8.2 The sun and the void

It will be very interesting to use the grand-canonical potential in Eq. 94 to analyze the average internal energy for a photon gas. With $\mu = 0$, Eq. 78 from Section 6.3 becomes

$$\langle E \rangle_{\text{ph}} = -T^2 \frac{\partial}{\partial T}\left[\frac{\Phi_{\text{ph}}}{T}\right] = \frac{\partial}{\partial \beta}\left[\beta \Phi_{\text{ph}}\right].$$

To begin, we will consider the energy density expressed as an integral over photon frequencies,

$$\frac{\langle E \rangle_{\text{ph}}}{V} = \int_0^\infty P(\omega)\, d\omega,$$

$E = \hbar\omega$

where the function $P(\omega)$ is known as the *spectral density*, or simply the *spectrum*. (It's not the pressure!) What is the spectrum for a photon gas?

$$\frac{\langle E \rangle_{\text{ph}}}{V} = \frac{1}{c^3\pi^2}\int_0^\infty d\omega\, \omega^2 \frac{\partial}{\partial \beta}\log\left[1 - e^{-\beta\hbar\omega}\right] = \frac{1}{c^3\pi^2}\int_0^\infty d\omega\ \omega^2 \frac{+e^{-\beta\hbar\omega}(+\hbar\omega)}{1-e^{-\beta\hbar\omega}}$$

$$= \frac{\hbar}{c^3\pi^2}\int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega}-1}\, d\omega = \int_0^\infty P(\omega)\, d\omega$$

$$\Rightarrow P(\omega) = \frac{\hbar}{c^3\pi^2}\frac{\omega^3}{e^{\beta\hbar\omega}-1}$$

You should find

$$P(\omega) = \left(\frac{\hbar}{c^3\pi^2}\right)\frac{\omega^3}{e^{\beta\hbar\omega}-1}, \tag{95}$$

which is known as the Planck spectrum, named after Max Planck. We can equally well consider the Planck spectrum $P(\lambda)$ as a function of the wavelength $\lambda = 2\pi c/\omega$, by changing variables in the expression above:

$$\frac{\langle E \rangle_{\text{ph}}}{V} = \frac{\hbar}{c^3\pi^2}\int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega}-1}\, d\omega = \frac{-(2\pi c)^4\hbar}{c^3\pi^2}\int_\infty^0 \frac{d\lambda}{\lambda^5\left(e^{2\pi\beta\hbar c/\lambda}-1\right)}$$

$\omega = \frac{2\pi c}{\lambda}$ $\quad d\omega = \frac{-2\pi c}{\lambda^2}d\lambda$

$\omega \to 0$ as $\lambda \to \infty$

$\omega \to \infty$ as $\lambda \to 0$

$$= 16\pi^2\hbar c\int_0^\infty \frac{d\lambda}{\lambda^5\left(e^{2\pi\beta\hbar c/\lambda}-1\right)} = \int_0^\infty P(\lambda)\, d\lambda$$

$$P(\lambda) = \left(\frac{16\pi^2\hbar c}{\lambda^5}\right)\frac{1}{e^{2\pi\beta\hbar c/\lambda}-1}$$

