3/1/11

Pions isospin, Generalization of highest-weight procedure 3/1/11

Consider pions: spinless bosons with approximately equal masses

Treat as isotriplet $\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$ π^+ highest weight due to $\{\frac{1}{2}\}\otimes\{\frac{1}{2}\} = \{1\}$

Example: $d+d \to He^4 + \pi^0$ (conserves electric charge)

$d+d$ is isosingle, π^0 is isosinglet with weight 0

$He^4 \in \{\frac{1}{2}\}\otimes\{\frac{1}{2}\}\otimes\{\frac{1}{2}\}\otimes\{\frac{1}{2}\}$ with weight zero; total isospin either 2, 1, 0.

Since only one He^4 (and no Li^4), conclude He^4 is isosinglet

Conclude that $d+d \to He^4 + \pi^0$ does not conserve isospin $(0 \to 1)$

but must proceed by ~~electro~~weak or electric interaction $\Rightarrow$ small σ.

Example: $p+p \to d+\pi^+$ is isotriplet $\to$ isotriplet, so can go strongly

$p+n \to d+\pi^0$ can go strongly or weakly/electrically

Can relate isotriplet $p+n$ state to $p+p$ state by Wigner-Eckart theorem

$\langle d+\pi^+ | H | p+p\rangle$ vs. $\langle d+\pi^0 | H | p+n\rangle$

Since we can distinguish p & n, write $|p+n\rangle = |p\rangle|n\rangle = |\text{projectile}\rangle|\text{target}\rangle$

Recall $\frac{1}{\sqrt{2}}(|p\rangle|n\rangle + |n\rangle|p\rangle)$ is $|\text{isotriplet}\rangle$

$\frac{1}{\sqrt{2}}(|p\rangle|n\rangle - |n\rangle|p\rangle)$ is $|\text{isosinglet}\rangle$

$|p\rangle|n\rangle = \frac{1}{\sqrt{2}}(|\text{isotriplet}\rangle + |\text{isosinglet}\rangle)$

So $\langle d+\pi^+ | H | p+p\rangle = A$, $\langle d+\pi^0 | H | p+n\rangle = \frac{1}{\sqrt{2}} A$ since $N_j = 1$

Predict $\sigma(p+p \to d+\pi^+) = 2\sigma(p+n \to d+\pi^0)$

Now generalize SU(2) highest-weight procedure to arbitrary simple group & rep

Assume we know f_{abc}, ~~...~~ & diagonalize as many generators as possible

First: "Find a maximal commuting set of generators" $\{H_i\}$ $i=1,\dots,m$

m is the rank of algebra, $\{H_i\}$ form the Cartan subalgebra

Imagine we have an irrep D (only interested in finite-dim irreps)

$H_i^\dagger = H_i$, $[H_i, H_j] = 0$, so as matrices $\mathrm{Tr}[H_i H_j] = k_D \delta_{ij}$

$\hookrightarrow$ possible because of $\uparrow$

Now we can diagonalize $\{H_i\}$ and label the basis states $|\mu, X, D\rangle$

~~μ~~ are eigenvalues of $\{H_i\}$, X are ~~..~~ other rep-related ~~...~~ labels, D is irrep

$H_i |\mu, X, D\rangle = \mu_i |\mu, X, D\rangle$, $\{\mu_i\}$ are called weights, $\vec{\mu}$

We proved that the adjoint rep is an irrep, so let's consider it

3/3/11
3/1/11

General adjoint rep, Roots, Raising and lowering

Adjoint rep has dimension equal to order of algebra (number of generators)

Label states $|X_a\rangle$ and demand $\alpha|X_a\rangle + \beta|X_b\rangle = |\alpha X_a + \beta X_b\rangle$

Required since linear combinations of states must be in vector space

Define inner product $\langle X_a|X_b\rangle = \frac{1}{\lambda}\mathrm{Tr}[X_a^\dagger X_b]_{adj}$ (basis independent)

Consider $\langle X_c|X_a|X_b\rangle = -if_{acb} = if_{abc} \Rightarrow X_a|X_b\rangle = if_{abc}|X_c\rangle = |[X_a, X_b]\rangle$

