$$E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 \langle N\rangle_f}{V}\right)^{2/3}$$

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8.6 Type-Ia supernovas

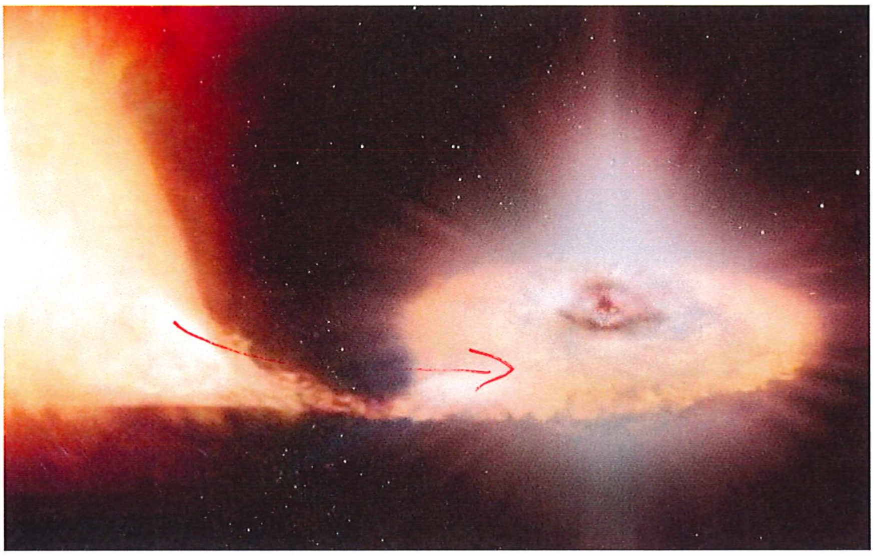
The positive pressure that remains for a fermion gas even at zero temperature, Eq. 104, is known as the *degeneracy pressure*. (This use of the word 'degeneracy' is unrelated to its other use describing multiple energy levels with the same value of the energy.) The degeneracy pressure plays a key role in a certain type of supernova explosions of stars — a famous astrophysical phenomenon.

To begin considering this topic, we'll remark that the temperature doesn't need to be exactly zero in order for the degeneracy pressure to be significant. The temperature just needs to be small compared to the Fermi energy, $T \ll E_F$. From Eq. 101 we can see that $E_F \propto \rho_\mathsf{f}^{2/3}$ increases for larger densities $\rho_\mathsf{f} = \langle N\rangle_\mathsf{f}/V$. Not surprisingly, the densities of stars can be very large indeed, due to the enormous amount of matter that is being squeezed together by gravitational attraction. Everyday matter has densities around 10^{28}–10^{29} atoms per cubic metre (roughly Avogadro's number per cubic centimetre), which corresponds to Fermi energies $E_F \sim 10^4$ K, very large compared to everyday temperatures. Fermi energies for particularly dense stars known as *white dwarfs* are a hundred thousand times larger still, $E_F \sim 10^9$ K, with densities around 10^{35}–10^{36} atoms per cubic metre roughly equivalent to a mass density of one tonne per cubic centimetre. This is around a million times more dense than our sun — while white dwarf stars have a mass similar to our sun's $M_\odot$, their radius is a hundred times smaller, comparable to the radius of the earth.

White dwarf stars are so dense because they have exhausted the hydrogen and helium 'fuel' for the nuclear fusion that generates heat and light — and therefore radiation pressure — in stars such as our sun. For actively 'burning' stars, this radiation pressure counteracts the gravitational attraction of the star's enormous mass. Without nuclear fusion, white dwarfs end up gravitationally compressed into much denser and more compact objects. The degeneracy pressure, Eq. 104, coming from the (fermionic) electrons in the white dwarf is what stabilizes these stars and prevents them from collapsing further into even denser objects such as black holes.

It is remarkable that even under these extreme conditions the electrons in white dwarf stars are well described by the non-interacting ideal fermion gas we analyzed above. In particular, it is crucial that white dwarfs' Fermi energies are so large, $E_F \sim 10^9$ K. Even though white dwarfs have burned all their nuclear fuel, their interiors remain quite hot by everyday standards, roughly ten million kelvin ($T \sim 10^7$ K). It is only due to their large densities and Fermi energies that $T \ll E_F$ and white dwarfs can be treated as zero-temperature objects to a good approximation.

