$\boxed{\text{RECAP}}$ Lecture 31/1/2020

E: Experiment

ω: events "roll a die" "world"

$\Downarrow$

$X(\omega)$: measurment "Maths"

$\{1, 2, 3, 4, 5, 6\}$

A: output space

F: event space $\{$even, odd$\}$

any subset of A

$\Downarrow$

Probabilities

LLN: Experiment A: $\{X_1 \dots X_{100}\} \rightarrow \mu, \sigma$

Experiment B: repeat A n-times

$n=4$: $A_B = \{X_1 X_{10} X_3 X_2, X_{11} X_3 X_1 X_1, \dots\}$

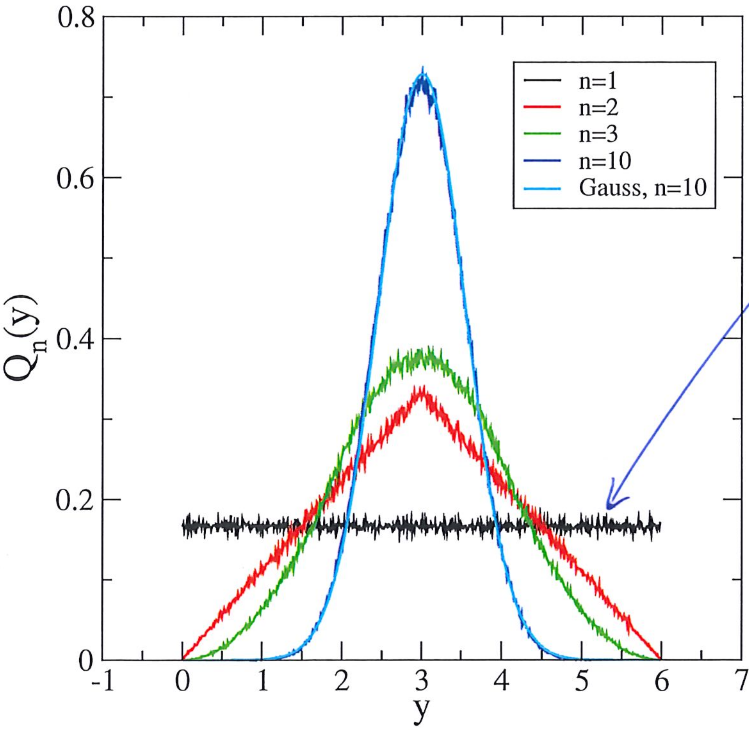
$n \to \infty$: $\frac{1}{n} \sum_{i=1}^{n} X^{(i)} = \mu$

CLT

Addendum page 10: Roulette – European Tables

The standard European table has 18 black, 18 red and one green pocket (numbered 0) making 37 pockets in all.

Addendum page 15: Central limit theorem



uniformly distributed random number $\in [0,6]$

RECAP end

Lecture continues ...

5 steps, $x=3$

RRRRL, RRRLR,

RRLRR, RLRRR, LRRRR

each event: $p^4 q^1$

$\binom{5}{4}$ elements $= \frac{5!}{4!\,1!} = 5$

If $P(x)$ is the probability that we find ourselves at x, we here find:

$P(x=3) = 5 \cdot p^4 q^1$ 5 steps

N steps: n times to the "Right"

$P(x) = \binom{N}{n} p^n q^{N-k}$ $x = 2n-N$

For the general case of N steps and k steps to the right, we need to distribute k steps R to N slots (and fill the remaining places with Ls). There are

$$\left(\begin{array}{c} N \\ k \end{array}\right) = \frac{N!}{k!\,(N-k)!}$$

possibilities to do so. Hence, the probability for k steps to the right (no matter in which order) is:

$$p_k = \left(\begin{array}{c} N \\ k \end{array}\right) p^k\, q^{N-k}\,.$$

average position after N steps

We now can answer the above question for average position and variance:

$$\langle x\rangle = \sum_{k=0}^{N}(2k-N)\,p_k, \qquad \langle x^2\rangle = \sum_{k=0}^{N}(2k-N)^2\,p_k. \tag{4}$$

