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Recap:

$d=1$ Ising model exact solution

↳ Same # of links as sites → transfer matrix

$\langle m \rangle \propto \frac{\partial F}{\partial H} = 0$ for all β

No phase transition

- Mean-field approx. fails
- General considerations overlooked domain walls

The problematic argument in Section 9.2 was our claim that *all* the degenerate micro-states of the first excited energy level correspond to flipping a single spin to oppose the full alignment of the ground state. This is true for any $d > 1$ simple cubic lattice, and even in one dimension this does account for $2N$ of the micro-states with next-to-minimal energy E_1. However, uniquely in one dimension there are additional micro-states with E_1.

Fixing $d = 1$, suppose we start from the ground state and flip spin s_j to reach the first excited energy level. Relative to the ground-state energy $E_0 = -N$, the energy of this micro-state is increased to $E_1 = -N + 4$ due to the positive contributions from the $s_{j-1}s_j$ and $s_j s_{j+1}$ links. But if we now consider *also* flipping spin s_{j+1}, the $s_j s_{j+1}$ link goes back to providing a negative contribution while the positive contribution shifts to the $s_{j+1}s_{j+2}$ link. This gives us an additional $2N$ micro-states featuring a flipped nearest-neighbour *pair* of spins, with the same energy E_1 but a smaller magnetization $|m| = 1 - \frac{4}{N}$. And we can continue this process, finding more degenerate micro-states with a flipped block of *any* number of neighbouring spins up to $N-1$, and hence any magnetization, including $m = 0$.

Therefore, for $d = 1$ only, our argument that the first excited energy level of the Ising model corresponds to the unique magnetization $|m| = 1 - \frac{2}{N} \to 1$ was incorrect. As we have now seen from our exact solution, there is actually no ordered phase in one dimension. A useful way to visualize this sort of behaviour is to think of all these micro-states in the first excited energy level as consisting of two *domains* — one in which the spins point up and the other in which they point down. The two **domain walls** separating these domains are able to move freely through the lattice without changing the energy, but as the domain walls move the magnetization samples the full range of values $-\left(1 - \frac{2}{N}\right) \leq m \leq 1 - \frac{2}{N}$.

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9.4.2 Two-dimensional critical temperature

While the zero-field Ising model on the $d = 2$ square lattice has also been exactly solved, both Onsager's original calculation and subsequent re-derivations using simpler techniques go substantially beyond this module. However, a few years before Onsager published his famous result, Hans Kramers and G. H. Wannier were able to determine the exact $d = 2$ critical temperature in 1941. They did this by identifying a relation between two Ising model partition functions, without actually evaluating either sum over micro-states:

$$\frac{Z(\beta)}{2^N \cosh^{2N}\beta} = \frac{Z(\widetilde{\beta})}{2e^{2N\widetilde{\beta}}}, \qquad (121)$$

where the two inverse temperatures β and $\widetilde{\beta}$ are related by

$$\sinh(2\beta) = \frac{1}{\sinh(2\widetilde{\beta})}. \qquad (122)$$

This relation is now known as Kramers–Wannier duality, and the general concept of duality has become a powerful tool in modern theoretical physics. Note that small β implies large $\widetilde{\beta}$ and vice versa — the duality relates one $d = 2$ Ising model at a high temperature to another one at a low temperature.

Although it can be instructive to explicitly compare such high- and low-temperature partition functions, by computing series expansions as we did for the non-interacting spin system in Section 3.4 and for the Einstein solid in a tutorial, to keep this section under control I'll skip that exercise. Those who are interested can find related discussions in Sections 5.3.2 and 5.3.3 of David Tong's *Lectures on Statistical Physics* (the first item in the list of further reading on page 5). Some of the manipulations below, which may seem to come out of thin air, can be motivated by considering these expansions.