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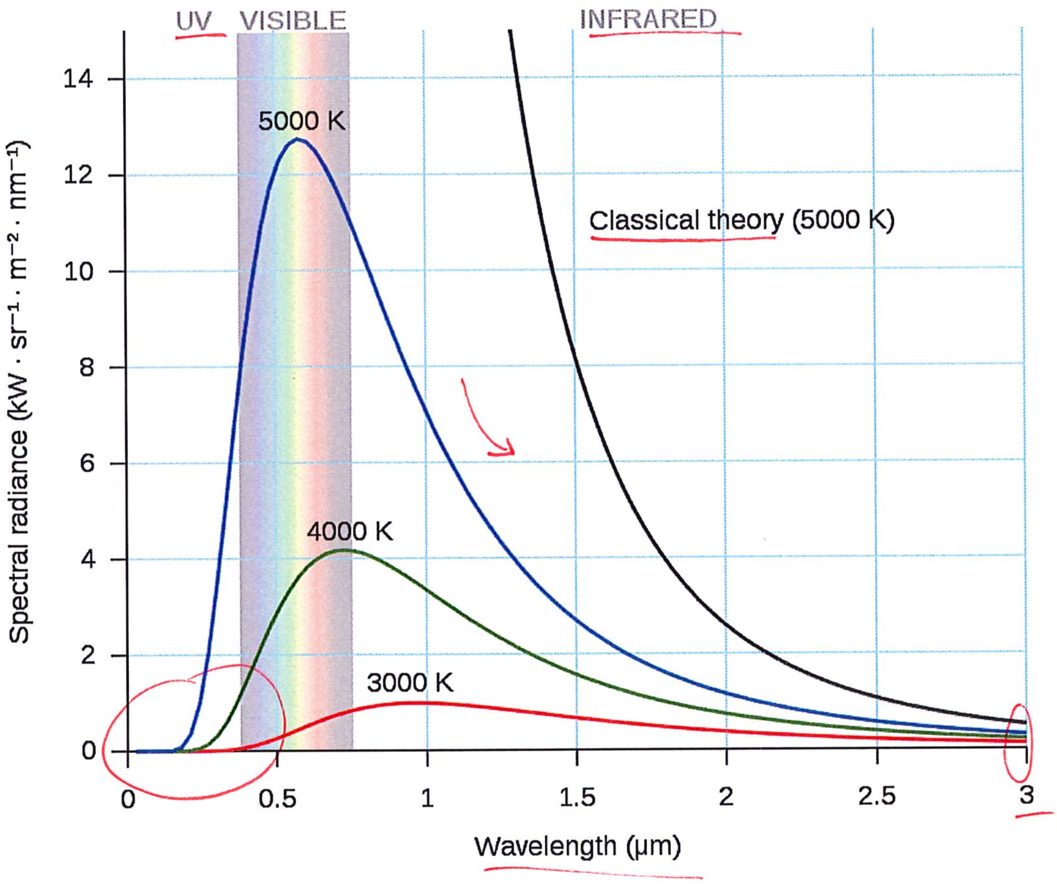
You should find

$$P(\lambda) = \left(\frac{16\pi^2\hbar c}{\lambda^5}\right)\frac{1}{e^{2\pi\beta\hbar c/\lambda} - 1}, \tag{96}$$

which is plotted for three temperatures $T = 1/\beta$ in the figure below, which comes from Wikimedia Commons. (The plot divides our $P(\lambda)$ by 4π steradian and multiplies it by c to convert from a spectral density to a spectral power per unit area per unit of solid angle. For our purposes only the functional form is significant.)



