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Recap:

Interacting systems → Ising model

Phase transitions signalled by discontinuity in order parameter or derivatives

Ordered phase vs. disordered phase

$\langle m \rangle \neq 0$ $\langle m \rangle = 0$ for $N \to \infty$

Magnetization $\langle m \rangle = \frac{1}{N\beta} \frac{\partial}{\partial H} \log Z$

Mean-field approx. based on $H_{eff} = 2d\langle m \rangle + H$

→ mean of nearest-neighbours

In light of this result, it isn't surprising that the mean-field approximation producing Eq. 115 makes it very easy to compute the corresponding canonical partition function

$$\begin{aligned}
Z_{\text{MF}} &= \sum_{\{s_n\}} \exp\left[-\beta E(s_n)\right] = \exp\left[-\beta d \cdot N \langle m\rangle^2\right] \left(\sum_{s_1=\pm 1} \cdots \sum_{s_N=\pm 1} \exp\left[-x \sum_{n=1}^{N} s_n\right]\right) \\
&= \exp\left[-\beta d \cdot N \langle m\rangle^2\right] \left(2\cosh\left[\beta H_{\text{eff}}\right]\right)^N \\
&= \exp\left[-\beta d \cdot N \langle m\rangle^2\right] \left(2\cosh\left[\beta\left(2d\langle m\rangle + H\right)\right]\right)^N, \qquad (116)
\end{aligned}$$

where we defined $x \equiv -\beta H_{\text{eff}}$ to put the sums into the same form as in Eq. 41. Although this factorized result is far simpler than the partition function for the full Ising model, it does involve some complicated dependence on $\langle m\rangle$ — especially when we recall that $\langle m\rangle$ itself is related to a derivative of $\log Z_{\text{MF}}$. With

$$\log Z_{\text{MF}} = N \log\cosh\left[\beta\left(2d\langle m\rangle + H\right)\right] + \{H\text{-independent terms}\},$$

the relation we derived above gives us

$$\langle m\rangle = \frac{1}{N\beta}\frac{\partial}{\partial H}\log Z_{\text{MF}} = \frac{1}{\beta}\frac{1}{\cosh\left[\beta\left(2d\langle m\rangle + H\right)\right]}\frac{\partial}{\partial H}\cosh\left[\beta\left(2d\langle m\rangle + H\right)\right].$$

Simplifying, we obtain a **self-consistency condition** for the Ising model magnetization in the mean-field approximation:

$$\langle m\rangle = \tanh\left[\beta\left(2d\langle m\rangle + H\right)\right]. \qquad (117)$$

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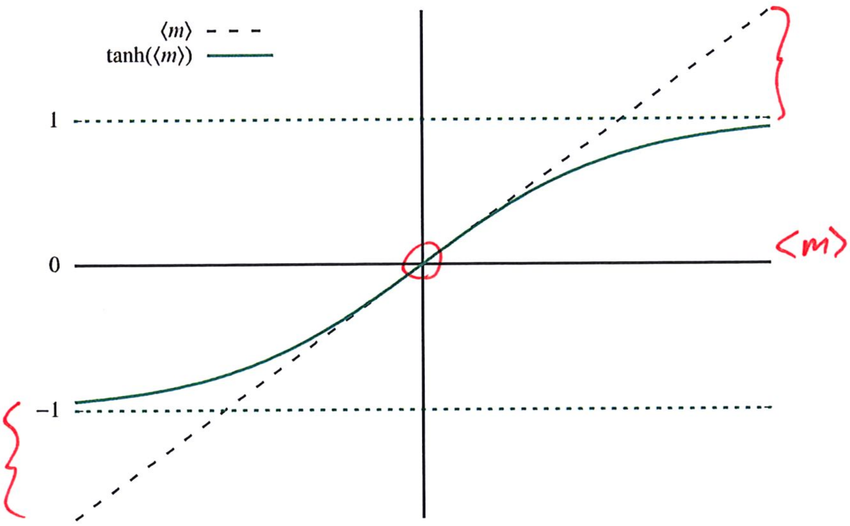
Solving this equation for $\langle m\rangle$ is equivalent to finding the roots of the equation $\tanh\left[\beta(2d \cdot x + H)\right] - x = 0$.

