

28 March

Recap:

Radiation pressure

→ photon gas

$$PV = \frac{\zeta(4)}{\zeta(3)} NT = \frac{\pi^4}{90 \zeta(3)} NT \approx 0.9 NT$$

1.202

Non-rel. $E = \frac{p^2}{2m} \quad \times 2 \text{ spin}$

$$\frac{\langle N \rangle_F}{V} \propto \int_0^\infty F(E) N E dE$$

$\left\{ \frac{1}{e^{\beta(E-\mu)} + 1} \right.$

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You should find a result proportional to $\mu^{3/2}$ but independent of T . The temperature independence turns out to be the leading-order behaviour of a more general result that can be organized in powers of the small temperature, $T/\mu \ll 1$, through a method known as the *Sommerfeld expansion* (named after Arnold Sommerfeld). The $\mu^{3/2}$ dependence on the chemical potential is something we could have predicted before doing the explicit calculation. This is because the step function in Eq. 100 corresponds to a single fermion occupying each and every energy level with $E_\ell < \mu$, while all energy levels with $E_\ell > \mu$ are unoccupied. Since $E(k) \propto k^2$, summing over all $k_{x,y,z}$ for which $E(k) < \mu$ corresponds to computing (a portion of) the volume of a sphere of radius $r_k = \sqrt{\mu}$. This volume is proportional to $r_k^3 = \mu^{3/2}$, in agreement with our result above. If we invert that result, we obtain the so-called Fermi energy as a function of the average particle number density,

$$E_F = \mu = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 \langle N \rangle_f}{V} \right)^{2/3}. \quad (101)$$

Like the step-function approximation to the Fermi function, this equality between the Fermi energy and the chemical potential is only exact at zero temperature, with $T > 0$ introducing small corrections that take some work to compute.

Now we can consider the average energy density of the non-relativistic fermion gas at low temperatures. Rather than taking another derivative of the grand-canonical potential, we can note from Eq. 85 and from our work on the photon gas in Section 8.3 that

$$\frac{\langle E \rangle_f}{V} = \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\infty \langle E \rangle F(E) \sqrt{E} dE. \quad (102)$$

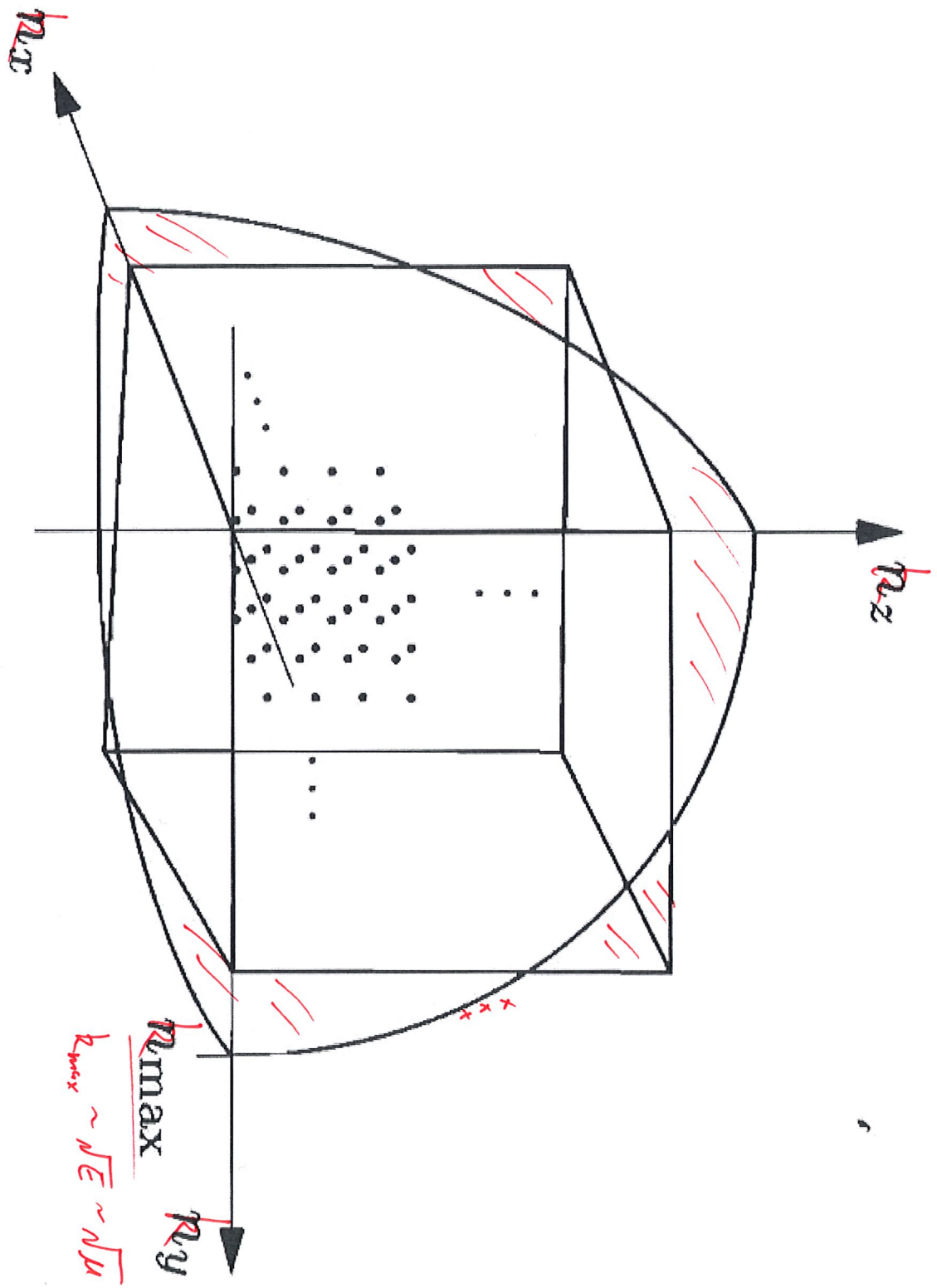
That is, instead of simply counting the number of fermions in the system, we need to add up their energies, introducing an extra factor of E compared to Eq. 99. Still using the low-temperature step-function approximation for the Fermi function in Eq. 100, what is the average energy density?