Obvious because the adjoint is a rep of the algebra, but we see above the specific basis constructed to guarantee this

The weights of the adjoint rep are called roots

1) Start with $|H_i\rangle$ in the Cartan subalgebra: $H_i|H_j\rangle = |[H_i, H_j]\rangle = 0$

$\therefore$ need X in $|\mu, X, D\rangle$ when rank >1 since $\vec{\mu} = \vec{0}\ \forall X \in \{H_i\}$

Converse also true: $H_i|X\rangle = 0 \Rightarrow [H_i, X] = 0\ \forall H_i \Rightarrow X \in \{H_i\}$

2) $H_i|E_\alpha\rangle = \alpha_i|E_\alpha\rangle = |[H_i, E_\alpha]\rangle = |\alpha_i E_\alpha\rangle \Rightarrow [H_i, E_\alpha] = \alpha_i E_\alpha$

Like $J^\pm$, E_α may not be hermitian, so $-[H_i, E_\alpha^\dagger] = \alpha_i E_\alpha^\dagger$

(Just labelling "E_α" linear combinations of X_a)

$\alpha_i \in \mathbb{R}$ since $H_i^\dagger = H_i$. Conclude $\pm\alpha_i$ appear in pairs as H_i e.v.s

Label $E_\alpha^\dagger = E_{-\alpha}$, two generators for each nonzero root vector

Recall states with different weights are orthogonal

$\langle E_\alpha|H_i|E_\beta\rangle = \beta_i\langle E_\alpha|E_\beta\rangle = \alpha_i\langle E_\alpha|E_\beta\rangle \Rightarrow \langle E_\alpha|E_\beta\rangle = 0$ if $\vec{\alpha} \neq \vec{\beta}$

↳ in adjoint rep

adjoint → algebra

Roots are special weights because related only to algebra ($[H_i, E_\alpha]$)

Identify $E_{\pm\alpha}$ as raising and lowering operators, because

arbitrary rep!

$H_i E_{\pm\alpha}|\mu, X, D\rangle = ([H_i, E_{\pm\alpha}] + \mu_i E_{\pm\alpha})|\mu, X, D\rangle = (\mu_i \pm \alpha_i)E_{\pm\alpha}|\mu, X, D\rangle$

3/1/11

3/3/11

Continue to use adjoint rep to constrain possible algebras

$E_\alpha|E_{-\alpha}\rangle$ has weight zero, so must correspond to some $|\vec{\beta}\cdot\vec{H}\rangle$

$E_\alpha|E_{-\alpha}\rangle = \beta_i|H_i\rangle = |[E_\alpha, E_{-\alpha}]\rangle$

Adjoint $\Rightarrow [E_\alpha, E_{-\alpha}] = \beta\cdot H$

Compute β_j from $\langle H_j|E_\alpha|E_{-\alpha}\rangle = \beta_j = \frac{1}{\lambda}\mathrm{Tr}[H_j[E_\alpha, E_{-\alpha}]]$

$= \mathrm{Tr}[E_{-\alpha}[H_j, E_\alpha]]/\lambda$

General SU(2) sub-algebras and resulting constraints 3/3/11

So we have $\beta_j = \left(\alpha_j/\lambda\right) Tr\left[E_{-\alpha} E_\alpha\right] = \alpha_j \langle E_\alpha | E_\alpha \rangle = \alpha_j$

$\therefore \; [E_\alpha, E_{-\alpha}] = \alpha \cdot H$

To summarize: (For convenience define $E_3 = \alpha \cdot H / |\alpha|^2$, $E^\pm = E_{\pm\alpha} / \sqrt{|\alpha|^2}$)

$[E_3, E^\pm] = \pm E^\pm$ $\quad [E^+, E^-] = E_3$ exactly SU(2)

We have identified SU(2) subalgebras of every general algebra with any $\alpha \neq 0$

