Fundamental reps, SU(3), Uniqueness of highest weight 3/24/11

Highest weights are linear combinations of fundamental weights (m of them) with non-negative integer coefficients

Simplest case: Fund. weights themselves $[1,0,\cdots 0]$, $[0,1,0,\cdots 0]$, ... $[0,0,\ldots,0,1]$

These are the m fundamental reps

Then can take tensor products to find more reps

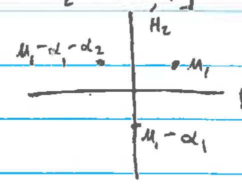
Example: SU(3) O—O

Choose $\alpha_1 = \frac{1}{2}(1, \sqrt{3})$ $\quad \alpha_2 = \frac{1}{2}(1, -\sqrt{3})$ (fixes basis)

Fund. weights: $\mu_1 = \frac{1}{2}(1, \frac{1}{\sqrt{3}})$ $\quad \mu_2 = \frac{1}{2}(1, \frac{-1}{\sqrt{3}})$

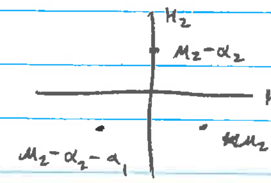
$\mu_1 = [1, 0]$ $\quad \mu_2 = [0, 1]$ by definition

Fund. reps: $[1, 0] \xrightarrow{\alpha_1} [-1, 1] \xrightarrow{\alpha_2} [0, -1]$



For μ_1 "3"

$[0, 1] \to [1, -1] \to [-1, 0]$



For μ_2 "$\bar{3}$"

Can an irrep have more than one highest weight?

Any state in the irrep can be written $E_{\phi_1} \cdots E_{\phi_n} |\mu\rangle$ where all ϕ_i are negative. $\vec{\phi}_i = \sum_j k_{ji}(-\alpha_j)$, so all states can be written $E_{-\alpha_1} \cdots E_{-\alpha_n} |\mu\rangle$

Suppose $|\mu\rangle$ and $|\mu'\rangle$ are distinct ($\langle \mu | \mu' \rangle = 0$) highest weights

Then $\langle \mu | E_{-\alpha_1} \cdots E_{-\alpha_n} | \mu' \rangle = 0$ since $|\mu\rangle$ can't be raised

Implies that $|\mu\rangle$ and $|\mu'\rangle$ generate different invariant subspaces so that the group would not be simple

By the same argument, any state obtainable by lowering from the highest weight in a unique way is unique

$\therefore$ both fundamental irreps of SU(3) are three-dimensional

Irreps inherit symmetry from SU(2) subalgebras associated with simple roots...

Weyl group, Complex conjugate rep $\bar{D}$ 3/24/11

For any positive root $\vec{\alpha}$, there is an SU(2) subalgebra with

$E_3|\mu\rangle = \vec{\alpha}\cdot\vec{H}/\vec{\alpha}^2|\mu\rangle = \vec{\alpha}\cdot\vec{\mu}/\vec{\alpha}^2|\mu\rangle = \frac{1}{2}(q-p)|\mu\rangle$

Immediately have $|\mu'\rangle$ with $E_3|\mu'\rangle = -\vec{\alpha}\cdot\vec{\mu}/\vec{\alpha}^2|\mu'\rangle = \frac{1}{2}(p-q)|\mu'\rangle$

(negated only for this SU(2) subalgebra, $|\mu'\rangle \neq |-\mu\rangle$)

$\therefore$ weight $\vec{\mu}$ produces weight $\vec{\mu}-(q-p)\vec{\alpha} = \vec{\mu}-\left(2\vec{\alpha}\cdot\vec{\mu}/\vec{\alpha}^2\right)\vec{\alpha}$

Draw on root diagram: reverses sign of $\vec{\mu}$ in direction $\hat{\alpha}$

(reflection about plane perpendicular to $\vec{\alpha}$)

"Weyl reflection": root diagram must be symmetric around reflections associated with all $\vec{\alpha}$