$\beta = \frac{1}{T}$

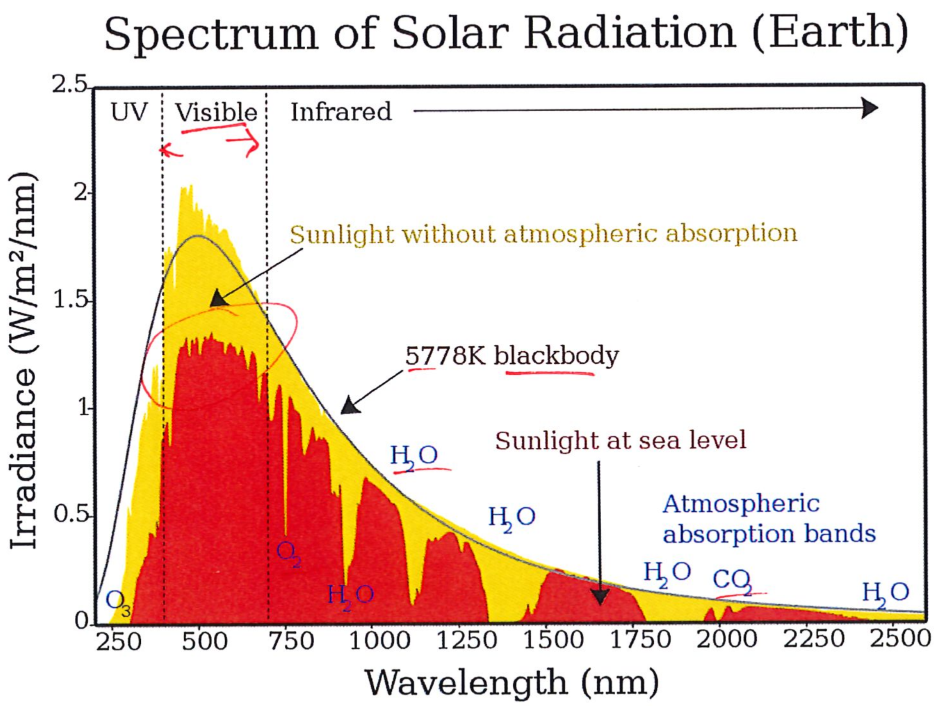
Considering first the high-energy ultraviolet (UV) limit of small wavelengths λ, we can see from Eq. 96 that $P(\lambda)$ is exponentially suppressed, which overwhelms the diverging factor $\propto 1/\lambda^5$ in parentheses. In the low-energy infrared limit, the large λ has the same effect that a large temperature ($\beta \ll 1$) would have: $e^{2\pi\beta\hbar c/\lambda} - 1 \approx 2\pi\beta\hbar c/\lambda$ and

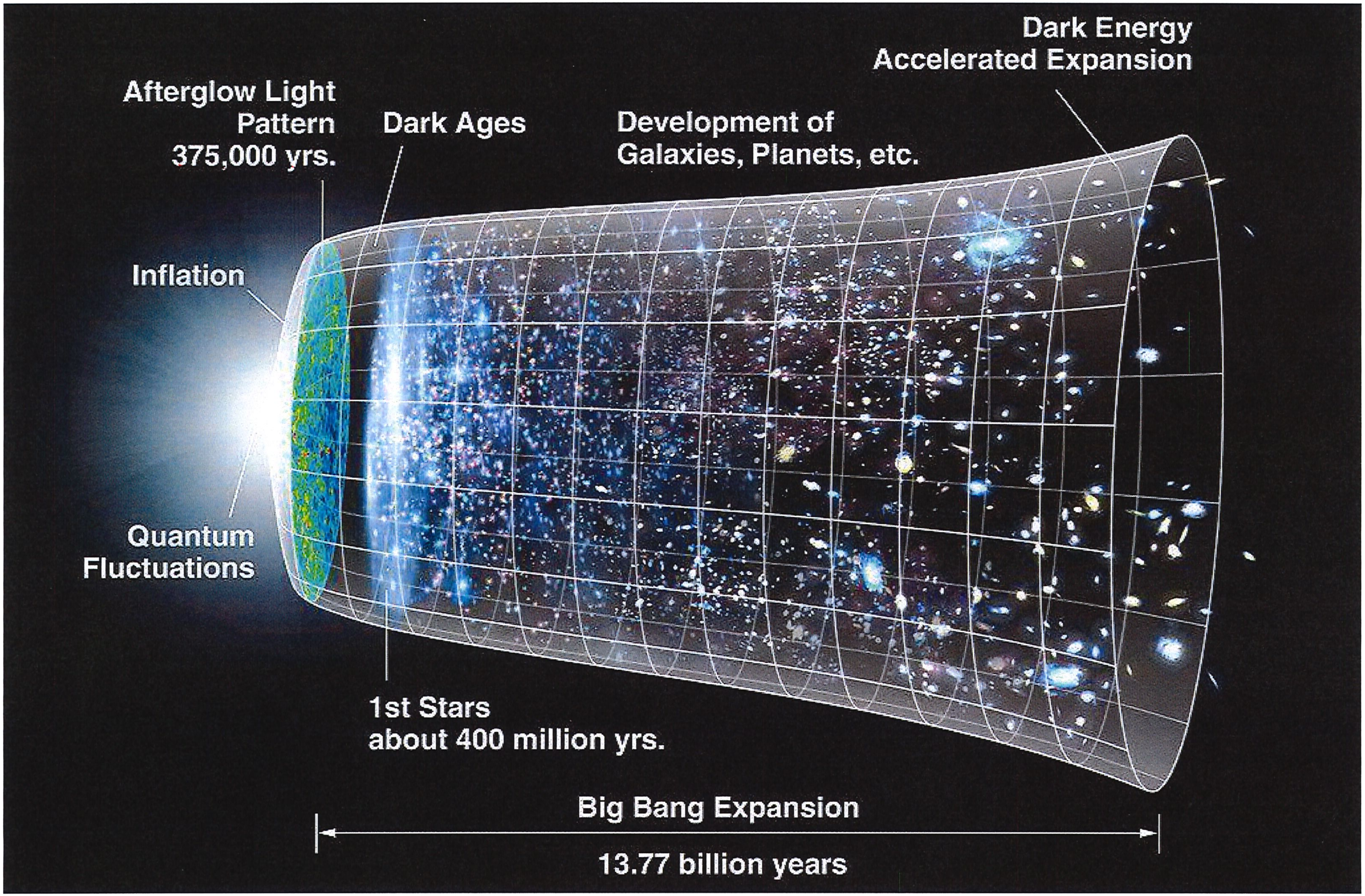
$$P(\lambda) \approx \left(\frac{16\pi^2\hbar c}{\lambda^5}\right)\frac{\lambda}{2\pi\beta\hbar c} = \frac{8\pi T}{\lambda^4}.$$