Irreps of full algebra correspond to (generally reducible) reps of SU(2) sub-

Hugely constraining, will give easy labelling scheme

Thm: There is only one generator with root $\vec{\alpha}$

Proof: Assume there are two, E_α and $E_\alpha' \neq E_\alpha$. (statement about algebra)

Work in adjoint rep, basis with H_i diagonal

Diagonalize two-dimensional subspace with same $\vec{\alpha}$, $\langle E_\alpha | E_\alpha' \rangle = 0$

$Tr[E_\alpha^\dagger E_\alpha']/\lambda = Tr[E_{-\alpha} E_\alpha']/\lambda = 0 \Rightarrow Tr[E^-, E_\alpha'] = 0$

Consider $E^- | E_\alpha' \rangle = | \beta \cdot H \rangle$ since weight is zero

As above, $\beta_j = \langle H_j | [E^-, E_\alpha'] \rangle = Tr[H_j [E^-, E_\alpha']]/\lambda = Tr[E^- [H_j, E_\alpha']]/\lambda$

$= -\alpha_j Tr[E^-, E_\alpha']/\lambda = 0! \Rightarrow E^- | E_\alpha' \rangle = 0$

So $| E_\alpha' \rangle$ is the lowest weight state of the $E_3, E^\pm$ subalgebra

$E_3 | E_\alpha' \rangle = \frac{1}{|\alpha|^2} \alpha \cdot H | E_\alpha' \rangle = | E_\alpha' \rangle \Rightarrow$ lowest weight is $1 > 0! \Rightarrow\Leftarrow$

There is no such state $\square$

Thm: If $\vec{\alpha}$ is a root, no multiple of $\vec{\alpha}$ (other than $\pm\vec{\alpha}$) is a root.

Proof: As above, assume there's state in the adjoint rep corresponding to $k\vec{\alpha}$

This state has weight k under the $E_3, E^\pm$ subalgebra $\Rightarrow$ k is negative half-integer

Then applying raising E^+ would, if k is negative integer,

give state with weight α, forbidden above

If k is a negative half-odd integer, we could reverse

hech the argument: $2\vec{\alpha}$ forbidden implies $\vec{\alpha}/2$ forbidden.

Can now reach significant conclusion regarding weights in an arbitrary irrep

$|\mu, X, D\rangle$ must be part of a rep of the $E_3, E^\pm$ subalgebra

$E_3 |\mu, X, D\rangle = \frac{\vec{\alpha} \cdot \vec{\mu}}{|\vec{\alpha}|^2} |\mu, X, D\rangle$ in SU(2) $\Rightarrow$ $2\frac{\vec{\alpha} \cdot \vec{\mu}}{|\vec{\alpha}|^2}$ is an integer!

3/8/11
3/3/11

Master Formula, SU(3) defining rep and Cartan subalgebra

Because $|\mu, X, D\rangle$ is part of a rep of the SU(2) subalgebra,
this subalgebra rep must have a highest weight state

$(E^+)^{p+1}|\mu, X, D\rangle = 0$ (can raise p times) $p \geq 0$ int

$(E^-)^{q+1}|\mu, X, D\rangle = 0$ (can lower q times) $q \geq 0$ int.

So $(E^+)^p|\mu, X, D\rangle$ is the highest weight state of some SU(2) irrep, say $\{j\}$

$E_3 (E^+)^p |\mu, X, D\rangle = j (E^+)^p |\mu, X, D\rangle \Rightarrow j = \vec{\alpha}\cdot(\vec{\mu}+p\vec{\alpha})/|\vec{\alpha}|^2 = p + \vec{\alpha}\cdot\vec{\mu}/|\vec{\alpha}|^2$

Similarly $-j = -q + \vec{\alpha}\cdot\vec{\mu}/|\vec{\alpha}|^2$

Adding, we find $\dfrac{\vec{\alpha}\cdot\vec{\mu}}{|\vec{\alpha}|^2} = \dfrac{-(p-q)}{2}$ "master formula" true for arbitrary rep

Apply master formula to adjoint rep ($\vec{\mu}$ is root $\vec{\beta}$)