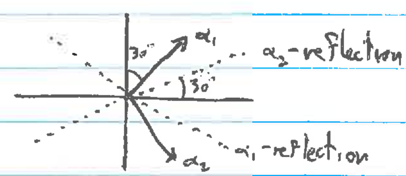
Weyl reflections plus identity form a (discrete) "Weyl group" - property of group

For SU(2), Weyl group is Z_2

For SU(3), three positive roots $\Rightarrow$ ~~at least~~ four-element Weyl group

Weyl group will be convenient way of counting states with same weights

For SU(3), planes are



α_1-reflection followed by α_2-reflection gives 120° rotation

$\Rightarrow$ every SU(3) rep will have triangular or hexagonal root diagram

Notes weights of 3 and $\bar{3}$ SU(3) reps are negatives of each other.

If an irrep has weights $\{\vec{\mu}\}$, then there is an irrep with weights $\{-\vec{\mu}\}$

Proof: Consider an irrep D of a unitary group (compact)

Then T_a hermitian and $[T_a, T_b] = i f_{abc} T_c$ with f_{abc} real

Complex conjugate: $[T_a^*, T_b^*] = -i f_{abc} T_c^*$

$[-T_a^*, -T_b^*] = i f_{abc}(-T_c^*)$

So $\{T_a\}$ being a rep $\Rightarrow \{-T_a^*\}$ is also a rep (same dimension)

Call it $\bar{D}$, the complex conjugate rep (possible $\bar{D}=D$)

If $D\sim\bar{D}$ (all weights the same), D is "real"

Highest weight of D is $-$lowest weight of $\bar{D}$

Irreducibility of D $\Rightarrow$ irreducibility of $\bar{D}$ (no invariant subalgebra)

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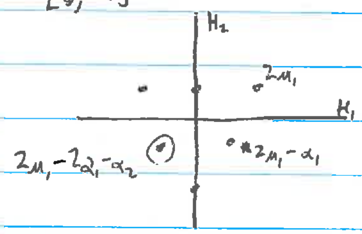
Example SU(3) irreps, Tensor methods in SU(3)

For SU(3) rep with Dynkin indices $[m, n]$ (highest weight $\vec{\mu} = m\vec{\mu}_1 + n\vec{\mu}_2$)
the lowest weight is $-n\vec{\mu}_1 - m\vec{\mu}_2$ $\Rightarrow$ $\bar{D}$ highest weight $[n, m]$

Example: SU(3) rep with $\mu = [2,0]$, $\vec{\mu} = 2\vec{\mu}_1 = (1, \frac{1}{\sqrt{3}})$

$[2,0]$	$2\mu_1$
$[0,1]$	$2\mu_1 - \alpha_1$
$[1,-1]$, $[-2,2]$	$2\mu_1 - \alpha_1 - \alpha_2$, $2\mu_1 - 2\alpha_1$
$[-1,0]$	$2\mu_1 - 2\alpha_1 - \alpha_2$ — Not a unique path...
$[0,-2]$	$2\mu_1 - 2\alpha_1 - 2\alpha_2$



Since this must be symmetric under 120° rot.
conclude $2\mu_1 - 2\alpha_1 - \alpha_2$ unique

∴ this is a six-dimensional representation, "6"
$\bar{6}$ has highest weight $\mu' = [0,2]$, upside-down triangle

Example: SU(3) rep with $\mu = [1,1]$ $\vec{\mu} = \vec{\mu}_1 + \vec{\mu}_2$ gives

$[1,1]$	
$[2,-1]$, $[-1,2]$	Need to check whether $E_{-\alpha_1} E_{-\alpha_2} \vert\mu\rangle$, $E_{-\alpha_2} E_{-\alpha_1} \vert\mu\rangle$
$[0,0]$	are linearly independent...
$[1,-2]$, $[-2,1]$	Turns out to be adjoint rep
$[-1,-1]$	(can see from knowing weights)