The connection to large temperatures indicates that this is what classical statistics would predict for the spectrum of light. It is known as the Rayleigh–Jeans spectrum, named after the third Baron Rayleigh and James Jeans. Recall that the classical approach sums over all possible energies for each degree of freedom, corresponding to a light-emitting object (historically known as a *black body*) emitting light of all wavelengths λ. According to the classical Rayleigh–Jeans spectrum, in the limit $\lambda \to 0$ this light would carry an infinite amount of energy, an obvious problem that became known as the *ultraviolet catastrophe*. Planck described his 1900 derivation of the UV-suppressed $P(\lambda)$ as "an act of desperation" to avoid this problem; it turned out to be one of the first steps towards the quantum physics.

Another noteworthy feature of the Planck spectrum shown above is that as the temperature increases, the maximum of $P(\lambda)$ moves to shorter wavelengths and correspondingly larger energies. The fact that the peak of the spectrum for $T \approx 5000$ K falls within the wavelengths of visible light (roughly 400–700 nm) is not a coincidence. As shown in the figure below, also from Wikimedia Commons, the amount of sunlight that reaches the surface of the earth is also maximized around visible wavelengths, which are visible to us because we have evolved to make the most efficient use of this sunlight.

Taking into account the absorption of some sunlight by molecules in the atmosphere, we can see from the figure below that the energy spectrum of the sunlight reaching the top of the atmosphere is quite close to a Planck (or 'blackbody') spectrum with temperature $T \approx 5778$ K. The agreement isn't perfect, which is to be expected since the Planck spectrum relies on the non-trivial assumption of an ideal gas of non-interacting particles. Despite that caveat, numerically fitting the measured sunlight to the Planck spectrum is how this 'effective' surface temperature of the sun is determined. This same fitting procedure can even be done for distant stars, with red stars corresponding to relatively low temperatures $T \lesssim 3500$ K and blue stars corresponding to relatively high temperatures $T \gtrsim 10{,}000$ K.

