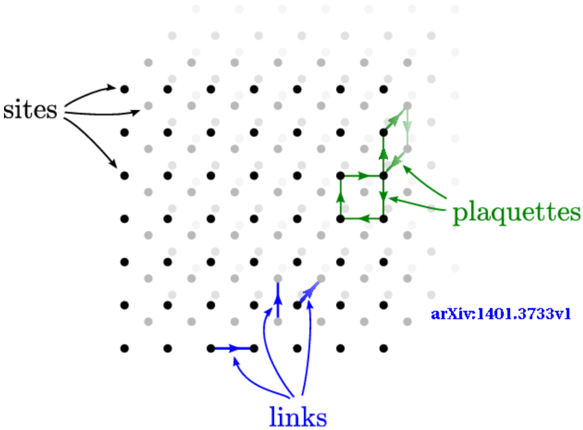
**The Large N Limit of Superconformal field theories and supergravity** hep-th/9711200

Juan Maldacena

In principle, we can use this duality to give a definition of M/string theory on flat spacetime as (a region of) the large $N$ limit of the field theories. Notice that this is a non-perturbative proposal for defining such theories, since the corresponding field theories can, *in principle*, be defined non-perturbatively.

# Supersymmetry must be broken on the lattice (I)

| | | | | |
|---|---|---|---|---|
| Typically | fermions $\longleftrightarrow$ sites | while | gauge fields $\longleftrightarrow$ links |
| | scalars $\longleftrightarrow$ sites | but no doubling problem | |



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Typically fermions $\longleftrightarrow$ sites while gauge fields $\longleftrightarrow$ links

scalars $\longleftrightarrow$ sites but no doubling problem

Broken supersymmetry $\longrightarrow$ relevant susy-violating operators

$\longrightarrow$ typically $\mathcal{O}(10)$ parameters to fine-tune for correct continuum limit

# Supersymmetry must be broken on the lattice (II)

"What if a sufficiently clever discretization could match up all superpartners?"

Supersymmetry is a space-time symmetry, $(\mathrm{I} = 1, \cdots, \mathcal{N})$

adding spinor generators $Q_\alpha^{\mathrm{I}}$ and $\overline{Q}_{\dot{\alpha}}^{\mathrm{I}}$ to translations, rotations, boosts

Super-Poincaré algebra includes $\left\{Q_\alpha^{\mathrm{I}}, \overline{Q}_{\dot{\alpha}}^{\mathrm{J}}\right\} = 2\delta^{\mathrm{IJ}}\sigma^{\mu}_{\alpha\dot{\alpha}}P_\mu$

broken in discrete space-time

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broken in discrete space-time

Continuous-time (hamiltonian) formulation [Polchinski; Banks–Windey]

can preserve $P_0 = H \sim i\partial_t \longrightarrow$ partial supersymmetry

But then fine-tuning needed to recover Lorentz invariance in continuum limit!

# Supersymmetry must be broken on the lattice (III)

"What if we adapt $\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$

to use the finite difference $\partial\phi(x) \longrightarrow \Delta\phi(x) = \frac{1}{a}\left[\phi(x+a) - \phi(x)\right]$?"

**Full supersymmetry requires Leibniz rule** $\partial\left[\phi\psi\right] = \left[\partial\phi\right]\psi + \phi\partial\psi$

Quantum mechanics: $\delta_i S \propto \int \left[(\partial_t\psi_i)\frac{\mathrm{d}W}{\mathrm{d}\phi} + \psi_i\frac{\mathrm{d}^2W}{\mathrm{d}\phi^2}\partial_t\phi\right] dt = \int \partial_t\left[\psi\frac{\mathrm{d}W}{\mathrm{d}\phi}\right] dt$

Doesn't hold for $\Delta\left[\phi\psi\right] = a^{-1}\left[\phi(x+a)\psi(x+a) - \phi(x)\psi(x)\right]$

$$= \left[\Delta\phi\right]\psi + \phi\Delta\psi + a\left[\Delta\phi\right]\Delta\psi$$

# Supersymmetry must be broken on the lattice (III)

**Full supersymmetry requires Leibniz rule** $\partial\left[\phi\psi\right] = \left[\partial\phi\right]\psi + \phi\partial\psi$

Doesn't hold for **any** local finite difference at non-zero lattice spacing $a > 0$

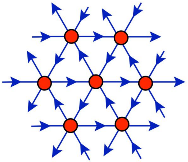
Supersymmetry vs. locality 'no-go' theorems
by Kato–Sakamoto–So [arXiv:0803.3121] and Bergner [arXiv:0909.4791]

