21 March

Recap:

Quantum gases $\begin{cases} \text{Grand-canonical ensemble} \\ \text{Quantum statistics} \end{cases}$

$\left.\begin{array}{l} \text{Bose-Einsteins statistics} \\ \text{Fermi-Dirac statistics} \end{array}\right\} \longrightarrow$ Maxwell-Boltzmann statistics (classical)

$-\mu \gg T \gg E_e$

x and p_x are defined. Since the particle is within a volume $V = L^3$, we know $\Delta x \lesssim L$. Therefore the uncertainty principle requires $\Delta p_x \gtrsim \hbar/L$, which is only possible if p_x is non-zero, corresponding to $k_x \geq 1$. Note that smaller lengths L imply larger momenta and therefore larger energies.

With this adjusted ansatz, $k_{x,y,z} \geq 1$, we can adapt an exercise from Section 4.1 and ask: What are the lowest energies and the degeneracies of the corresponding energy levels?

Ground state: $E_0 = 3\varepsilon$ $\quad \vec{k} = (1,1,1)$ $\quad$ #: 1

$E = 6\varepsilon$ $\quad$ #: 3 $\quad \vec{k} = (2,1,1), (1,2,1), (1,1,2)$

$E = 9\varepsilon$ $\quad$ #: 3

$E = 11\varepsilon$ $\quad$ #: 3 $\quad (3,1,1)$ $\qquad E = 12\varepsilon$ $\quad$ #: 1 $\quad (2,2,2)$

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8.1.2 Massless photons

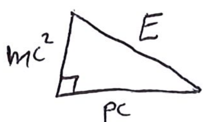
Now we will build on our experience with massive bosons to consider a gas of *photons*, massless bosonic quantum particles of light. For our purposes, with no prior knowledge of particle physics, we can define photons simply by specifying their energy levels. Clearly $E \propto 1/m$ from Eq. 91 is problematic for massless particles with $m = 0$. Instead, a photon's energy is proportional to the magnitude of its momentum,

$$E_{\text{ph}}(p) = c\sqrt{p_x^2 + p_y^2 + p_z^2} \equiv cp.$$

Here the speed of light c is really just a unit conversion factor (like the Boltzmann constant) that we could set to $c = 1$ by working in appropriate units.

This relation is connected to the non-relativistic energy $E = \frac{p^2}{2m}$ that we considered in Section 4.1 through the general expression

$$E^2 = \left(mc^2\right)^2 + \left(pc\right)^2,$$



which is sometimes called Einstein's triangle. When $m = 0$, or $m \ll p/c$ more generally, this reproduces the *ultra-relativistic* relation above. For stationary particles with $p = 0$ it reduces to the famous 'mass-energy' $E = mc^2$, while the non-relativistic kinetic energy is recovered for $m \gg p/c$:

$$E = mc^2\sqrt{1+\frac{p^2c^2}{m^2c^4}} = mc^2\left(1 + \frac{p^2}{2m^2c^2} + \mathcal{O}\left(\frac{p^4}{m^4c^4}\right)\right)$$

$$= mc^2 + \frac{p^2}{2m} + \mathcal{O}\left(\frac{p^4}{m^3c^2}\right)$$

Another feature of photons' energy levels is that each momentum (p_x, p_y, p_z) corresponds to two degenerate energy levels with the same energy $E(p)$. This arises from photons' connection to electric and magnetic fields, which allows each photon to be polarized in two different ways. If there is interest we can discuss this further in a tutorial, but it is not relevant to the statistical physics of photons, for which we can take this double degeneracy as input. Note that this factor of two multiplies all other degeneracies, for instance from permutations of (p_x, p_y, p_z).