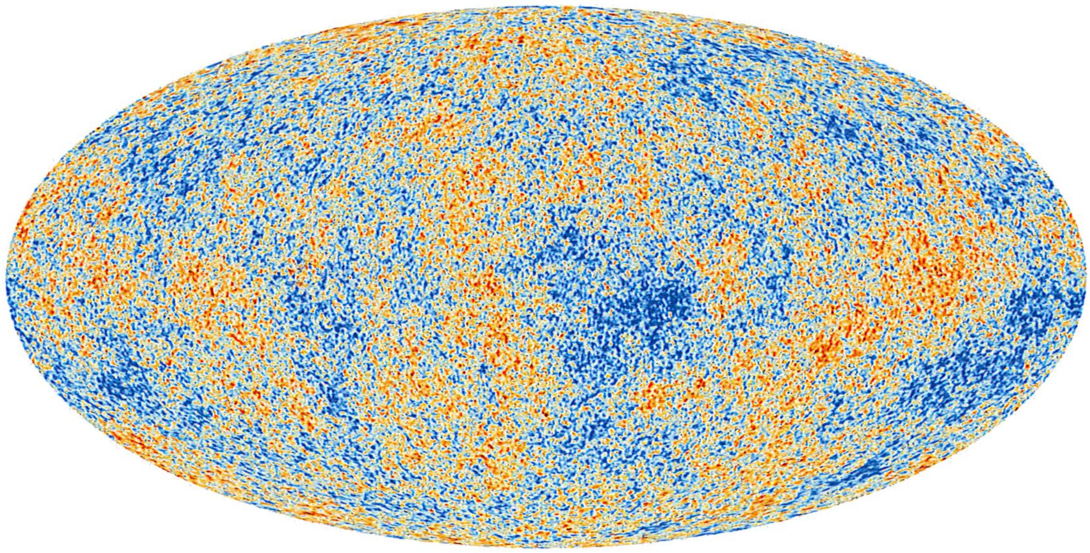




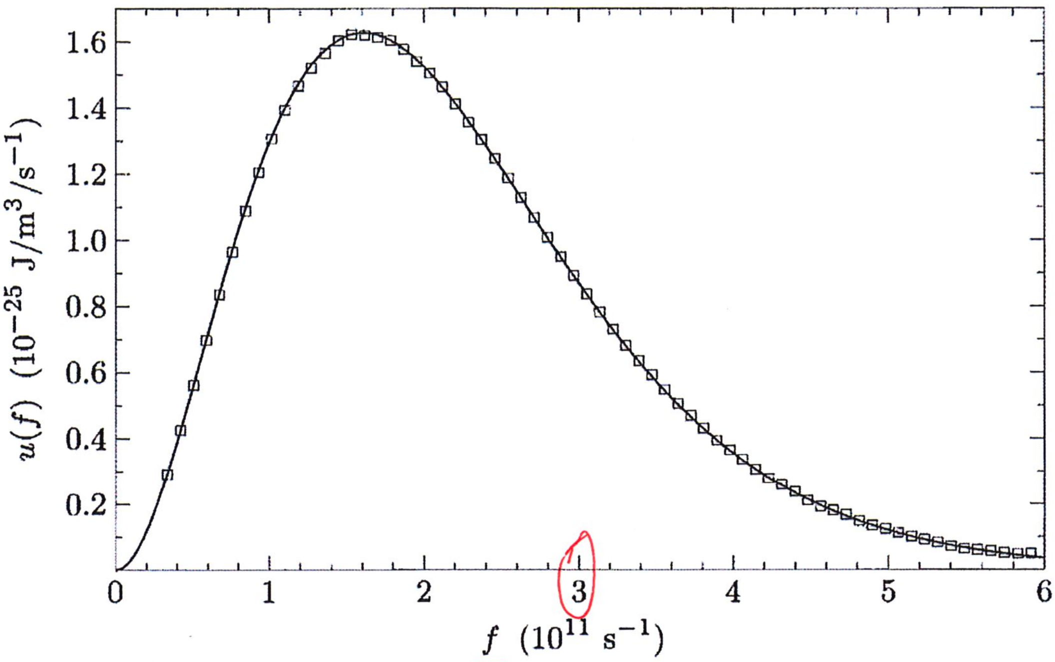
commons.wikimedia.org/wiki/File:CMB_Timeline300_no_WMAP.jpg

Even more remarkably, we can use the Planck spectrum to determine the temperature of intergalactic space. Rather than being empty, these voids are actually permeated by a very low-temperature photon gas left over from the Big Bang roughly 14 billion years ago. This photon gas is known as the cosmic microwave background (CMB), and carries information about the early evolution of the universe, including some of the strongest evidence for the existence of dark matter.