$\vec{\alpha}\cdot\vec{\beta} = -\vec{\alpha}^2(p-q)/2$

But since $\vec{\beta}$ is a root, can find another relation $\vec{\beta}\cdot\vec{\alpha} = -\vec{\beta}^2(p'-q')/2$

Product: $\dfrac{(\vec{\alpha}\cdot\vec{\beta})^2}{\vec{\alpha}^2\vec{\beta}^2} = \cos^2\theta_{\alpha\beta} = \dfrac{(p-q)(p'-q')}{4} \leftarrow$ integer

Five possibilities:

$(p-q)(p'-q')$:	0	1	2	3	4
$\theta_{\alpha\beta}$:	$90°$	$60°, 120°$	$45°, 135°$	$30°, 150°$	$\vec{\beta}=\vec{\alpha}$ redundant

Small set of possible angles implies will be easy to label

3/3/11
3/8/11

Develop in parallel the general case and a specific case, SU(3)

SU(3) defining rep is set of special unitary 3x3 matrices

Exponential map relates group elements to generators, $U = e^{i\alpha\cdot X} \approx 1 + i\alpha\cdot X$

$U^\dagger U = 1 \Rightarrow X^\dagger = X$

$\det U = 1$ in basis where $U = \mathrm{diag}(e^{i\lambda_1}, e^{i\lambda_2}, e^{i\lambda_3}) \Rightarrow \lambda_1+\lambda_2+\lambda_3 = 0 = \mathrm{Tr}\,X$

Number of linearly independent generators: $18-9-1 = 8$ $T^A = \frac{1}{2}\lambda^A$

Very clear SU(2) subalgebra T_1, T_2, T_3 (isospin) $\mathrm{Tr}[T^A T^B] = \frac{1}{2}\delta^{AB}$

Instead of going through structure constants, consider roots and weights

check Maximal commuting set is $\{T_3, T_8\}$ (can see from Gell-Mann matrices)

Rank 2, $\{H_1, H_2\}$ in basis with $\mathrm{Tr}[H_i] = 0$, $\mathrm{Tr}[H_i^2] = \frac{1}{2}$

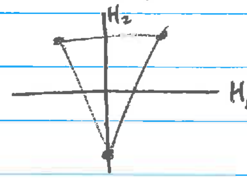
What are weights of defining rep?

SU(3) roots, Weight diagrams, "Highest" weight 3/8/11

Weights in defining rep from $\frac{1}{2}$ Gell-Mann matrices

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = \left(T_3\begin{pmatrix}1\\0\\0\end{pmatrix}, T_8\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right) \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} = \left(\frac{-1}{2}, \frac{1}{2\sqrt{3}}\right) \qquad \begin{pmatrix}0\\0\\1\end{pmatrix} = \left(0, \frac{-1}{\sqrt{3}}\right)$$

Plot:



(should be equilateral triangle)

Roots correspond to generators of SU(2) subalgebra, which
move from one weight to ~~the~~ another (three lines in plot)
Have six roots since two generators in Cartan subalgebra
$(\pm 1, 0)$ $\left(\pm\frac{1}{2}, \pm\frac{1}{2}\sqrt{3}\right)$ (all four combinations)

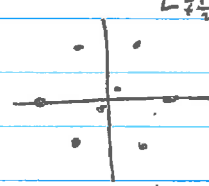
Must be defining rep

In general, not every difference of weights is a root
Depends on commutation relation, numbers of roots and differences

Corresponding generators: $E_{\pm 1, 0} = \frac{1}{\sqrt{2}}(T_1 \pm i T_2)$

$E_{\pm\frac{1}{2}, \pm\sqrt{3}/2} = \frac{1}{\sqrt{2}}(T_4 \pm i T_5)$ $E_{\mp\frac{1}{2}, \pm\sqrt{3}/2} = \frac{1}{\sqrt{2}}(T_6 \pm i T_7)$

Plot on same plane



(should be regular hexagon, two at (0,0))

Weight diagrams for rank-2 algebras are easy, rank-3 harder

These H_i and E are the algebra — can deduce structure constants if desired
Can see irrep (no invariant subspace) from diagram