For photons in a volume $V = L^3$, only the same discrete momenta as in the massive case are allowed,

$$p = \hbar\frac{\pi}{L}\sqrt{k_x^2 + k_y^2 + k_z^2} \equiv \hbar\frac{\pi}{L}k \qquad\qquad k_{x,y,z} = 1, 2, \cdots,$$

so that the quantized photon energies are

$$E_{\text{ph}}(k) = \hbar c\frac{\pi}{L}k. \tag{92}$$

Because light is an electromagnetic wave, it is convenient to work in terms of photons' wavelength λ and angular frequency $\omega = 2\pi f$ (not to be confused with generic micro-states ω_i). Together, the wavelength and frequency determine the speed of the wave's propagation, in this case the speed of light

$$c = \frac{\lambda\omega}{2\pi}.$$

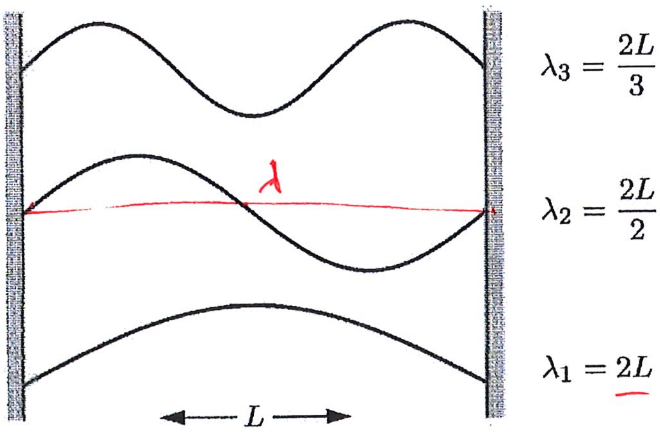
In quantum physics, a particle's momentum is related to its de Broglie wavelength, implying that in a volume $V = L^3$ the wavelengths are also quantized as illustrated in the picture below (from Schroeder's *Introduction to Thermal Physics*). Specifically, the length L must be an integer multiple of half a wavelength,

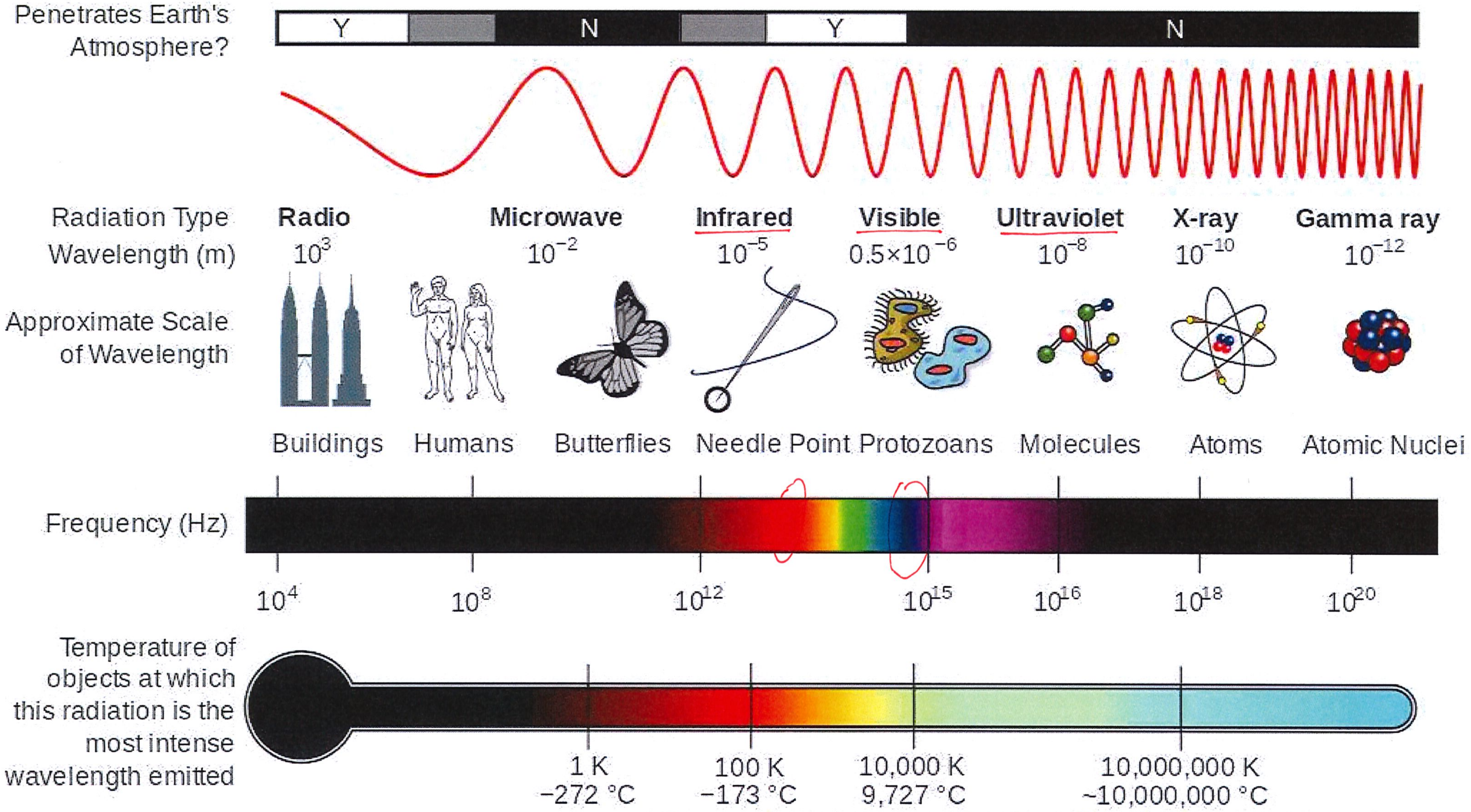
$$L = k\frac{\lambda}{2} \qquad \Longrightarrow \qquad c = \frac{L}{k}\frac{\omega}{\pi} = \frac{\omega}{\frac{\pi}{L}k}, \qquad \omega = c\frac{\pi}{L}k$$