$$\begin{aligned} \frac{\langle E \rangle_f}{V} &= \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\mu F(E) E^{3/2} dE = \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \int_0^\mu E^{3/2} dE = \frac{\sqrt{2m^3}}{\pi^2 \hbar^3} \left(\frac{E^{5/2}}{5/2} \right) \Big|_0^\mu \\ &= \frac{(2m\mu)^{3/2}}{5\pi^2 \hbar^3} \mu = \frac{3}{5} \frac{\langle N \rangle_f}{V} \mu \end{aligned}$$

You should find

$$\langle E \rangle_f = \frac{3}{5} \mu \langle N \rangle_f, \quad (103)$$

which means that the average energy of each fermion in a low-temperature ideal gas, $\langle E \rangle_f / \langle N \rangle_f$, is three-fifths of the Fermi energy $E_F = \mu$.



In particular, because Eq. 103 is independent of the temperature, we find that non-interacting quantum fermions retain a positive energy even as the temperature approaches absolute zero, $T \rightarrow 0$.

This can be understood by recalling that the lowest-energy pair of degenerate energy levels can each only hold a single fermion, forcing any additional fermions to 'fill' energy levels with larger energies $E_\ell > 0$, up to the Fermi energy set by the chemical potential. This is a stark contrast to the $\langle E \rangle = \frac{3}{2}NT$ we found for classical ideal gases in the canonical ensemble in Eq. 52, as well as the $\langle E \rangle_{\text{ph}} \approx 2.7 \langle N \rangle_{\text{ph}} T \propto T^4$ we more recently computed in Eq. 97 for a grand-canonical quantum gas of photons. In both of those cases the average energy vanishes in the zero-temperature limit. This is because all the particles in those classical and bosonic systems are able to occupy the lowest energy level at low temperatures, with only exponentially small probabilities $\propto e^{-E_\ell/T}$ for particles to occupy any energy levels with $E_\ell > E_0$.

This picture of fermions filling energy levels up to the Fermi energy also clarifies why the chemical potential for a fermion gas must be positive at low temperatures. Recalling Eq. 81 for the chemical potential,

$$\mu = \left. \frac{\partial E}{\partial N} \right|_{S,V}, > 0$$

which we derived from the generalized thermodynamic identity in Section 6.3, we can consider what happens when we increase the number of particles in a zero-temperature fermion gas. In this limit $T \rightarrow 0$, there is only the single quantum micro-state described above, with all energy levels filled below the Fermi energy and empty above the Fermi energy. Adding particles, $\Delta N > 0$, doesn't increase the number of accessible micro-states, and therefore doesn't increase the entropy $S_f = -\sum_{i=1}^M p_i \log p_i = 0$, satisfying the constant-entropy condition required by this equation. However, this does necessarily increase the energy, because the added particles must fill the first available energy levels, just above the Fermi energy. That is, $\Delta E = E_F \Delta N > 0$, and we find $\mu = E_F > E_0 > 0$ as claimed earlier in this section. It is an interesting but lengthy exercise (discussed in Section 7.3 of Schroeder's *Introduction to Thermal Physics*) to show that controlling the number of fermions in the gas requires the chemical potential to become negative as the temperature increases and we approach the classical limit.

