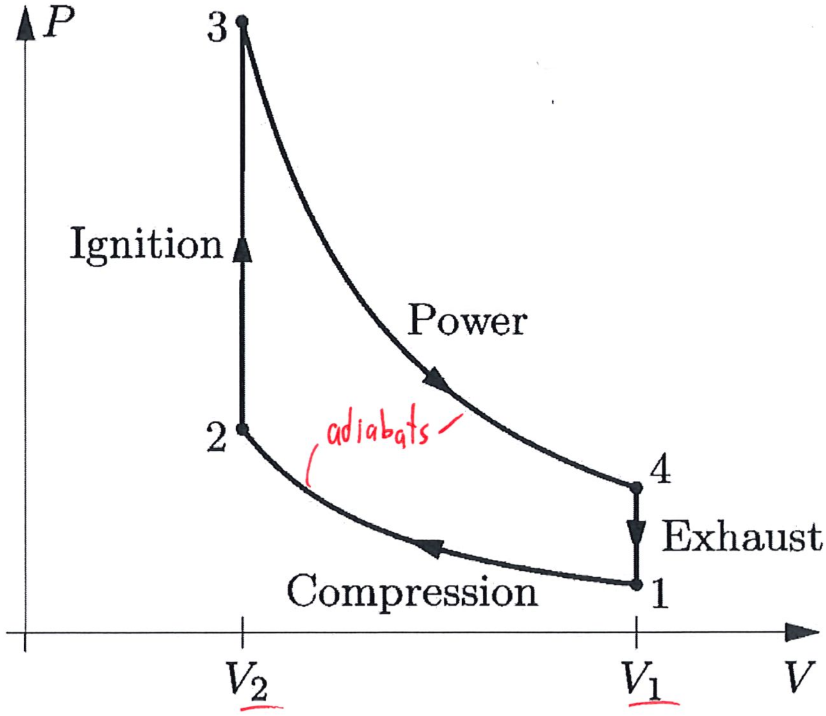
MATH327: Statistical Physics, Spring 2022

Tutorial problem — Otto cycle

The figure below shows the 'Otto cycle' that describes an idealized petrol engine. The compression and expansion ('power') stages are adiabatic, while the volume is fixed at V_2 for the 'ignition' stage that burns the fuel to produce heat, and at $V_1 > V_2$ for the 'exhaust' stage that replaces the burnt fuel with cooler, fresh gas. The **compression ratio** is defined as $r \equiv V_1/V_2 > 1$.



The efficiency η of the Otto cycle depends *only* on the compression ratio r. What is this efficiency? How does it compare to the efficiency of the Carnot cycle? How should V_1 and V_2 be chosen to maximize the efficiency?

Hint: Given the labels in the diagram above, T_1 would be the low temperature of the cold reservoir while T_3 would be the high temperature of the hot reservoir. The corresponding Carnot cycle efficiency is therefore $\eta_{\text{Carnot}} = 1 - \frac{T_1}{T_3}$, and the comparison is easiest if the Otto cycle efficiency is expressed in terms of temperatures rather than volumes.

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Tutorial comments — Otto cycle

The best starting point to compute the efficiency

$$\eta = \frac{W_{\text{out}} - W_{\text{in}}}{Q_{\text{in}}}$$

is to determine which stages of the cycle contribute to W_{out}, W_{in} and Q_{in}.

- The system does work on its surroundings during the 'power' stage of adiabatic expansion. Because this stage is adiabatic,

$$W_{\text{out}} = -W_{34} = -\Delta\langle E\rangle = -\frac{3}{2}N\Delta T = \frac{3}{2}N\left(T_3 - T_4\right) > 0.$$

- Similarly, work has to be done on the system by its surroundings in order to reduce its volume during the adiabatic compression stage, for which

$$W_{\text{in}} = W_{12} = \Delta\langle E\rangle = \frac{3}{2}N\Delta T = \frac{3}{2}N\left(T_2 - T_1\right) > 0.$$

- Finally, heat only enters the system when the fuel is burnt during the 'ignition' stage. Because the volume is fixed during ignition, $W_{23} = 0$ and

$$Q_{\text{in}} = Q_{12} = \Delta\langle E\rangle = \frac{3}{2}N\Delta T = \frac{3}{2}N\left(T_3 - T_2\right) > 0.$$

Putting things together,

$$\eta = \frac{T_3 - T_4 - T_2 + T_1}{T_3 - T_2} = 1 - \frac{T_4 - T_1}{T_3 - T_2}.$$

We want to end up with an efficiency that depends only on the compression ratio $r = V_1/V_2$. We can make progress towards this goal by applying the condition of constant entropy obeyed during both the adiabatic compression and expansion stages, which relates

$$V_1 T_1^{3/2} = V_2 T_2^{3/2} \qquad\qquad V_1 T_4^{3/2} = V_2 T_3^{3/2}$$

Rearranging, we find

$$\frac{T_1}{T_2} = \left(\frac{V_2}{V_1}\right)^{2/3} = \frac{1}{r^{2/3}} = \frac{T_4}{T_3} \qquad \Longrightarrow \qquad \frac{T_1}{T_4} = \frac{T_2}{T_3}.$$