and we can rewrite Eq. 92 as

$$E_{\text{ph}}(\omega) = \hbar\omega. \tag{93}$$

With $\lambda \propto c/\omega$, this incorporates the relation between length and energy scales that we noted above. Low (*infrared*) frequencies correspond to small energies and long wavelengths, while high (*ultraviolet*) frequencies correspond to large energies and short wavelengths.





commons.wikimedia.org/wiki/File:EM_Spectrum_Properties_edit.svg

We are now ready to write down the grand-canonical potential for a photon gas:

$$\Phi_{\text{ph}} = T \sum_{\ell=0}^{\mathcal{L}} \log\left[1 - e^{-\beta(E_\ell - \mu)}\right] = 2T \sum_{\vec{k}} \log\left[1 - e^{-\beta(E_{\text{ph}}(k) - \mu)}\right],$$ $\mu < E_\ell$

where the factor of 2 in the final expression accounts for the doubly degenerate energy levels. We can simplify this expression by appreciating that photons are easy to create and destroy. Every time a light is switched on, it begins emitting a constant flood of photons (with wavelengths of several hundred nanometres). Food in a microwave oven gets hot by absorbing many lower-energy photons (with longer wavelengths around 12 centimetres). In both cases an enormous number of photons is required to make even a small change in energy, so that Eq. 81 implies the chemical potential of a photon gas must be negligible,

$$\mu = \left.\frac{\partial E}{\partial N}\right|_S \approx 0 \qquad \Longrightarrow \qquad \Phi_{\text{ph}} \approx 2T \sum_{\vec{k}} \log\left[1 - e^{-\beta E_{\text{ph}}(k)}\right].$$

Since we have $k_{x,y,z} \geq 1$, the strictly positive energies $E_{\text{ph}}(k) \propto k/L > 0$ ensure Bose–Einstein statistics is still convergent even with $\mu = 0$.

Another simplification comes from considering the photon gas in a large volume, so that the energies $E_{\text{ph}}(k) \propto k/L$ are very closely spaced and we can approximate the sum over integer $k_{x,y,z}$ by integrals over continuous real $\widehat{k}_{x,y,z}$,

$$\Phi_{\text{ph}} \approx 2T \int d\widehat{k}_x \, d\widehat{k}_y \, d\widehat{k}_z \, \log\left[1 - e^{-\beta E_{\text{ph}}(\widehat{k})}\right].$$