Would like to find all irreps
How to define highest weight for rank-2 algebra?
No "natural" choice, but all choices equivalent
Definition: weight is "positive" if its first non-zero component is positive
(depends on choice of Cartan subalgebra)
μ is ~~positive~~ negative if $-\mu$ is positive (nonzero roots in $\pm$ pairs)
$\mu > \nu$ if $\mu - \nu$ is positive
Highest weight is greater than all other weights

Simple roots and their properties 3/8/11

A simple root is a positive root that is not the sum of positive roots with positive integer coefficients

Can reconstruct entire algebra (uniquely) from simple roots

Rank-m algebras have m simple roots

Let's prove some properties of simple roots

Thm: If α and β are different simple roots, then $\alpha-\beta$ is not a root

Proof: Assume $\beta > \alpha$ (just labelling). Then $\beta-\alpha$ is a positive vector

If $\beta-\alpha$ were a root (positive), then $\beta = (\beta-\alpha)+\alpha$ and β is not simple $\Rightarrow\Leftarrow$

Thm: $E_{-\alpha}|E_\beta\rangle = 0 = E_{-\beta}|E_\alpha\rangle$ α, β as above lowest weights!

Proof: LTR, follows from above

Therefore $|E_\beta\rangle$ is the lowest weight state of the α $SU(2)$ & vice-versa

Apply master formula: $q=0$ so $2\vec{\alpha}\cdot\vec{\beta}/\alpha^2 = -p$ } both integers

similarly $2\alpha\cdot\beta/\beta^2 = -p'$

So $\cos\theta_{\alpha\beta} = -\sqrt{pp'}/2$ $\beta^2/\alpha^2 = p/p'$

Negative cosine and positive roots implies $\frac{\pi}{2} \le \theta_{\alpha\beta} < \pi$

Thm: The simple roots are linearly independent

Proof: Let $\vec{\gamma} = \sum_i c_i \vec{\alpha}_i = 0$ with not all $c_i = 0$

Because all α_i are positive vectors, some $c_i \le 0$, others $c_i \ge 0$

$$\vec{\gamma} = \sum_{c_i>0} c_i\vec{\alpha}_i - \sum_{c_i<0} (-c_i)\vec{\alpha}_i = \vec{P}_+ - \vec{P}_-$$

Both P_+ and P_- are positive sums of simple roots $\Rightarrow \cos_{+-} < 0$

$\therefore \gamma^2 = P_+^2 + P_-^2 - 2P_+\cdot P_-$ is three positive terms

So both P_+ and P_- must vanish separately, which requires all $c_i = 0$.

Thm: Any positive root can be written $\vec{\phi} = \sum k_i \vec{\alpha}_i$ $k_i \ge 0$ integer

Proof: Trivial if $\vec{\phi}$ is simple. Otherwise $\vec{\phi}$ is sum as described in definition above $\phi = \phi_1 + \phi_2$, ϕ_1 & ϕ_2 both positive. Etc.

3/10/11
3/8/11

Simple roots, Reconstructing algebra

Thm: The simple roots are complete (precisely m of them spanning space)

Proof: If not, then $\exists\ \vec{\xi}$ s.t. $\vec{\xi}\cdot\vec{\alpha}_i = 0\ \forall\ \vec{\alpha}_i$ simple roots

By previous theorem, $\vec{\xi}\cdot\vec{\phi} = 0\ \forall\ \vec{\phi}$ positive $\Rightarrow \vec{\xi}\cdot\vec{\phi} = 0\ \forall$ roots

So $[\xi\cdot H, E_\phi] = 0$ for all generators $\rightarrow$ algebra not simple
$= \xi\cdot\phi E_\phi$ (not even semi-simple)
($\xi\cdot H$ would be invar. subalg)

So simple roots are a complete basis for all (semi) simple algebra (but not orthonormal)

3/8/11
3/10/11

Let's prove that we can reconstruct all roots (& all the algebra) from the simple roots. Only need to worry about positive roots

How to decide whether $\phi_k = \sum_i k_i \vec{\alpha}_i$ is or isn't a root?