So far we haven't encountered supernovas. In isolation, white dwarfs will happily cool for trillions of years, supported by their degeneracy pressure, until they reach thermal equilibrium with the low-temperature cosmic microwave background radiation we discussed in Section 8.2. Things become more interesting for a white dwarf in a binary system with a companion star. If this companion star is still burning hydrogen or helium through nuclear fusion, it will emit matter that is then captured by the white dwarf, slowly increasing the white dwarf's mass. Such a binary system is pictured below, in an artist's illustration provided by the European Space Agency.



As the white dwarf accumulates the matter emitted by its companion, its mass and its density will steadily increase. As the mass of the white dwarf approaches a value roughly 40% larger than the mass of our sun, known as the *Chandrasekhar limit* (named after Subrahmanyan Chandrasekhar), its density becomes large enough for new types of nuclear fusion reactions to occur. Instead of hydrogen or helium, which the white dwarf has already burned, these new fusion reactions involve carbon and oxygen, which remain present in abundance. In the space of just a few seconds, these fusion reactions run away, increase the temperature of the white dwarf to billions of kelvin, and blast it apart in a supernova explosion about five billion times brighter than the sun.

For obscure historical reasons, these particular stellar explosions are known as *type-Ia* ("one-A") *supernovas*. They rely on the degeneracy pressure (Eq. 104) of a low-temperature gas of non-interacting fermions, which allows a specific amount of mass to build up before the explosion is triggered. The specificity of the process results in a great deal of regularity among type-Ia supernovas, which is very useful to astronomers. In particular, these type-Ia supernovas play a key role in demonstrating that the expansion of the universe is accelerating (a phenomenon popularly called 'dark energy'), which was awarded the 2011 Nobel Prize in Physics.

8.7 Relativistic ideal fermion gas

Although we will discuss them more briefly, gases of relativistic fermions also play important roles in nature. In fact, by changing units we can see that the white dwarf Fermi energy discussed above, $E_F \sim 10^9$ K ~ 0.3 MeV is comparable to the 0.511 MeV rest-energy of electrons, suggesting that relativistic effects may be non-negligible in white dwarfs. Such relativistic effects are indeed crucial to the computation of the Chandrasekhar limit mentioned above.

While the full calculations required to analyze massive relativistic particles are beyond the scope of this module, we can take advantage of our earlier analyses of gases of massless photons to briefly consider similar gases of massless fermions. *Neutrinos* (denoted 'ν') are physical examples of fermions whose masses are so small that they can be very well approximated as massless. In fact, for many years neutrinos were thought to be exactly massless — the discovery that neutrinos have non-zero masses was awarded the 2015 Nobel Prize in Physics.

In the same way as photons, massless fermions would travel at the speed of light, c, and have energies $E = cp$ determined by their angular frequencies,

$$E_\nu = \hbar\omega.$$

In a volume $V = L^3$, these energies are quantized as usual,

$$\omega = \frac{2\pi c}{\lambda} = c\frac{\pi}{L}k,$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$ and $k_{x,y,z} > 0$ are positive integers. Just as for the non-relativistic case considered in Section 8.4, for each distinct $\vec{k}$ typical massless fermions, including neutrinos, have two degenerate energy levels with the same energy $E(k)$ but opposite spin.

The computations required to analyze a gas of massless fermions are very similar to the work we recently did for photon gases. In particular, massless fermions are also easy to create and destroy, and therefore well described by a vanishing chemical potential, $\mu \approx 0$. Again approximating sums over discrete integer $k_{x,y,z}$ by integrals over continuous real $\widehat{k}_{x,y,z}$, and changing variables to integrate over the angular frequency, we end up with the grand-canonical potential

$\mu = \frac{\partial E}{\partial N}$

$$\Phi_\nu = -\frac{VT}{c^3\pi^2}\int_0^\infty d\omega\,\omega^2 \log\left[1 + e^{-\beta\hbar\omega}\right]. \tag{106}$$

The only changes here compared to Eq. 94 for the photon Φ_{ph} are a couple of negative signs, precisely as we would expect from comparing the Bose–Einstein and Fermi–Dirac grand-canonical potentials in Section 7.5.