The first manipulation is to express the zero-field partition function as

$$Z = \sum_{\{s_n\}} \exp\left[\beta \sum_{(jk)} s_j s_k\right] = \sum_{\{s_n\}} \prod_{(jk)} \exp\left[\beta s_j s_k\right] = \sum_{\{s_n\}} \prod_{(jk)} \left[\cosh\beta + s_j s_k \sinh\beta\right],$$

which relies on the fact $s_j s_k = \pm 1$ for the Ising model. It's easy to check the relation $e^{\beta s_j s_k} = \cosh\beta + s_j s_k \sinh\beta$ for both cases:

$s_j s_k = 1$

$e^{\beta} = \frac{1}{2}\left(e^{\beta} + e^{-\beta} + e^{\beta} - e^{-\beta}\right)$

$e^{\beta} = e^{\beta}$ ✓

$s_j s_k = -1$

$e^{-\beta} = \frac{1}{2}\left(e^{\beta} + e^{-\beta} - e^{\beta} + e^{-\beta}\right)$

$= e^{-\beta}$ ✓

Next, we write the sum over the $\cosh$ and $\sinh$ as an explicit summation,

$$Z = \sum_{\{s_n\}} \prod_{(jk)} \left[\cosh\beta + s_j s_k \sinh\beta\right] = \sum_{\{s_n\}} \prod_{(jk)} \sum_{p_{jk}=0,1} C_{p_{jk}}(\beta) (s_j s_k)^{p_{jk}},$$

raising $s_j s_k$ to the corresponding power p_{jk} while defining $C_0(\beta) \equiv \cosh\beta$ and $C_1(\beta) \equiv \sinh\beta$. Recall that the sum over nearest-neighbour pairs (jk) corresponds to summing over all $2N$ links in the $d = 2$ lattice. To make the language a little less awkward, we can say that $p_{jk} = 1$ corresponds to link jk being 'on' while $p_{jk} = 0$ when it is turned 'off'. The product and sum above account for all possible configurations of links that are turned on and off, which we can more conveniently represent as another configuration sum,

$$\sum_{\{p\}} \equiv \sum_{p_1=0,1} \cdots \sum_{p_{2N}=0,1} .$$

Introducing this configuration sum lets us isolate and then trivially rearrange the final product,

$$Z = \sum_{\{s_n\}} \sum_{\{p\}} \prod_{(jk)} C_{p_{jk}}(\beta) (s_j s_k)^{p_{jk}} = \sum_{\{s_n\}} \sum_{\{p\}} \left[\prod_{(jk)} C_{p_{jk}}(\beta)\right] \left[\prod_{(jk)} s_j^{p_{jk}} s_k^{p_{jk}}\right].$$

The final factor can now be converted from a product over links to a product over sites. Any given spin s_n will appear four times in the product, once for each of the four links connected to it in two dimensions. The product of these four factors can be rewritten

$$\prod_{(nk)} s_n^{p_{nk}} = s_n^{P_n}, \qquad \text{defining } P_n \equiv \sum_{(nk)} p_{nk}.$$

We are now left with a product over individual s_n:

$$Z = \sum_{\{p\}} \prod_{(jk)} C_{p_{jk}}(\beta) \sum_{\{s_n\}} \prod_{n=1}^{N} s_n^{P_n}.$$

$s_n = -1, 1$

$(1)^{P_n} + (-1)^{P_n}$

Although each s_n is raised to a power that depends on p_{nk} for all four of its links to its nearest neighbours, and can consider what happens when we sum over the two values $s_n = \pm 1$. There are two possibilities: If P_n is odd, then the $s_n = 1$ and $s_n = -1$ contributions cancel — making the entire product over sites vanish! Otherwise, if P_n is even, they add up to 2. In other words, we have

$$Z = \sum_{\{p\}} \prod_{(jk)} C_{p_{jk}}(\beta) \prod_{n=1}^{N} 2\delta_2(P_n) = 2^N \sum_{\{p\}} \prod_{(jk)} C_{p_{jk}}(\beta) \prod_{n=1}^{N} \delta_2\left(\Sigma_{(nk)} p_{nk}\right), \qquad (121)$$

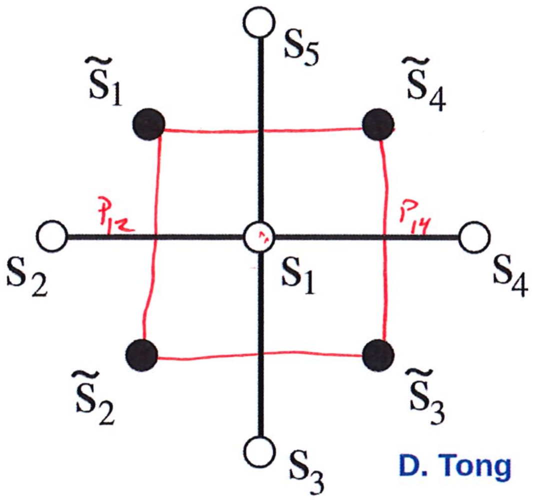
where the 'mod-2' Kronecker delta $\delta_2(P_n)$ vanishes if P_n is odd and equals one if P_n is even.