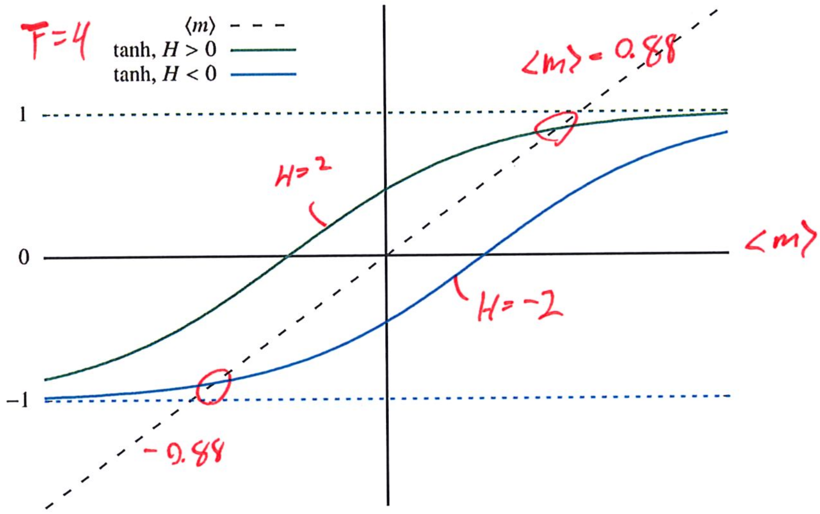
A straightforward way to inspect such solutions is by plotting both

$$f(\langle m\rangle) = \langle m\rangle \qquad\qquad g(\langle m\rangle) = \tanh\left[\beta(2d\langle m\rangle + H)\right]$$

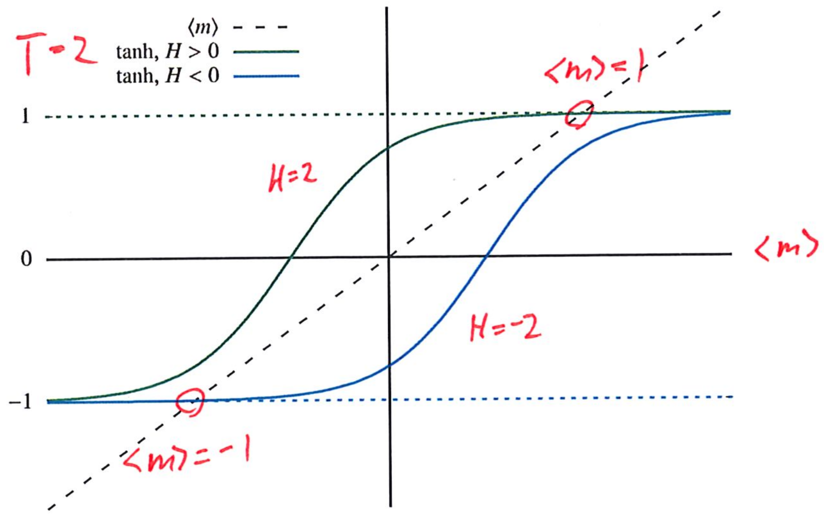
and monitoring the intersections of these two functions. Fixing $d = 2$ dimensions, the plot below considers the simplest case $\beta = \frac{1}{4}$ and $H = 0$ for which $g(\langle m\rangle) = \tanh\left[\langle m\rangle\right]$ (the solid line). There is only a single intersection between this function and $f(\langle m\rangle)$ (the dashed line), at $\langle m\rangle = 0$, which we should interpret as a disordered phase.



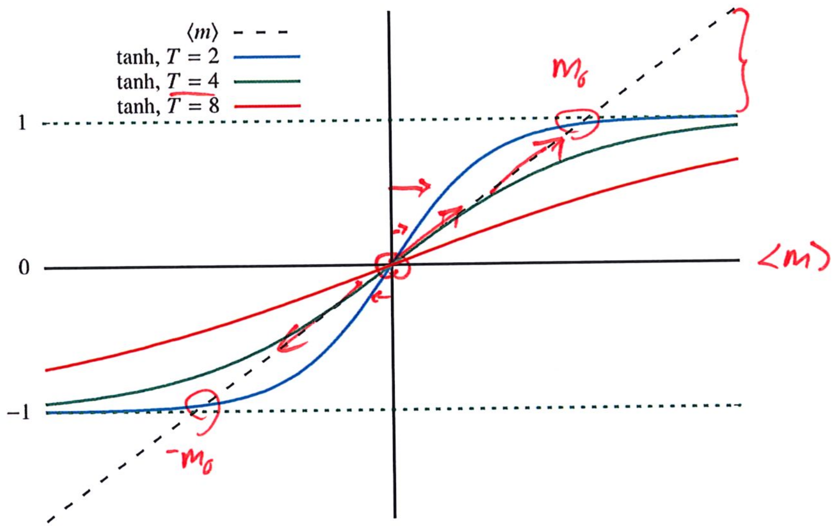
To confirm our interpretation of this result, let's check how the intersections depend on β and H. In the next plot below we keep the same temperature $T = 1/\beta = 4$ while comparing two non-zero values for the external magnetic field. A positive $H = 2$ simply shifts $g(\langle m \rangle)$ to the left (the green line), while a negative $H = -2$ shifts it to the right (the blue line). In both cases there is still only a single intersection, at $\langle m \rangle \approx \pm 0.88$ for $H = \pm 2$. We can interpret this non-zero result as an indication that the system is in an ordered phase where the spins tend to align with the external field.