Complicated constructions to balance locality vs. supersymmetry

Non-ultralocal product operator $\longrightarrow$ lattice Leibniz rule but not gauge invariance
D'Adda–Kawamoto–Saito, arXiv:1706.02615

Cyclic Leibniz rule $\longrightarrow$ partial lattice supersymmetry but only (0+1)d QM so far
Kadoh–Kamei–So, arXiv:1904.09275

# Checkpoint



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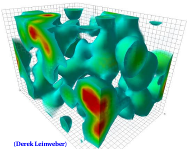


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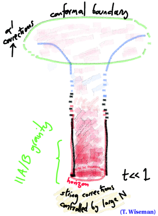



Remaining challenges: Super-QCD; Sign problems

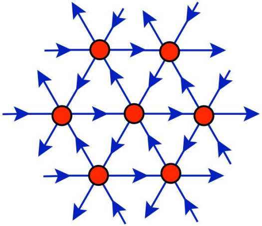
**Questions?**

"It's better to uncover a little than to cover a lot"

# Supersymmetry need not be **completely** broken on the lattice

Preserve susy sub-algebra in discrete lattice space-time
$\Longrightarrow$ correct continuum limit with little or no fine tuning

Equivalent constructions from 'topological' twisting and dim'l deconstruction



Review:
Catterall–Kaplan–Ünsal,
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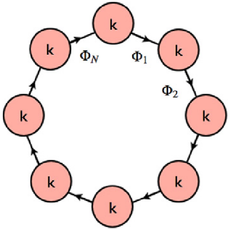


Orbifolding came first; twisting steps easier to follow

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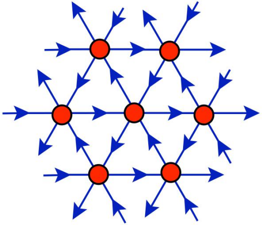

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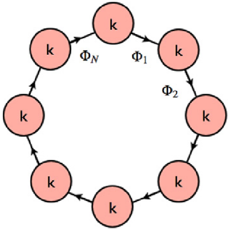


Need $2^d$ supersymmetries in $d$ dimensions
$\Longrightarrow$ Only $\mathcal{N} = 4$ super-Yang–Mills (SYM) for $d = 4$

# $\mathcal{N} = 4$ SYM in a nutshell

Arguably simplest non-trivial 4d QFT $\longrightarrow$ dualities, amplitudes, ...

SU($N$) gauge theory with $\mathcal{N} = 4$ fermions $\Psi^{\mathrm{I}}$ and 6 scalars $\Phi^{\mathrm{IJ}}$,
all massless and in adjoint rep.

$$
\begin{aligned}
|\Omega_1\rangle & \qquad 1 \longrightarrow A_\mu \\
Q^{\mathrm{I}}|\Omega_1\rangle & \qquad 1/2 \longrightarrow \Psi^{\mathrm{I}} \\
Q^{\mathrm{J}}Q^{\mathrm{I}}|\Omega_1\rangle & \qquad 0 \longrightarrow \Phi^{\mathrm{IJ}} \\
Q^{\mathrm{K}}Q^{\mathrm{J}}Q^{\mathrm{I}}|\Omega_1\rangle & \qquad -1/2 \longrightarrow \Psi^{\mathrm{I}} \\
Q^{\mathrm{L}}Q^{\mathrm{K}}Q^{\mathrm{J}}Q^{\mathrm{I}}|\Omega_1\rangle & \qquad -1 \longrightarrow A_\mu
\end{aligned}
$$

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SU($N$) gauge theory with $\mathcal{N} = 4$ fermions $\Psi^{\rm I}$ and 6 scalars $\Phi^{\rm IJ}$,
all massless and in adjoint rep.

**Symmetries** relate coefficients of kinetic, Yukawa and $\Phi^4$ terms

Maximal 16 supersymmetries $Q^{\rm I}_{\alpha}$ and $\overline{Q}^{\rm I}_{\dot{\alpha}}$ $\quad {\rm I} = 1, \cdots, 4$
transform under global SU(4) $\sim$ SO(6) R symmetry

Conformal $\longrightarrow$ $\beta$ function is zero for all values of $\lambda = g^2 N$

# Twisting $\mathcal{N} = 4$ SYM — main idea

**Intuitive picture — expand $4 \times 4$ matrix of supersymmetries**

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}} \gamma_5$$

R-symmetry index $\times$ Lorentz index $\Longrightarrow$ reps of 'twisted rotation group'

$$\text{SO}(4)_{\text{tw}} \equiv \text{diag}\left[\text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R\right] \qquad \text{SO}(4)_R \subset \text{SO}(6)_R$$