The picture below is a famous visualization of the CMB, provided by the European Space Agency and produced from measurements taken by their 'Planck' satellite. To produce this image, for each point in the sky the satellite measures the photon spectrum reaching it from that direction. The contributions coming from stars and galaxies are subtracted, and the remaining data is fit to the Planck spectrum to find the temperature of the intergalactic CMB photon gas at that point. From point to point, there are only small temperature fluctuations around the average $T_{\text{CMB}} \approx 2.725$ K. That average temperature is subtracted and the fluctuations themselves are shown below, with warmer red-coloured regions only $\Delta T \approx 0.0002$ K hotter than the cooler blue-coloured regions.



The final figure below illustrates such a fit of CMB data to the Planck spectrum, using measurements taken by the Cosmic Background Explorer (COBE) satellite and published in 1990. (This version of the figure is adapted from that publication, and copied from Schroeder's *Introduction to Thermal Physics*.) The squares are the measured data, and their size represents a cautious estimate of uncertainties. They are plotted with the frequency $f = \omega/(2\pi)$ on the horizontal axis, with $f \approx 3 \times 10^{-11}$ s^{-1} corresponding to a low-energy wavelength $\lambda = c/f \approx 1$ mm, roughly 1000 times longer than the wavelengths of visible light. The solid line is a fit to the data that produces $T_{\text{CMB}} = 2.735 \pm 0.060$ K. While more recent satellites have increased the precision with which we know T_{CMB}, this first result was awarded the 2006 Nobel Prize in Physics.



$\lambda \sim mm$

Even though we derived the Planck spectrum by assuming an ideal gas of non-interacting photons, we see that it provides an excellent mathematical model for real physical systems, stretching from the hottest to the coldest places in the universe.

8.3 Equation of state for the photon gas

Having looked in some detail at the integrand for the photon gas energy density, Eq. 95, let's complete the integration, which involves a famous integral related to the Riemann zeta function:

$$I_4 = \int_0^\infty \frac{x^3}{e^x - 1} dx = \overset{6}{\Gamma(4)}\zeta(4) = \frac{\pi^4}{15}.$$

Using this result, what is the average energy density for an ideal photon gas?

$$\frac{\langle E \rangle_{\text{ph}}}{V} = \frac{\hbar}{c^3\pi^2} \int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega} - 1} d\omega = \frac{\hbar}{c^3\pi^2}\left(\frac{T}{\hbar}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1} dx$$

$x = \beta\hbar\omega = \hbar\omega / T$

$d\omega = \left(\frac{T}{\hbar}\right) dx$

$$= \frac{\pi^2 T^4}{15 \hbar^3 c^3}$$

$E = \frac{3}{2} NT$

You should find a result proportional to T^4, which appears significantly more complicated than Eq. 52 for the energy of an N-particle non-relativistic ideal gas in the canonical ensemble. This is related to the fluctuating particle number now that we are working in the grand-canonical ensemble. It's possible to simplify the current situation by computing the average photon number from Eq. 76,

$$\langle N \rangle_{\text{ph}} = -\frac{\partial}{\partial \mu} \Phi_{\text{ph}} \bigg|_{\mu=0} = -\frac{VT}{c^3 \pi^2} \int_0^\infty d\omega\, \omega^2 \frac{\partial}{\partial \mu} \log\left[1 - e^{-\beta \hbar \omega} e^{\beta \mu}\right] \bigg|_{\mu=0},$$

recalling $\mu = 0$ for photon gases. The calculation is quite similar to that for the average internal energy density, now involving the integral

$$I_3 = \int_0^\infty \frac{x^2}{e^x - 1} dx = \Gamma(3)\zeta(3) = 2\zeta(3).$$

Using this result, what is the average particle number density ideal photon gas?