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More complicated notation will make multiplicity more transparent...
Equivalent to simplifying Wigner-Eckhart computations

Tensor methods: start with $T_a = \frac{1}{2}\lambda_a$ as basis of 3 of SU(3)
"Translate" back from matrices to the vector space language

$\begin{pmatrix}1\\0\\0\end{pmatrix} \to \vert \frac{1}{2}, \frac{\sqrt{3}}{6}\rangle \equiv \vert 1\rangle$ $\quad \begin{pmatrix}0\\1\\0\end{pmatrix} \to \vert -\frac{1}{2}, \frac{\sqrt{3}}{6}\rangle \equiv \vert 2\rangle$ $\quad \begin{pmatrix}0\\0\\1\end{pmatrix} \to \vert 0, -\frac{1}{\sqrt{3}}\rangle \equiv \vert 3\rangle$

$T_a \vert i\rangle = \vert j\rangle [T_a]^j{}_i$ write as $T_a \vert_i\rangle = \vert_j\rangle [T_a]^j{}_i$ $\quad [T_a]^j{}_i = \frac{1}{2}[\lambda_a]_{ji}$

For $\bar{3}$, $\vert -\frac{1}{2}, \frac{-\sqrt{3}}{6}\rangle = \vert^1\rangle$ $\quad \vert \frac{1}{2}, \frac{-\sqrt{3}}{6}\rangle = \vert^2\rangle$ $\quad \vert 0, \frac{1}{\sqrt{3}}\rangle = \vert^3\rangle$
Distinguishes 3 and $\bar{3}$ (highest weight)

Magnetic moments from SU(6), Classification of Lie groups SU(N), SO(2N), SO(2N+1), Sp(2N) 5/5/11 5/3/11

1 Example: $\langle p, \tfrac{1}{2} | Q\sigma_3 | p, \tfrac{1}{2} \rangle = \frac{\sqrt{2}}{6} \{ \frac{2}{3}$

$-\frac{2}{3}$ $\langle n, \tfrac{1}{2} | Q\sigma_3 | n, \tfrac{1}{2} \rangle = \frac{\sqrt{2}}{6} \{ -\frac{1}{3}$

So predict $\mu_p/\mu_n = -3/2$. Experiment: $\mu_p = 2.79$, $\mu_n = -1.91$, $\mu_p/\mu_n = -1.46$ ✓

5/3/11

5/5/11

Review the families of compact Lie algebras we have seen

SU(N) is rank N-1

$[\vec{\nu}^j]_m = [H_m]_{jj} = (2m(m+1))^{-1/2} \{ \sum_{k=1}^{m} \delta_{jk} - m\,\delta_{j,m+1} \}$ $j = 1, \ldots, N-1$

$\alpha^j = \nu^j - \nu^{j+1}$

SO(2N) is rank N

Break up Cartan elements into 2x2 blocks $H_k = \begin{pmatrix} 0 & & 0 \\ & \sigma_2 & \\ 0 & & 0 \end{pmatrix}$

$\vec{e}$ is n-dim unit vector — Roots are $\pm\vec{e}^j \pm \vec{e}^k$, positive $\vec{e}^j \pm \vec{e}^k$ $j<k$

Simple roots are $e^j - e^{j+1}$ $j = 1, \ldots, N-1$

and $e^{N-1} + e^N$

Next, SO(2N+1) is also rank N (can't have antisymmetry in last single component)

So choose same H_k as for SO(2N)

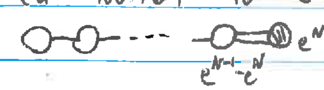
However, weights are different: 0 in addition to $\pm\vec{e}^k$

∴ roots are differences $\pm\vec{e}^j \pm \vec{e}^k$ and $\pm\vec{e}^k$ (2N more)

Positive roots are $\vec{e}^j \pm \vec{e}^k$ for $j<k$ and $+\vec{e}^k$

$e^{N-1} + e^N$ is no longer simple, but e^N is (in addition to $e^j - e^{j+1}$)