**Change of variables $\longrightarrow$ $\mathcal{Q}$s transform with integer 'spin' under $\text{SO}(4)_{\text{tw}}$**

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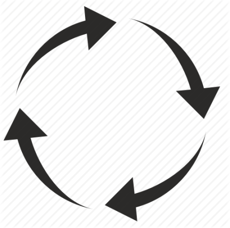
## Discrete space-time

Can preserve closed sub-algebra

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$$



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## Discrete space-time

Can preserve closed sub-algebra

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# Twisting the fields

The four Majorana $\Psi^I$ behave just like the supercharges — no spinors remain!

$$\begin{pmatrix} \Psi^1 & \Psi^2 & \Psi^3 & \Psi^4 \end{pmatrix} \longrightarrow \left(\eta,\ \psi_\mu,\ \chi_{\mu\nu},\ \overline{\psi}_\mu,\ \overline{\eta}\right)$$

Under $SO(4)_R \subset SO(6)_R$ the six scalars $\Phi^{IJ} \longrightarrow \left(B_\mu, \phi, \overline{\phi}\right)$

Organize into 5-component complexified gauge fields

$$\mathcal{A}_a = (A_\mu,\ \phi) + i(B_\mu,\ \overline{\phi}) \qquad \overline{\mathcal{A}}_a = (A_\mu,\ \phi) - i(B_\mu,\ \overline{\phi})$$

# Complexified gauge field

Organize $4 + 6$ bosons into 5-component complexified gauge fields

$$\mathcal{A}_a = (A_\mu,\ \phi) + i(B_\mu,\ \overline{\phi}) \qquad \overline{\mathcal{A}}_a = (A_\mu,\ \phi) - i(B_\mu,\ \overline{\phi})$$

## Why?

Easiest to see for $\text{SO}(5)_{\text{tw}} = \text{diag}\left[\text{SO}(5)_{\text{euc}} \otimes \text{SO}(5)_R\right]$ in five dimensions

$$\begin{aligned} A_a &\sim \text{vector} \otimes \text{scalar} &= \text{vector} \\ \Phi^a &\sim \text{scalar} \otimes \text{vector} &= \text{vector} \end{aligned}$$

Dimensionally reduce $\mathcal{A}_a = A_a + i\Phi^a \longrightarrow (A_\mu, \phi) + i(B_\mu, \overline{\phi})$

# Completing the twist

Complexified $\{\mathcal{A}_a, \overline{\mathcal{A}}_a\}$ produce $\mathrm{U}(N) = \mathrm{SU}(N) \otimes \mathrm{U}(1)$ gauge theory

Similarly combine

$$\psi_a = (\psi_\mu,\ \overline{\eta}) \qquad \chi_{ab} = (\chi_{\mu\nu},\ \overline{\psi}_\mu)$$

$$\mathcal{Q}_a = (\mathcal{Q}_\mu,\ \overline{\mathcal{Q}}) \qquad \mathcal{Q}_{ab} = (\mathcal{Q}_{\mu\nu},\ \overline{\mathcal{Q}}_\mu)$$

Check $\mathcal{Q}$ interchanges bosonic $\longleftrightarrow$ fermionic d.o.f. with $\mathcal{Q}^2 = 0$

$$\mathcal{Q}\ \mathcal{A}_a = \psi_a \qquad \mathcal{Q}\ \psi_a = 0$$

$$\mathcal{Q}\ \chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad \mathcal{Q}\ \overline{\mathcal{A}}_a = 0$$

$$\mathcal{Q}\ \eta = d \qquad \mathcal{Q}\ d = 0$$

$\nwarrow$ bosonic auxiliary field with e.o.m. $d = \overline{\mathcal{D}}_a \mathcal{A}_a$

# Twisted $\mathcal{N} = 4$ SYM

✓ $\mathcal{Q}$ interchanges bosonic $\longleftrightarrow$ fermionic d.o.f. with $\mathcal{Q}^2 = 0$

$$\mathcal{Q}\,\mathcal{A}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$
$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{A}}_a = 0$$
$$\mathcal{Q}\,\eta = d \qquad\qquad \mathcal{Q}\,d = 0$$

Action is $S = \int \frac{N}{4\lambda} \text{Tr}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{A}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de}\right]$

Supersymmetric $(\mathcal{Q}S = 0)$ from $\mathcal{Q}^2 \cdot = 0$ and Jacobi identity $\epsilon_{abcde}\,\overline{\mathcal{F}}_{ab}\overline{\mathcal{D}}_c = 0$