Apparently, we need to calculate sums fo the type:

$$\sum_{k=0}^{N} k^n\,p_k\,.$$

There is an excellent trick in stochastics to do this. We define the generating function

$$T(\theta) = \sum_{k=0}^{N} e^{\theta\,k}\,p_k. \tag{5}$$

We easily check (*do it!*) that

$$\lim_{\theta\to 0}\frac{d^nT(\theta)}{d\theta^n} = \sum_{k=0}^{N} k^n\,p_k\,. \tag{6}$$

Fortunately, we can calculate the closed form of $T(\theta)$ for our example here:

$$T(\theta) = \sum_k e^{\theta k}\underbrace{\binom{N}{k}p^k q^{N-k}}_{p_k}$$

$$= \sum_k \binom{N}{k}\big(\underbrace{e^\theta p}_{=a}\big)^k \underbrace{q}_{=b}^{\,N-k} = (a+b)^N$$

$$= (e^\theta p + q)^N$$

and finally obtain:

$$T(\theta) = (e^\theta\,p + q)^N\,.$$

We thus obtain:

$$\langle k\rangle = \sum_{k=0}^{N} k\,p_k \;=\; \lim_{\theta\to 0}\frac{dT(\theta)}{d\theta} \;=\; N\,p\lim_{\theta\to 0} e^{\theta}\left(e^{\theta}p+q\right)^{N-1} \;=\; N\,p\,(p+q)^{N-1} \;=\; N\,p\,,$$

$$\langle k^2\rangle = \sum_{k=0}^{N} k^2\,p_k \;=\; \lim_{\theta\to 0}\frac{d^2T(\theta)}{d\theta^2} \;=\; N^2p^2 + N\,p\,q\,.$$

We hence obtain

$$\boxed{\begin{aligned}\langle x\rangle &= \langle 2k-N\rangle = 2\langle k\rangle - N = 2Np - N = N(2p-1)\\ \langle x^2\rangle-\langle x\rangle^2 &= \langle (2k-N)^2\rangle - N^2(2p-1)^2\\ &= \langle 4k^2-4Nk+N^2\rangle - N^2(2p-1)^2\\ &= 4\langle k^2\rangle - 4N\langle k\rangle + N^2 - N^2(2p-1)^2 = 4Npq\end{aligned}}$$

In summary, we find: t

$$\langle x\rangle = \frac{\Delta t}{}N\,(2p-1)\,, \qquad \langle x^2\rangle - \langle x\rangle^2 = 4\,N\,p\,q\,. \tag{7}$$

Δt

We can interpret these finding by using time t (3) instead of time. We firstly find that the average changes linearly with time:

$$\langle x\rangle = v_{\rm dr}\,t, \qquad v_{\rm dr} = (2p-1)/\Delta t\,,$$

where $v_{\rm dr}$ is called the drift velocity. If we quantify the uncertainty in the position Δx by the standard deviation, i.e.,

$$\Delta x = \left[\langle x^2\rangle - \langle x\rangle^2\right]^{1/2},$$

and if we assume that the drift velocity is not vanishing, we find:

$$\Delta x/\langle x\rangle = \frac{\sigma}{\langle x\rangle} = \frac{\sqrt{\langle x^2\rangle - \langle x\rangle^2}}{\langle x\rangle} = \frac{\sqrt{4pq}\sqrt{N}}{(2p-1)\cdot N} = \frac{\sqrt{4qp}}{2p-1}\frac{1}{\sqrt{N}}$$

$p \neq 1/2$

We observe that the uncertainty over the drift vanishes for large time.

Note that for $p = 1/2$, the drift vanishes. This is intuitively clear since we step left and right with the same probability $1/2$. It is then most likely to find the person (who steps) at the origin (the probability peaks there). However, in this case, the standard deviation Δx then quantifies the distance in which we can expect the person to find with some measure of probability. How does the standard deviation depend on time?