$$\frac{\langle N \rangle_{\text{ph}}}{V} = -\frac{T}{c^3 \pi^2} \int_0^\infty d\omega\, \omega^2 \frac{\partial}{\partial \mu} \log\left[1 - e^{-\beta \hbar \omega} e^{\beta \mu}\right] \bigg|_{\mu=0}$$

$$= \frac{-\cancel{T}}{c^3\pi^2} \int_0^\infty \frac{-\omega^2 e^{-\beta\hbar\omega} \cancel{\beta} e^{\beta\mu}}{1 - e^{-\beta\hbar\omega} e^{\beta\mu}} d\omega \bigg|_{\mu=0} = \frac{1}{c^3\pi^2} \int_0^\infty \frac{\omega^2}{e^{\beta\hbar\omega} - 1} d\omega$$

$$= \frac{1}{c^3\pi^2} \left(\frac{T}{\hbar}\right)^3 \int_0^\infty \frac{x^2}{e^x - 1} dx = \frac{2\zeta(3) T^3}{\pi^2 \hbar^3 c^3}$$

You should find a result proportional to $T^3 \propto \langle E \rangle_{\text{ph}} / T$, so that

$$\langle E \rangle_{\text{ph}} = \frac{\pi^2}{15 \hbar^3 c^3} V T^4 = \frac{\pi^4}{30 \zeta(3)} \langle N \rangle_{\text{ph}} T. \tag{97}$$

The functional form is the same as Eq. 52, with a larger numerical factor

$$\frac{\pi^4}{30\zeta(3)} = \frac{\Gamma(4)\zeta(4)}{\Gamma(3)\zeta(3)} \approx 2.7$$

compared to the $\frac{3}{2}$ for the classical non-relativistic case.

To get the rest of the way to the equation of state for the photon gas, we need to compute the *radiation pressure*

$$P_{\text{ph}} = -\frac{\partial}{\partial V} \langle E \rangle_{\text{ph}} \bigg|_{S_{\text{ph}}},$$

which requires first figuring out the condition of constant entropy S_{ph} for a photon gas. From Eq. 79 with $\mu = 0$, we have

$$S_{\text{ph}} = \frac{\langle E \rangle_{\text{ph}} - \Phi_{\text{ph}}}{T}.$$

Looking back to Eq. 94 for the grand-canonical potential, we see

$$\frac{\Phi_{\text{ph}}}{T} = \frac{V}{c^3\pi^2}\int_0^\infty d\omega\, \omega^2 \log\left[1 - e^{-\beta\hbar\omega}\right] = \frac{VT^3}{\hbar^3 c^3 \pi^2}\int_0^\infty dx\, x^2 \log\left[1 - e^{-x}\right],$$

changing variables to $x = \beta\hbar\omega = \hbar\omega/T$. The final factor in this expression is yet another delightful integral,

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$$\int_0^\infty dx\, x^2 \log\left[1 - e^{-x}\right] = -2\zeta(4) = -\frac{\pi^4}{45}.$$

Since this gives us $S \propto VT^3$, we can conclude that the condition of constant entropy for a photon gas is $VT^3 = \text{constant}$, in contrast to the $VT^{3/2}$ dependence of Eq. 53 for classical non-relativistic particles.

At this point it is straightforward to take the derivative of the average internal energy if we express the constant-entropy condition as $T = bV^{-1/3}$, with b a constant:

$$P_{\text{ph}} = -\frac{\partial}{\partial V}\left\langle E\right\rangle_{\text{ph}}\Big|_{S_{\text{ph}}} = -\frac{\partial}{\partial V}\frac{\pi^2}{15\hbar^3 c^3}VT^4\Big|_{S_{\text{ph}}} =$$

For the resulting equation of state for the photon gas, you should find

$$P_{\text{ph}}V = \frac{1}{3}\left\langle E\right\rangle_{\text{ph}} = \frac{\pi^4}{90\zeta(3)}\left\langle N\right\rangle_{\text{ph}} T. \tag{98}$$

The functional form is the same as the (classical, non-relativistic) ideal gas law, with just an additional numerical factor of

$$\frac{\pi^4}{90\zeta(3)} = \frac{\zeta(4)}{\zeta(3)}.$$