# Lattice $\mathcal{N} = 4$ SYM

**Lattice theory looks nearly the same despite breaking $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$**

Covariant derivatives $\longrightarrow$ finite difference operators

Complexified gauge fields $\{\mathcal{A}_a, \overline{\mathcal{A}}_a\}$ $\longrightarrow$ gauge links $\{\mathcal{U}_a, \overline{\mathcal{U}}_a\} \in \mathfrak{gl}(N, \mathbb{C})$

with gauge-invariant flat measure $D\mathcal{U}\, D\overline{\mathcal{U}}$

✓ $\mathcal{Q}$ interchanges bosonic $\longleftrightarrow$ fermionic d.o.f. with $\mathcal{Q}^2 = 0$

$$\mathcal{Q}\,\mathcal{A}_a \longrightarrow \mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$

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with gauge-invariant flat measure $D\mathcal{U}\, D\overline{\mathcal{U}}$

Need $\mathcal{U}_a \to \mathbb{I}_N + \mathcal{A}_a$ to recover continuum covariant derivative

Lattice action $S_{\text{lat}} = \sum \frac{N}{4\lambda_{\text{lat}}} \text{Tr}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de}\right]$

still supersymmetric from $\mathcal{Q}^2 \cdot = 0$ and finite-difference Bianchi identity

# Lattice $\mathcal{N} = 4$ SYM — gauge invariance

Lattice action $S = \frac{N}{4\lambda_{\text{lat}}}\text{Tr}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\ \chi_{ab}\overline{\mathcal{D}}_c\ \chi_{de}\right]$

**Gauge invariance $\longleftrightarrow$ trace over closed loops**

Fixes orientations of lattice variables and finite-difference operators

| Site variables | Link variables | Plaquette variables |
|---|---|---|
| $G(n)\ \eta(n)\ G^{\dagger}(n)$ | $G(n)\ \psi_a(n)\ G^{\dagger}(n+\widehat{\mu}_a)$ | $G(n+\widehat{\mu}_a+\widehat{\mu}_b)\ \chi_{ab}(n)\ G^{\dagger}(n)$ |
| | $G(n)\ \mathcal{U}_a(n)\ G^{\dagger}(n+\widehat{\mu}_a)$ | |
| | $G(n+\widehat{\mu}_a)\ \overline{\mathcal{U}}_a(n)\ G^{\dagger}(n)$ | |

# Lattice $\mathcal{N} = 4$ SYM — gauge invariance

**Site variables**

$$G(n)\,\eta(n)\,G^\dagger(n)$$

**Link variables**

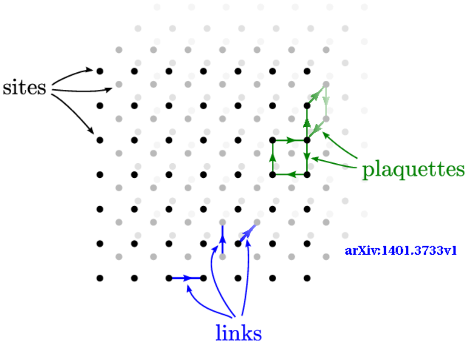
$$G(n)\,\psi_a(n)\,G^\dagger(n+\widehat{\mu}_a)$$

$$G(n)\,\mathcal{U}_a(n)\,G^\dagger(n+\widehat{\mu}_a)$$

$$G(n+\widehat{\mu}_a)\,\overline{\mathcal{U}}_a(n)\,G^\dagger(n)$$

**Plaquette variables**

$$G(n+\widehat{\mu}_a+\widehat{\mu}_b)\,\chi_{ab}(n)\,G^\dagger(n)$$

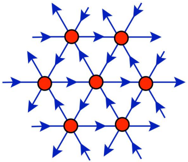



Examples:

$$\text{Tr}\left[\eta\overline{\mathcal{U}}_a\psi_a\right]$$

$$\text{Tr}\left[\chi_{ab}\mathcal{U}_a\psi_b\right]$$

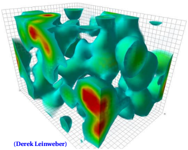
# Next time



Overcoming challenges opens many opportunities
for lattice studies of supersymmetric QFTs

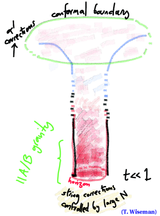
Motivation, background, formulation

- ✓ Supersymmetry breaking in discrete space-time
- Supersymmetry preservation — wrap up




Applications with significant recent progress

- Maximal $\mathcal{N} = 4$ super-Yang–Mills
- Lower dimensions $d < 4$
- Minimal $\mathcal{N} = 1$ super-Yang–Mills




Remaining challenges: Super-QCD; Sign problems