Something interesting has happened in Eq. 121: there is no longer any reference to our original spin degrees of freedom, s_n. We have successfully summed over all spin configurations $\{s_n\}$, at the cost of introducing a different sum over all configurations of on/off links ($p_{jk} = 1$ or 0, respectively). And there is a very tricky set of N inter-dependent constraints coming from the product of δ_2 factors, which require that an even number of links be turned on at *every* lattice site, in order to get a non-zero contribution to the partition function. In effect, this informs us that our new variables p_{jk} aren't really independent of each other — which makes sense, since we have $2N$ of them, but started off with only N degrees of freedom.

What we can do about this is essentially to try working backwards. We introduced the $p_{jk} = 0, 1$ in order to manipulate a configuration sum over $s_n = \pm 1$, so we can guess that introducing a different set of $\widetilde{s}_n = \pm 1$ can have an effect on the resulting configuration sum over p_{jk}. We want the ± 1 values of $\widetilde{s}_n$ to be related to the $\{0, 1\}$ values of p_{jk}, which we can achieve by implicitly defining $\widetilde{s}_n$ via

$$p_{12} = \frac{1 - \widetilde{s}_1 \widetilde{s}_2}{2} \qquad p_{13} = \frac{1 - \widetilde{s}_2 \widetilde{s}_3}{2}$$
$$p_{14} = \frac{1 - \widetilde{s}_3 \widetilde{s}_4}{2} \qquad p_{15} = \frac{1 - \widetilde{s}_1 \widetilde{s}_4}{2}$$

and so on for all $2N$ links. A convenient way to keep track of the subscripts above is to identify these $\widetilde{s}_n$ with the **dual lattice** drawn on the next page. Each $\widetilde{s}_n$ is identified with one of the N plaquettes of the original lattice, and pairs $\widetilde{s}_a$ and $\widetilde{s}_b$ determine p_{jk} for the link passing between them.



These relations confirm the claim above that p_{jk} aren't independent variables — both p_{12} and p_{15} depend on $\widetilde{s}_1$, both p_{12} and p_{13} depend on $\widetilde{s}_2$, etc. Delightfully, these patterns of dependence are precisely what we need to handle the δ_2 factors that are still in our partition function. If we consider

$$\begin{aligned} P_1 = p_{12} + p_{13} + p_{14} + p_{15} &= 2 - \frac{\widetilde{s}_1\widetilde{s}_2 + \widetilde{s}_2\widetilde{s}_3 + \widetilde{s}_3\widetilde{s}_4 + \widetilde{s}_1\widetilde{s}_4}{2} \\ &= 2 - \frac{(\widetilde{s}_1 + \widetilde{s}_3)(\widetilde{s}_2 + \widetilde{s}_4)}{2} \in \{0, 2, 4\}, \end{aligned}$$

we see that working in terms of $\widetilde{s}_n$ automatically turns on an even number of links at every site, producing $\prod_{n=1}^{N} \delta_2(P_n) = 1$!

In addition to eliminating odd P_1, you can confirm that all possible ways of obtaining even P_1 are accounted for as configurations of the dual spins $\widetilde{s}_n$. In fact, they all appear twice, as we can see by checking the simple example $P_1 = 4$:

$$P_1 = 4 = 2 - \frac{1}{2}(\widetilde{s}_1 + \widetilde{s}_3)(\widetilde{s}_2 + \widetilde{s}_4) \rightarrow (\widetilde{s}_1 + \widetilde{s}_3)(\widetilde{s}_2 + \widetilde{s}_4) = -4 = (2)(-2)$$

$$\{\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3, \widetilde{s}_4\} = \{1, -1, 1, -1\}; \quad \{-1, 1, -1, 1\} \qquad \widetilde{s}_n \rightarrow -\widetilde{s}_n$$

leave all p_{jk} unchanged

This generalizes to the full N-spin system,[15] so converting the partition function from the configuration sum over $\{p\}$ in Eq. 121 to a configuration sum over $\{\widetilde{s}\}$ gives us

$$Z = \frac{1}{2} 2^N \sum_{\{\widetilde{s}\}} \prod_{(jk)} C_{(1 - \widetilde{s}_j\widetilde{s}_k)/2}(\beta).$$