Dynkin diagram only differs at right end:



$|\vec{e}^{N-1} - \vec{e}^N| = 2$, while $|\vec{e}^N| = 1$

Next Sp(2N) is defined by 2Nx2N matrices: write as (2x2)⊗(NxN) product

Generators are $\mathbb{1} \otimes A$, $\sigma_a \otimes S_a$ (A is anti-sym, S_1, S_2, S_3 are symmetric)

All traceless, so $\det[e^{i\alpha G}] = 1$. All hermitian for unitarity

Consider commutators: since A are generators of SO(N), $\mathbb{1} \otimes A$ is closed

& $[\sigma_a \otimes S_a, \sigma_a \otimes S_a] \in \mathbb{1} \otimes A$ while $[\sigma_a \otimes S_a, \sigma_b \otimes S_b] \in \sigma_c \otimes S_c$ (check)

Choose Cartan subalgebra by considering SU(N) subgroup

$\mathbb{1} \otimes A$, $\sigma_3 \otimes S_3$ for $\mathrm{Tr}[S_3] = 0$

These give $\begin{pmatrix} T_a & \\ & T_a^* \end{pmatrix}$ (blocks NxN) corresponding to $N \oplus \bar{N}$ rep of SU(N)

$Sp(2N)$, Classification theorem 5/5/11

Put in $Sp(2N)$ Cartan subalgebra the $N-1$ generators of the $SU(N)$ subalgebra

There is one additional element in the Cartan subalgebra

$$H_N = (2N)^{-1/2}\, \sigma_3 \otimes 1$$

$H_N \vec{\nu}^j = 0$, so extend $SU(N)$ weights $\pm \nu^j$ with 0 in Nth component

Those are the weights of the Cartan subalgebra... what are those of other generators? $(\sigma_1 \pm i\sigma_2) \otimes S_{k\ell}$ are differences

Take $[S_{k\ell}]_{ij} = \delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell}$ giving root vectors

These have eigenvalues $\pm\sqrt{2/N}$ for H_N, $\pm(\nu^k + \nu^\ell)$ for H_m $m = 1, \dots, N-1$

Defining ν^{N+1} to be the unit vector orthogonal to ν^j $j = 1, \dots, N$

So the roots of $Sp(2N)$ are

$\nu^j - \nu^k$ $j \neq k$ and $\pm\left[\nu^j + \nu^k + \sqrt{\frac{2}{N}}\, \nu^{N+1}\right]$

Positive roots are $\nu^j - \nu^k$ $j < k$ and $+\left[\nu^j + \nu^k + \sqrt{\frac{2}{N}}\, \nu^{N+1}\right]$

Simple roots are $\nu^j - \nu^{j+1}$ $j = 1, \dots, N-1$ and $2\nu^N + \sqrt{\frac{2}{N}}\, \nu^{N+1}$ ← longer than others

So the Dynkin diagram is 

Also a handful of "exceptional" simple compact Lie algebras in addition to these four infinite families

Classification theorem follows from geometry of simple roots

Recall simple roots are linearly independent and for each distinct pair

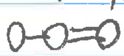
$$2\vec{\alpha} \cdot \vec{\beta} / |\vec{\alpha}|^2 = 0, -1, -2, -3$$

Finally, simple roots are "indecomposable" (cannot be divided into two mutually orthogonal sets) ⇒ Dynkin diagram connected, algebra simple

Following Dynkin, call any set of vectors satisfying these three conditions a "Π system" and enumerate them

Note that any connected subset of a Π system is also a Π system

Lemma 1: The only Π systems with three vectors are

 and  (very strong constraint!)