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{\sqrt{4pq}}{\sqrt{\Delta t}}\sqrt{N}\sqrt{\Delta t} = D \times \sqrt{t}$$

$p = \frac{1}{2}$

$v_{Dr} = 0$

In summary, we find the well know law of diffusion:

$$\Delta x = D\sqrt{t}\,, \qquad D = \sqrt{4pq/\Delta t}\,, \tag{8}$$

where Δx is sometime called the diffusion length, and D is the so-called diffusion constant.

For a large number N of steps (or for large times t), we can invoke the Central limit theorem. The elementary process is a step to the right with probability p and to the left with q. Let us say we find ourselves at postion X_1. We repeat this experiment N times and get displacements $X_2 \ldots X_N$.

We are interested where we are after N-steps. Hence, we are interested in the random variable X of the *total position* after N steps:

$$X = X_1 + X_2 + \ldots + X_N \, ,$$

and the corresponding probability distribution $p(x)$ of this variable. All what we need to do (see subsection 2.3) is to calculate the mean μ and the standard deviation σ^2 of the elementary process:

$$\boxed{\begin{aligned} \mu &= \langle x \rangle_1 = 1 \cdot P_R - 1 \cdot P_L = p - q = 2p - 1 \\ \langle x^2_{(1)} \rangle &= \langle x^2 \rangle_1 = 1^2 \cdot P_R + (-1)^2 P_L = p + q = 1 \\ \sigma^2 &= \langle x^2 \rangle_1 - \langle x \rangle_1^2 = 1 - (2p-1)^2 \\ &= 1 - 4p^2 + 4p - 1 = 4p(1-p) = 4p \cdot q ; \end{aligned}}$$

$F = \{L, R\}$

$P_L = q \quad P_R = p$

$(p + q = 1)$

We the obtain from the CLT:

$$\begin{aligned} p(x) &= \frac{1}{\sqrt{2\pi \, N \overset{\sigma^2}{4pq}}} \exp\left\{ - \frac{(x - \overset{\mu}{N(2p-1)})^2}{8N \, pq} \right\} \\ &= \frac{1}{\sqrt{2\pi \, t \, D^2}} \exp\left\{ - \frac{(x - v_{\mathrm{dr}} t)^2}{2t D^2} \right\} . \quad \sigma_N^2 \end{aligned} \tag{9}$$

Something remarkable has happened here: as long as for the elementary process is such that mean μ and the standard deviation σ exists (no matter what the probability distribution is for the elementary step), we will always end up with distribution $p(x)$ in (9) for sufficiently large times. This also means that as long as the conditions for the CLT are fulfilled, the relation between diffusion length and time, i.e.,