Due to these negative signs, when we compute derived quantities by taking derivatives of the potential,

$$\langle E\rangle_\nu = \frac{\partial}{\partial\beta}\left[\beta\Phi_\nu\right] \qquad\qquad \langle N\rangle_\nu = -\left.\frac{\partial}{\partial\mu}\Phi_\nu\right|_{\mu=0},$$

we will encounter slightly different but equally enjoyable integrals,

$$\int_0^\infty \frac{x^3}{e^x+1}dx = \left(1 - \frac{1}{2^3}\right)\Gamma(4)\zeta(4) = \frac{7\pi^4}{120}, \quad = 7\left(\frac{3}{4}\right)\frac{\pi^4}{90}$$

$$\int_0^\infty \frac{x^2}{e^x+1}dx = \left(1 - \frac{1}{2^2}\right)\Gamma(3)\zeta(3) = \frac{3}{2}\zeta(3).$$

$$\langle E\rangle_\nu = \langle N\rangle_\nu T\left(\frac{7\pi^2}{120}\right)\left(\frac{2\pi^2}{3\zeta(3)}\right) = \frac{7\pi^4}{180\zeta(3)}\langle N\rangle_\nu T$$

Using these results, what are the average internal energy density and the average particle number density for a gas of massless fermions?

$$\frac{\langle E\rangle_\nu}{V} = -\frac{1}{c^3\pi^2}\int_0^\infty d\omega\, \omega^2 \frac{\partial}{\partial\beta}\log\left[1+e^{-\beta\hbar\omega}\right] = -\frac{1}{c^3\pi^2}\int_0^\infty \frac{-\hbar\omega^3 e^{-\beta\hbar\omega}}{1+e^{-\beta\hbar\omega}}\, d\omega$$

$$\boxed{\begin{aligned} x &= \beta\hbar\omega = \hbar\omega/T \\ d\omega &= \left(\tfrac{T}{\hbar}\right)dx \end{aligned}}$$

$$= \frac{\hbar}{c^3\pi^2}\left(\frac{T}{\hbar}\right)^4\int_0^\infty \frac{x^3}{e^x+1}\,dx = \frac{7\pi^2 T^4}{120\,\hbar^3 c^3}$$

$$\frac{\langle N\rangle_\nu}{V} = \frac{T}{c^3\pi^2}\int_0^\infty d\omega\, \omega^2 \frac{\partial}{\partial\mu}\log\left[1+e^{-\beta\hbar\omega}e^{\beta\mu}\right]\bigg|_{\mu=0} = \frac{T\beta}{c^3\pi^2}\left(\frac{T}{\hbar}\right)^3\int_0^\infty \frac{x^2}{e^x+1}\,dx$$

$$= \frac{3\zeta(3)T^3}{2\pi^2\hbar^3c^3}$$

You should again find $\langle E\rangle_\nu \propto \langle N\rangle_\nu T \propto VT^4$, and by noting that

$7\pi^4/360 = \frac{7}{4}\zeta(4)$ // $x = \beta\hbar\omega$

$$\frac{\Phi_\nu}{T} = \frac{V}{c^3\pi^2}\left(\frac{T}{\hbar}\right)^3\int_0^\infty dx\, x^2\log\left[1+e^{-x}\right] \propto VT^3,$$

we can see that the entropy $S_\nu = (E_\nu - \Phi_\nu)/T$ is constant when $VT^3 = \text{constant}$. Applying this, what is the pressure for a gas of massless fermions?

$$P_\nu = -\frac{\partial}{\partial V}\langle E\rangle_\nu\bigg|_{S_\nu} = -\frac{7\pi^2}{120\hbar^3c^3}\frac{\partial}{\partial V}(VT^4)\bigg|_{S_\nu} = -\frac{7\pi^2}{120\hbar^3c^3}\frac{\partial}{\partial V}b^4V^{-1/3}$$

$$\boxed{T = bV^{-1/3}}$$

$$= \frac{1}{3V}\left(\frac{7\pi^2}{120\hbar^3c^3}b^4V^{-1/3}\right)$$

$$= \frac{1}{3}\frac{\langle E\rangle_\nu}{V}$$

You should find yet another equation of state with the usual functional form,

$$P_\nu V = \frac{1}{3}\langle E\rangle_\nu = \frac{7\pi^4}{540\zeta(3)}\langle N\rangle_\nu T,$$

and just a new numerical factor of

$$\frac{7\pi^4}{540\zeta(3)} = \frac{(7/8)\zeta(4)}{(3/4)\zeta(3)}. = \frac{7}{6}\frac{\zeta(4)}{\zeta(3)} \sim 1.05$$

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