Classification theorem 5/5/11

Proof of lemma 1: the sum of the three angles are less than $360°$
(otherwise vectors would lie in plane and not be linearly independent)
Can only have one $90°$ angle by indecomposability
$\therefore$ only have $90° + 120° + 120°$ and $90° + 120° + 135°$ $\square$

Corollary: No Π system can contain the subsets

Therefore the only Π system with a triple line is O≡O $= G_2$

Lemma 2: If a Π system contains two vectors $\vec{\alpha}$ and $\vec{\beta}$
connected by a single line, then the Dynkin diagram obtained
by merging those circles corresponds to a Π system

I.e.,

"shrinking" single lines
γ connected to everything α & β
were connected to

Proof: $|\vec{\alpha}|^2 = |\vec{\beta}|^2 \Rightarrow (\alpha+\beta)^2 = 2\alpha^2 + 2\alpha\cdot\beta = 2\alpha^2(1 + \alpha\cdot\beta/\alpha^2) = \alpha^2$
(single line) $\qquad$ $\alpha\cdot\beta/\alpha^2 = -1/2$ (single line)

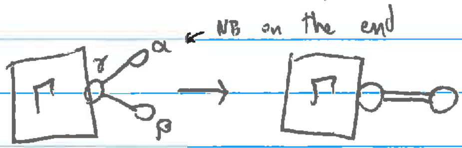
Any other $\delta \in \Gamma$ can be connected to at most one of α or β
So $\delta\cdot(\alpha+\beta) = \begin{cases}\delta\cdot\alpha \\ \delta\cdot\beta\end{cases}$ depending on which (if any) it is connected to
$\therefore$ the Dynkin diagram

is a Π system $\square$
(fewer vectors, same angles and lengths)

Corollaries:

1) No Π system has more than one double line
(or else shrink single lines to get forbidden diagram)
2) No closed loop in any Π system

Lemma 3:



is a Π system

Proof: $\alpha\cdot\beta = 0$ while $\frac{2\alpha\cdot\gamma}{\gamma^2} = \frac{2\alpha\cdot\gamma}{\alpha^2} = \frac{2\beta\cdot\gamma}{\gamma^2} = \frac{2\beta\cdot\gamma}{\beta^2} = -1$ $\quad \alpha^2 = \beta^2 = \gamma^2$

$\Rightarrow \frac{2\gamma\cdot(\alpha+\beta)}{\gamma^2} = -2$ while $\frac{2\gamma\cdot(\alpha+\beta)}{(\alpha+\beta)^2} = -1$ gives shrunken system as above

Classification theorem

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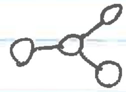
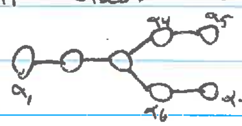
Corollaries:

1) Only allowed branch is  (each node connected to no more than three others)

2) Can never have more than one branch in a Π system

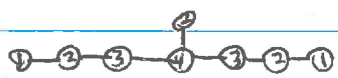
Now that we have constrained the components out of which we can build Π systems, consider some special cases

Special cases of forbidden ~~diagrams~~ Π systems

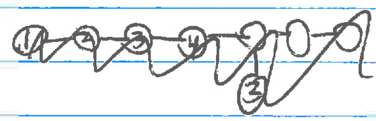


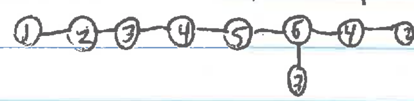
are not linearly independent

$$(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 + \alpha_7)^2 = 0$$



all these also not linearly independent





 (two possibilities)

Now can write all allowed Π systems (classification theorem)

 $= A_N \to SU(N+1)$

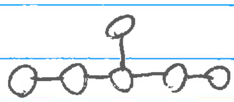
 $= B_N \to SO(2N+1)$

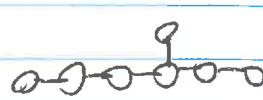
 $= C_N \to Sp(2N)$

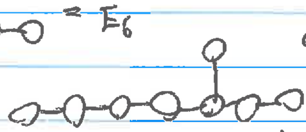
 $= D_N \to SO(2N)$

 $= G_2$

 $= F_4$

 $= E_6$

 $= E_7$

 $= E_8$

Trying to add any more nodes to the exceptional groups produces ~~something~~ a forbidden Π system