Since the energy $E_{\text{ph}}(\widehat{k})$ depends only on the magnitude $\widehat{k}$, we can profit from converting to spherical coordinates. When we do so, we have to keep in mind that $k_{x,y,z} > 0$ corresponds only to the positive octant of the sphere,

$$\int_0^\infty d\widehat{k}_x \int_0^\infty d\widehat{k}_y \int_0^\infty d\widehat{k}_z = \int_0^\infty d\widehat{k}\, \widehat{k}^2 \int_0^{\pi/2} d\theta \, \sin\theta \int_0^{\pi/2} d\phi = \frac{\pi}{2} \int_0^\infty d\widehat{k}\, \widehat{k}^2,$$

so that

$$\Phi_{\text{ph}} \approx \pi T \int_0^\infty d\widehat{k}\, \widehat{k}^2 \log\left[1 - e^{-\beta E_{\text{ph}}(\widehat{k})}\right].$$

We can finally change variables to integrate over the photon angular frequency $\omega = c\frac{\pi}{L}k$, with $E_{\text{ph}} = \hbar\omega$, to find

$$\begin{aligned} \Phi_{\text{ph}} &\approx \pi T \left(\frac{L}{c\pi}\right)^3 \int_0^\infty d\omega\, \omega^2 \log\left[1 - e^{-\beta\hbar\omega}\right] \\ &= \frac{VT}{c^3\pi^2} \int_0^\infty d\omega\, \omega^2 \log\left[1 - e^{-\beta\hbar\omega}\right], \end{aligned} \qquad (94)$$

recognizing $L^3 = V$. With this grand-canonical potential derived, we just need to take the appropriate derivatives to determine the thermodynamics and equation of state for the photon gas.

8.2 The sun and the void

It will be very interesting to use the grand-canonical potential in Eq. 94 to analyze the average internal energy for a photon gas. With $\mu = 0$, Eq. 78 from Section 6.3 becomes

$$\langle E \rangle_{\text{ph}} = -T^2 \frac{\partial}{\partial T}\left[\frac{\Phi_{\text{ph}}}{T}\right] = \frac{\partial}{\partial \beta}\left[\beta \Phi_{\text{ph}}\right].$$

To begin, we will consider the energy density expressed as an integral over photon frequencies,

$$\frac{\langle E \rangle_{\text{ph}}}{V} = \int_0^\infty P(\omega)\, d\omega,$$

where the function $P(\omega)$ is known as the *spectral density*, or simply the *spectrum*. (It's not the pressure!) What is the spectrum for a photon gas?

$$\frac{\langle E \rangle_{\text{ph}}}{V} = \frac{1}{c^3\pi^2}\int_0^\infty d\omega\, \omega^2 \frac{\partial}{\partial \beta}\log\left[1 - e^{-\beta\hbar\omega}\right] = \frac{1}{c^3\pi^2}\int_0^\infty d\omega\, \omega^2 \frac{+e^{-\beta\hbar\omega}(+\hbar\omega)}{1-e^{-\beta\hbar\omega}}$$

$$= \frac{\hbar}{c^3\pi^2}\int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega}-1}\, d\omega = \int_0^\infty P(\omega)\, d\omega$$

$$\Rightarrow P(\omega) = \frac{\hbar}{c^3\pi^2}\frac{\omega^3}{e^{\beta\hbar\omega}-1}$$

You should find

$$P(\omega) = \left(\frac{\hbar}{c^3\pi^2}\right)\frac{\omega^3}{e^{\beta\hbar\omega}-1}, \tag{95}$$

which is known as the Planck spectrum, named after Max Planck. We can equally well consider the Planck spectrum $P(\lambda)$ as a function of the wavelength $\lambda = 2\pi c/\omega$, by changing variables in the expression above:

$$\frac{\langle E \rangle_{\text{ph}}}{V} = \frac{\hbar}{c^3\pi^2}\int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega}-1}\, d\omega = \frac{-(2\pi c)^4\hbar}{c^3\pi^2}\int_\infty^0 \frac{d\lambda}{\lambda^5\left(e^{2\pi\beta\hbar c/\lambda}-1\right)}$$

$\omega = \frac{2\pi c}{\lambda}$ $\quad d\omega = \frac{-2\pi c}{\lambda^2}d\lambda$

$\omega \to 0$ as $\lambda \to \infty$

$\omega \to \infty$ as $\lambda \to 0$

$$= 16\pi^2\hbar c\int_0^\infty \frac{d\lambda}{\lambda^5\left(e^{2\pi\beta\hbar c/\lambda}-1\right)} = \int_0^\infty P(\lambda)\, d\lambda$$

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