To get the rest of the way to the low-temperature equation of state for ideal gases of non-relativistic fermions, we need to compute the pressure

$$P_f = - \left. \frac{\partial}{\partial V} \langle E \rangle_f \right|_{N, S_f}.$$

As we just discussed, the single accessible micro-state for $T \rightarrow 0$ automatically satisfies the condition of constant entropy, $S_f = 0$. Applying Eq. 101 relating the chemical potential to the average particle number density, we have

$$\langle E \rangle_f = \frac{3}{5} \mu \langle N \rangle_f = \frac{3}{5} \left(\frac{\hbar^2}{2m} \right) \left(\frac{3\pi^2}{V} \right)^{2/3} \langle N \rangle_f^{5/3}.$$

This is all we need to determine the pressure, which we can relate to the energy density, the Fermi energy $E_F = \mu$ and the particle number density:

$$\begin{aligned}
 P_f &= -\frac{\partial}{\partial V} \left[\frac{3}{5} \left(\frac{\hbar^2}{2m} \right) \left(\frac{3\pi^2}{V} \right)^{2/3} \langle N \rangle_f^{5/3} \right] = \frac{2}{3V} \left[\frac{3}{5} \left(\frac{\hbar^2}{2m} \right) \left(\frac{3\pi^2}{V} \right)^{2/3} \langle N \rangle_f^{5/3} \right] \\
 &= \frac{2}{3} \frac{\langle E \rangle_f}{V} = \frac{2}{5} \mu \frac{\langle N \rangle_f}{V} \\
 &= \frac{\hbar^2}{5m} \left(3\pi^2 \right)^{2/3} \left(\frac{\langle N \rangle_f}{V} \right)^{5/3}
 \end{aligned}$$

In particular, we can see that the pressure (like the energy) remains positive even as the temperature approaches absolute zero, with

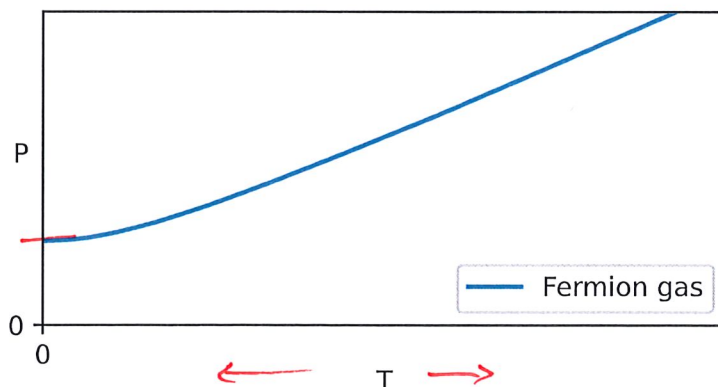
$$P_f = (3\pi^2)^{2/3} \frac{\hbar^2}{5m} \rho_f^{5/3}, \quad (104)$$

where we define the density $\rho_f = \langle N \rangle_f / V$. This positive pressure in the zero-temperature limit is not due to any direct force between the fermions, which remain non-interacting in this ideal gas. Instead, it is a purely quantum effect resulting from the Pauli exclusion principle.

As we saw earlier in this section, the temperature independence of the pressure P_f is due to approximating the low-temperature Fermi function as a step function in Eq. 100, and systematic corrections to this approximation can be computed through a Sommerfeld expansion. Even without getting into such detailed calculations, we know that in the high-temperature classical regime the quantum ideal gas of massive fermions will be well approximated by the classical ideal gas we considered in Section 4.4, with equation of state

$$\underline{PV = NT} \quad \implies \quad P = \frac{N}{V} T = \underline{\rho T}. \quad (105)$$

In words, at high temperatures the pressure depends linearly on the temperature, with the slope corresponding to the density ρ . The plot below (produced by [this Python code](#)) shows how the pressure changes from a positive constant as $T \rightarrow 0$ to this linear behaviour at higher temperatures.



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