From our work in the previous section, we can expect that the spins' alignment will increase — approaching the minimal-energy ground state — as the temperature decreases. Decreasing the temperature increases β, which causes the argument of the $\tanh$ to vary more rapidly with $\langle m \rangle$, making $g(\langle m \rangle)$ a steeper function that more rapidly approaches its limiting values ± 1. The plot below illustrates this for $T = 1/\beta = 2$, so that $\beta = \frac{1}{2}$ is doubled. Already for this temperature and magnetic field $H = \pm 2$, the intersection is $\langle m \rangle \approx \pm 1$ to a very good approximation. We can also appreciate that $-1 \leq \tanh x \leq 1$ ensures that the mean-field self-consistency condition can only ever be satisfied for $-1 \leq \langle m \rangle \leq 1$, reassuringly consistent with the definition of the magnetization.



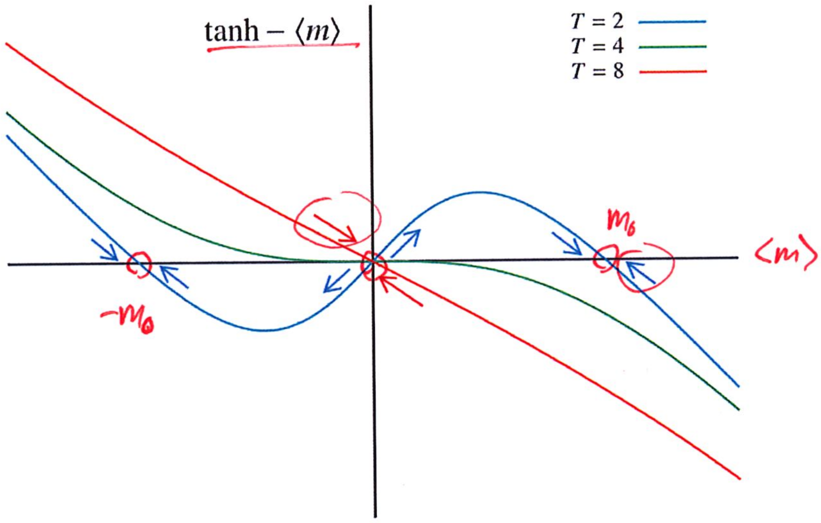
Also in the previous section, we saw that the Ising model spins should align at low temperatures, even without an external field to promote one direction over the other. We hope to see this behaviour captured by the mean-field approximation, which we can check by considering the self-consistency condition for various temperatures with $H = 0$. The plot below shows the results, considering a low temperature $T = 2$ with $\beta = \frac{1}{2}$ (the blue line), the same green curve for $T = 4$ shown in the first plot above, and a high temperature $T = 8$ with $\beta = \frac{1}{8}$ (the red line). While the $\langle m \rangle = 0$ expected in the disordered phase is always a possible solution, something interesting happens at lower temperatures, where the steeper $\tanh$ function introduces two additional solutions at non-zero $\langle m \rangle = \pm m_0$ corresponding to the ordered phase. As $T \to 0$, this magnetization approaches its maximum value $m_0 \to 1$.



When there are three solutions $\langle m \rangle = \{-m_0, 0, m_0\}$ at low temperatures, we can determine that the $\langle m \rangle = 0$ solution is actually *unstable*. Here we are venturing briefly into non-equilibrium territory, and thinking of the mean-field system as a 'blind' process that attempts to satisfy the self-consistency condition $\langle m \rangle = \tanh[2\beta d \langle m \rangle]$, based only on knowledge of whether the expectation value of the magnetization is too small or too large compared to the $\tanh$. Once the magnetization is self-consistent, the system can happily settle into thermodynamic equilibrium.