The final step is to express

$$C_0(\beta) = \cosh\beta \qquad\qquad C_1(\beta) = \sinh\beta$$

[15] This is discussed in more detail by Robert Savit, "Duality in field theory and statistical systems", *Reviews of Modern Physics* **52**:453, 1980

in terms of the dual variables $\widetilde{s}_n$ that we're now working with. It's easy to see that

$$C_p(\beta) = (\cosh\beta)\exp\left[p\log\tanh\beta\right] = (\cosh\beta)\exp\left[\frac{1-\widetilde{s}_j\widetilde{s}_k}{2}\log\tanh\beta\right],$$

substituting $p = (1-\widetilde{s}_j\widetilde{s}_k)/2$. Breaking up the exponential gives us

$$C_p(\beta) = (\cosh\beta\sinh\beta)^{1/2}\exp\left[-\frac{1}{2}\widetilde{s}_j\widetilde{s}_k\log\tanh\beta\right].$$

Inserting this into the partition function, the product over all links just provides $2N$ factors of the $\widetilde{s}_n$-independent first term, and we can then convert the product of exponentials into an exponential of the sum, producing

$$Z = \frac{1}{2}(2\cosh\beta\sinh\beta)^N\left(\sum_{\{\widetilde{s}\}}\exp\left[\underbrace{-\frac{\log\tanh\beta}{2}}_{\widetilde{\beta}}\sum_{(jk)}\widetilde{s}_j\widetilde{s}_k\right]\right).$$

If we define

$$\widetilde{\beta} \equiv -\frac{\log\tanh\beta}{2},$$

then we can recognize the sum over $\{\widetilde{s}\}$ configurations as simply a zero-field Ising model partition function $Z(\widetilde{\beta})$ as in Eq. 112. We can also recognize this definition of $\widetilde{\beta}$ as equivalent to Eq. 122:

$$\tanh\beta = e^{-2\widetilde{\beta}}$$

$$e^{2\widetilde{\beta}} - e^{-2\widetilde{\beta}} = \frac{1}{\tanh\beta} - \tanh\beta = \frac{\cosh\beta}{\sinh\beta} - \frac{\sinh\beta}{\cosh\beta} = \frac{1}{\cosh\beta\sinh\beta}$$

$$\cosh\beta\sinh\beta = \frac{1}{4}\left(e^{\beta}+e^{-\beta}\right)\left(e^{\beta}-e^{-\beta}\right) = \frac{1}{4}\left(e^{2\beta}-e^{-2\beta}\right) = \frac{1}{2}\sinh(2\beta)$$

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$$\sinh(2\widetilde{\beta}) = \frac{1}{\sinh(2\beta)}$$

Using similar manipulations, we can express the spin-independent prefactor in terms of either β or $\widetilde{\beta}$,

$$2\cosh\beta\sinh\beta = \sinh(2\beta) = \frac{1}{\sinh(2\widetilde{\beta})},$$

or in the mixed form that reproduces Eq. 121:

$$2\cosh\beta\sinh\beta = 2\cosh^2\beta\tanh\beta = \frac{2\cosh^2\beta}{e^{2\widetilde{\beta}}} \qquad \Longrightarrow \qquad \frac{Z(\beta)}{\left(2\cosh^2\beta\right)^N} = \frac{Z(\widetilde{\beta})}{2e^{2N\widetilde{\beta}}}.$$

We have successfully derived Kramers–Wannier duality! Now let's briefly interpret what it means. It's a worthwhile **exercise** to show that multiplying a partition function by an overall spin-independent factor, $Z(\beta) \to c(\beta)Z(\beta)$, has no