The last equality proves its use when we rewrite the efficiency as

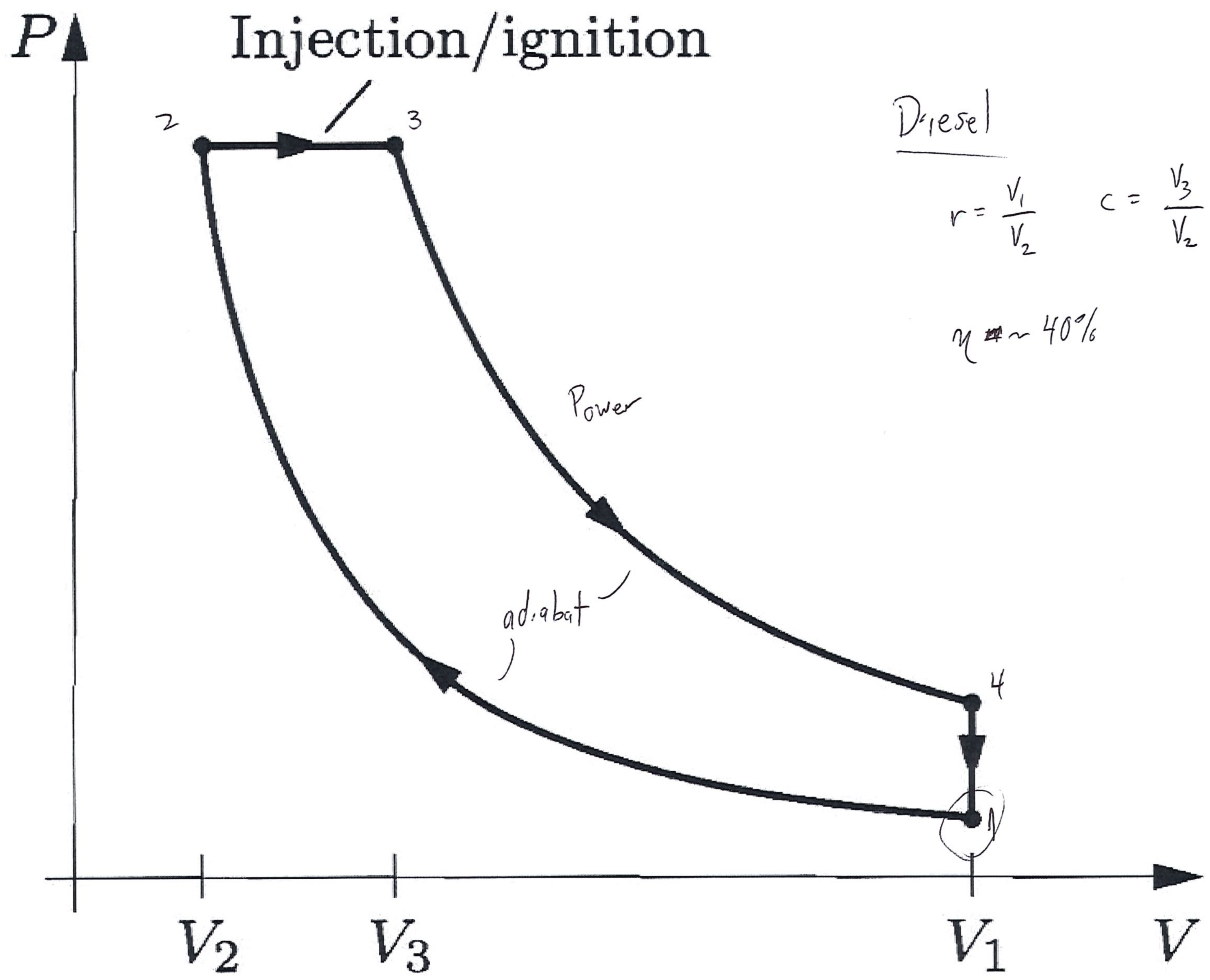
$$\eta = 1 - \frac{T_4}{T_3}\left(\frac{1 - T_1/T_4}{1 - T_2/T_3}\right) = 1 - \frac{T_4}{T_3} = 1 - \frac{1}{r^{2/3}}. \tag{1}$$

To compare this efficiency with the Carnot cycle, we can use another of the relations above to write

$$\eta = 1 - \frac{T_4}{T_3} = 1 - \frac{T_1}{T_2} < 1 - \frac{T_1}{T_3} = \eta_{\text{Carnot}}, \tag{2}$$

where the inequality results from $T_2 < T_3 \Longrightarrow \frac{1}{T_2} > \frac{1}{T_3}$. Recall that burning the fuel adds heat to raise the temperature from T_2 to T_3. As expected, we find that the efficiency of the Otto cycle is less than the theoretical maximum saturated by the Carnot cycle.

From Eq. 1, we can see that the efficiency is maximized by maximizing the compression ratio $r = V_1/V_2$. That is, increasing the amount by which we compress the fuel before igniting it, and thereby increasing the change in volume over which we extract work during the 'power' stage, improves the efficiency of petrol engines. There are practical limitations to this efficiency, which motivate considering diesel engines as an alternative.



Diesel

$r = \frac{V_1}{V_2}$ $c = \frac{V_3}{V_2}$

$\eta \sim 40\%$