Define the "level" of ϕ_k to be $\sum_i k_i$ (number of simple roots in sum)

$\alpha_1 + 2\alpha_2$ has level 3.

Level 1 are the simple roots

Induction hypothesis: assume we know the roots up to level ℓ

Can we determine all roots of level $\ell+1$?

Act with all raising operators E_α in adjoint rep, $\phi_\ell \rightarrow \phi_\ell + \alpha$ by $E_\alpha|\phi_\ell\rangle$

Master formula for α-SU(2): $\dfrac{2\alpha\cdot\phi_\ell}{\alpha^2} = q - p$

q is known by the induction hypothesis! (so is ϕ_ℓ, of course)

$\therefore$ we can calculate p

If $p=0$, $\phi_{\ell+\alpha}$ is not a root. otherwise $\phi_{\ell+\alpha}$ is not a root

Since simple roots known, all roots at level $\ell+\alpha$ now determined

Finally, prove no "dangling roots" that cannot be obtained in going from level ℓ to level $\ell+1$.

This would imply $E_\alpha|\phi_\ell+\alpha\rangle = 0\ \forall\ \alpha$

$|\phi_\ell+\alpha\rangle$ is lowest weight state for all SU(2)s, all $E_3|\phi_\ell+\alpha\rangle \leq 0$

$\alpha\cdot\phi_{\ell+1}/\alpha^2 \leq 0\ \forall\ \alpha$, but $\vec{\phi}_{\ell+1} = \sum_i k_i \vec{\alpha}_i$ k_i positive integers

$\therefore \vec{\phi}_{\ell+1}^{\,2} = \vec{\phi}_{\ell+1}\cdot\sum_i k_i \vec{\alpha}_i \leq 0 \Rightarrow\Leftarrow$

Reconstructing SU(3) and G_2 3/10/11

Example: reconstruct SU(3) from simple roots

Recall positive roots $(1,0)$, $(\frac{1}{2}, \sqrt{3}/2)$, $(\frac{1}{2}, -\sqrt{3}/2)$

Latter two should be simple. Check that they can't be written as sums of the other two. ✓ Call them $\vec{\alpha}_1$, $\vec{\alpha}_2$

Now determine p and p'. $\alpha_1^2 = 1$, $\alpha_2^2 = 1$, $\alpha_1 \cdot \alpha_2 = -1/2$

$\Longrightarrow p = p' = 1$

Each simple root can be raised once by the other, $\alpha_1 + \alpha_2 = (1,0)$

check Can check p values of $\alpha_1 + \alpha_2$ and see that it can't be raised

Simple roots determine algebra

Only need to specify their lengths and relative angles

Do it pictorially - Dynkin diagrams

Draw a circle for each simple roots (number of circles is rank)

p & p' determined by angle between simple roots

Possible angles: 90°, 120°, 135°, 150°

no line, —, =, ≡

(note n lines corresponds to $p \cdot p' = n$)

Examples: ○ is SU(2), only rank-1 algebra

Rank-2: ○—○ is SU(3)

Choosing $\alpha_1^2 = \alpha_2^2 = 1$, $\alpha_1 \cdot \alpha_2 = -\frac{1}{2} \Rightarrow (\frac{1}{2}, \frac{\sqrt{3}}{2})$ & $(\frac{1}{2}, \frac{-\sqrt{3}}{2})$

○≡○: choose $\alpha_1^2 = 1$, $\alpha_2^2 = 3$, $\alpha_1 \cdot \alpha_2 = -3/2$ (check) ✓

choose basis $(0,1) = \alpha_1$, $\alpha_2 = \sqrt{3}(\frac{1}{2}, \frac{-\sqrt{3}}{2})$

○≡○ is called G_2. Let's find all of its roots (# generators unknown)

Level 1: α_1 and α_2 cannot be lowered

$2\alpha_1 \cdot \alpha_2 / \alpha_1^2 = -p = -3$, so α_2 can be raised by α_1 three times

$2\alpha_1 \cdot \alpha_2 / \alpha_2^2 = -p' = -1$ so α_1 can be raised by α_2 once

This tells us the roots $\alpha_2 + \alpha_1$, $\alpha_2 + 2\alpha_1$, $\alpha_2 + 3\alpha_1$

Forbidden: $\alpha_1 + 2\alpha_2$, $\alpha_2 + 4\alpha_1$, $2\alpha_1$, $2\alpha_2$, $2(\alpha_1 + \alpha_2)$

Draw a picture to help be systematic...