From the figure above we can see that (with $H = 0$) we can have three solutions only when the slope of the $\tanh$ at $\langle m \rangle = 0$ is greater than 1. Any positive value $\langle m \rangle = \varepsilon > 0$ would then produce $\tanh[2\beta d \langle m \rangle] > \langle m \rangle$, which the system 'feels' as a magnetization that is too small to be self-consistent. This drives the system to continue increasing its magnetization, until it eventually settles at the non-zero solution $\langle m \rangle = m_0$. Similarly, any negative magnetization $\langle m \rangle = -\varepsilon < 0$ would drive $\langle m \rangle$ away from zero and to the $\langle m \rangle = -m_0$ solution.

This argument can be visualized more easily by plotting $\tanh[2\beta d \langle m \rangle] - \langle m \rangle$ vs. $\langle m \rangle$ as shown in the final plot below. Whenever this difference is negative, it implies $\langle m \rangle$ is larger than the self-consistency condition allows, driving the system to smaller $\langle m \rangle$ as shown by the arrows pointing to the left. Conversely, whenever the difference is positive, the system 'seeks' self-consistency by increasing $\langle m \rangle$ as shown by the arrows pointing to the right. For the low temperature $T = 2$, we see that the arrows move the system away from the unstable solution $\langle m \rangle = 0$ and to the stable solutions $\langle m \rangle = \pm m_0$.



So in the end we can conclude that the non-interacting mean-field approximation successfully captures at least the high- and low-temperature limits of the interacting zero-field Ising model that we determined in the previous section. For high temperatures the mean-field self-consistency condition demands $\langle m \rangle = 0$ consistent with the disordered phase, while for low temperatures it produces $|\langle m \rangle| = m_0 > 0$ consistent with the ordered phase.

Going further, now that we have a more tractable non-interacting system we can consider the value of the temperature at which the $\langle m \rangle = \pm m_0$ solutions appear and the $\langle m \rangle = 0$ solution becomes unstable. As described above, this occurs whenever the slope of the $\tanh$ function at $\langle m \rangle = 0$ is greater than 1. Let's call the corresponding temperature T_c, though it remains to be determined whether it is really a critical temperature of a true phase transition. Expanding $\tanh(x) = x + \mathcal{O}\left(x^3\right)$ for $x \approx 0$, what is T_c?

$$\frac{d}{dm}\left(\tanh(2\beta d\langle m\rangle) = 2\beta d\langle m\rangle + \mathcal{O}(\langle m\rangle^3)\right)\Big|_{\langle m\rangle=0}$$

$$2\beta d = 1 \longrightarrow T_c = 2d = \frac{1}{\beta_c}$$

You should find that the change from the high-temperature disordered phase to the low-temperature ordered phase occurs at $T_c = 2d$ in d dimensions, or equivalently $\beta_c = \frac{1}{2d}$ — corresponding to the green lines in the two figures above with

$d = 2$. In order to determine whether or not this is a true critical temperature, we need to check whether the order parameter $\langle m \rangle$ or its T-derivatives are discontinuous at T_c. We can do this by considering the self-consistency condition for a temperature T lower than but very near to $T_c = 2d$, which would produce $0 < |\langle m \rangle| \ll 1$ and allow us to expand $\tanh(x) = x - \frac{x^3}{3} + \mathcal{O}(x^5)$. What is the resulting prediction for $\langle m \rangle$? $2d = T_c$

$$\langle \cancel{m} \rangle = \tanh(2\beta d \langle m \rangle) = 2\beta d \langle \cancel{m} \rangle - \frac{1}{3}(2\beta d \langle m \rangle)^{3} + \mathcal{O}(\langle m \rangle^5)$$

$$\frac{1}{3}\left(\frac{T_c}{T}\right)^3 \langle m \rangle^2 = \frac{T_c}{T} - 1$$

$$\langle m \rangle \approx \pm\sqrt{3}\,\underbrace{\left(\frac{T}{T_c}\right)^{3/2}}_{\to 1}\left(\frac{T_c}{T} - 1\right)^{1/2} \approx \pm\sqrt{3}\left(1 - \frac{T}{T_c}\right)^{1/2}$$

Making the approximation $\left(\frac{T}{T_c}\right)^2 \approx 1$, your result should resemble

$$\langle m \rangle = \pm\sqrt{3}\left(\frac{T_c - T}{T_c}\right)^{1/2} \qquad \text{for } T \lesssim T_c.$$