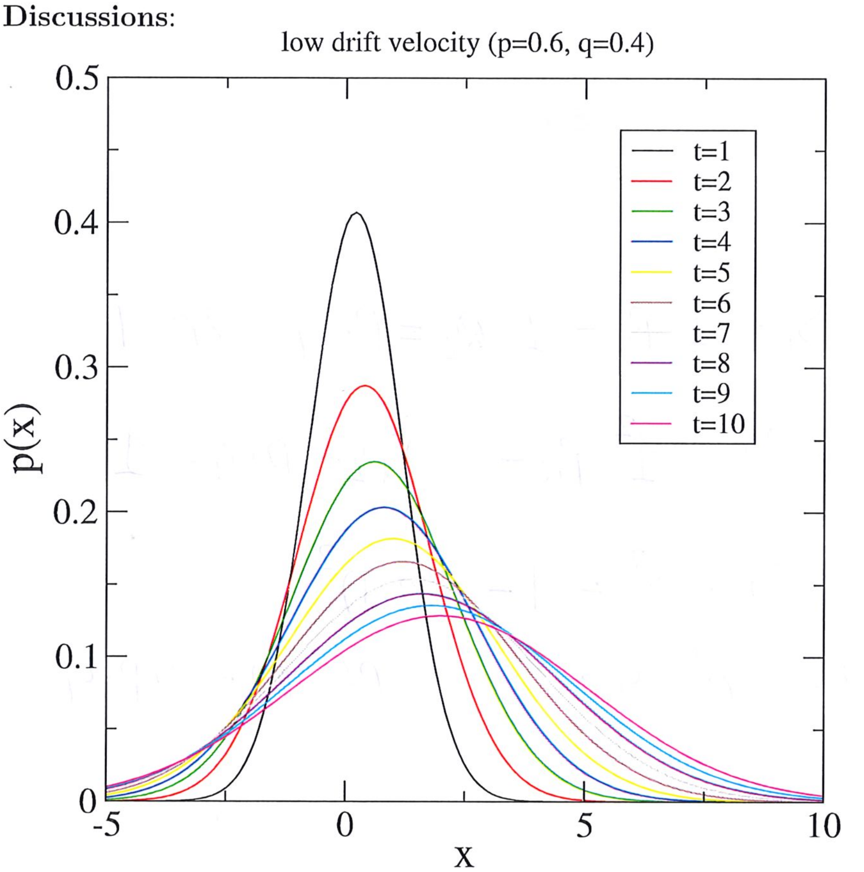
$$\Delta x \propto t^{1/2}$$

is universal. Below we will go to great length to find a different exponent than 1/2 and will discover what it takes to observe *anomalous diffusion*.

Lecture: 31/1/2020

9–10^{00}

Discussions:



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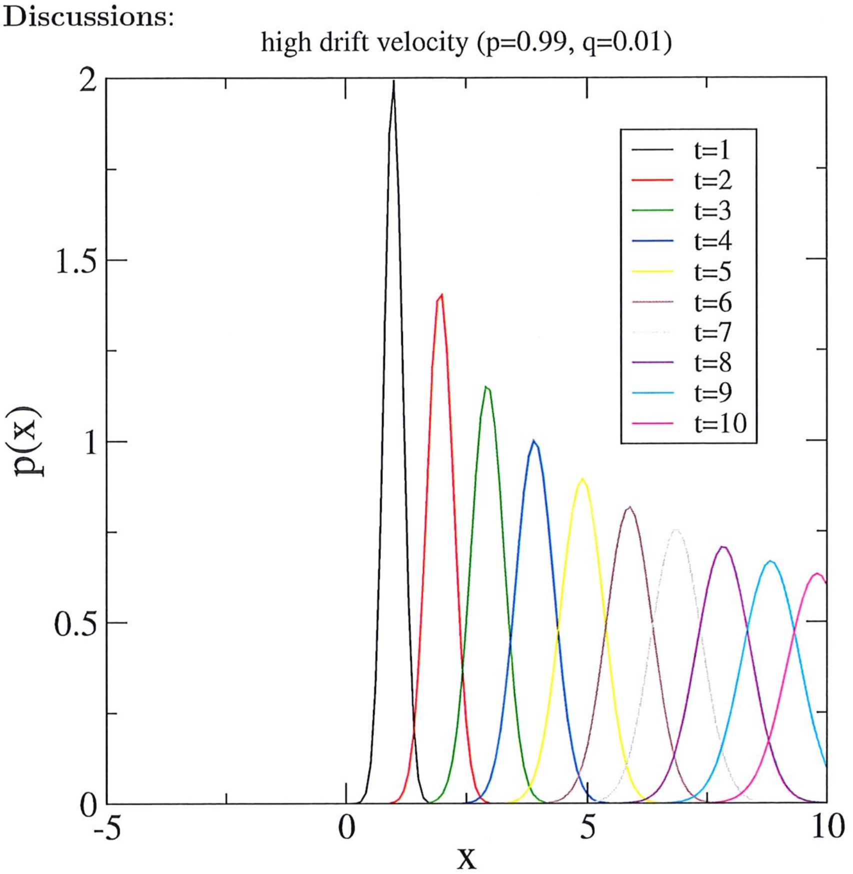

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