$\langle m \rangle \to 0$ as $T \to T_c$

From this, we can see that the order parameter $\langle m \rangle$ is continuous at T_c:

$$\langle m \rangle \propto \begin{cases} (T_c - T)^{1/2} & \text{for } T \lesssim T_c \\ 0 & \text{for } T \gtrsim T_c \end{cases}. \tag{118}$$

However, its first derivative

$$\frac{d\langle m \rangle}{dT} \propto \frac{1}{(T_c - T)^{1/2}}$$

diverges as $T \to T_c$ from below. This is the situation we discussed at the end of the previous section, which predicts a second-order phase transition with critical temperature $T_c = 2d$ in d dimensions. The power-law dependence $\langle m \rangle \propto (T_c - T)^b$ with *non-integer* b is a generic feature of second-order phase transitions. The power b is known as a **critical exponent**, in this case $b = 1/2$.

At this point we have invested some effort to find that the mean-field approximation of the d-dimensional Ising model, with $H = 0$, predicts a second-order phase transition at $T_c = 2d$ with critical exponent $1/2$. Let's wrap up this section with some quick comments on the reliability of the mean-field approximation and the accuracy of these results it has given us.

The accuracy of the mean-field results turns out to depend on the number of dimensions. For the one-dimensional Ising model that Ising himself solved, there is no phase transition at all, as we will derive in the next section. In other words, the mean-field approximation simply fails for $d = 1$.

The situation improves for the two-dimensional Ising model. Onsager's exact $H = 0$ solution features a second-order phase transition, at an inverse critical temperature $\beta_c = \frac{1}{2}\log\left(1+\sqrt{2}\right) \approx 0.44$ that had been exactly determined a few years before his work. For $T \lesssim T_c$, the magnetization vanishes as $\langle m \rangle \propto (T_c - T)^{1/8}$, corresponding to a critical exponent $1/8$. While the mean-field prediction of a second-order phase transition is now qualitatively correct, at a quantitative level its predicted $\beta_c = \frac{1}{2d} = 0.25$ is off by almost a factor of 2, while the mean-field critical exponent $b = 1/2$ is four times larger than the true $b = 1/8$.

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For higher dimensions $d \geq 3$ there is no known exact solution for the Ising model, but the existence of a second-order phase transition can be established and the corresponding critical temperature and critical exponents can be computed numerically, as we will discuss in Unit 10. In three dimensions the mean-field $T_c = 2d = 6$ and $b = 1/2$ are still significantly different from the true $T_c \approx 4.5$ and $b \approx 0.32$. The mean-field prediction for the critical exponent $b = 1/2$ turns out to be correct for $d \geq 4$, while the critical temperature $T_c = 2d$ gradually approaches the true value as the number of dimensions increases. Numerical computations find $T_c \approx 6.7$, 8.8, 10.8 and 12.9 for $d = 4$, 5, 6 and 7, respectively, so that the mean-field result improves from being $\sim$19% too high for $d = 4$ to only $\sim$9% too high for $d = 7$. Formally, the mean-field approximation exactly reproduces the Ising model in the abstract limit of infinite dimensions, $d \to \infty$. Roughly speaking, the greater reliability of the mean-field approach in higher dimensions is due to the larger number of nearest neighbours for each site, $2d$. The larger number of nearest-neighbour spins produces a more reliable approximation of the mean spin in the effective field seen by each site in the mean-field approach.

9.4 Supplement: Ising model exact results

If time permits, it is not too hard to prove some of the exact results mentioned above, for Ising models in one and two dimensions where the mean-field approximation is least reliable.

9.4.1 One-dimensional partition function and magnetization

The special property of the one-dimensional Ising model that helps us derive a closed-form expression for its partition function is the fact that it has exactly as many links as it has sites. Looking back to the illustration on page 134, we can rewrite the nearest-neighbour interaction term as

$$\sum_{(jk)} s_j s_k = \sum_{n=1}^{N} s_n s_{n+1},$$

where the periodic boundary conditions identify $s_{N+1} = s_1$. If we also rewrite $H\sum_{n=1}^{N} s_n = \frac{H}{2}\sum_{n=1}^{N}(s_n + s_{n+1})$, then the full internal energy is

$$E_i = -\sum_{n=1}^{N}\left[s_n s_{n+1} + \frac{H}{2}(s_n + s_{n+1})\right].$$