Reconstructing G_2, Notation for raising & lowering, Cartan matrix 3/22/11 3/16/11

Level (check that complete)

✓ 1

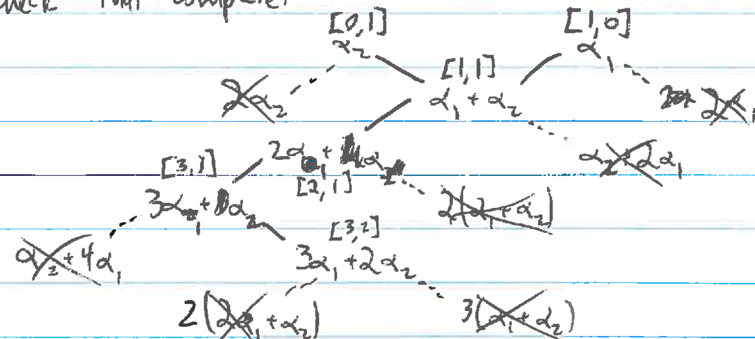
✓ 2

✓ 3

✓ 4

✓ 5

✓ 6

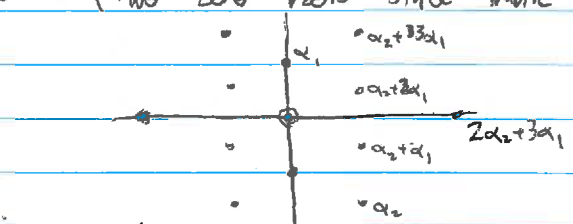


Check raising $3\alpha_1+\alpha_2$ by α_2: $2\alpha_2\cdot(3\alpha_1+\alpha_2)/\alpha_2^2 = \frac{2}{3}\left(3\cdot\frac{-3}{2}+3\right) = -1 = q-p$

Clear that $3\alpha_1+2\alpha_2$ cannot be raised

So G_2 has 14 roots (two zero roots since rank 2): $\dim(G_2)=14$

Root diagram:



Easy to change notation to [a, b] shown above - simpler

3/16/11

3/22/11

Try to keep $q-p$ values explicit at every stage

Note that simple roots are a non-orthonormal basis for the weight space

For each simple root α_i, $l_i = q_i - p_i = 2\alpha_i\cdot\phi/\alpha_i^2$ (integers) (Dynkin indices)

Choose basis vectors so that coefficients are l_i: go from l_i to ϕ

Define basis vectors μ_j so that $2\alpha_i\cdot\mu_j/\alpha_i^2 = \delta_{ij}$ (dual vectors)

Claim $\phi = \sum l_i \mu_i$ (Fundamental weights)

Proof: Clearly $2\alpha_i\cdot\phi/\alpha_i^2 = l_i \Rightarrow \phi = \sum l_i\mu_i + \Delta$

But $\alpha_i\cdot\Delta = 0 \ \forall \alpha_i \Rightarrow \Delta = 0$ since $\vec{\alpha}$ are complete L.I. basis

Choose $\{\vec{\mu}_1, \vec{\mu}_2\}$ as basis of G_2: $\vec{\alpha}_1 = [2,-1]$, $\vec{\alpha}_2 = [-3, 2] = q-p$

Got these from $2\alpha_i\cdot\mu_j/\alpha_i^2$; can easily check raising & lowering

On the next level, want to add $\vec{\alpha}_1+\vec{\alpha}_2 \ldots$

Define Cartan "matrix" $A_{ji} = \dfrac{2\vec{\alpha}_j\cdot\alpha_i}{\alpha_i^2} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

The jth row of the Cartan matrix is $\vec{\alpha}_j$ in the μ_j basis

(no obvious information in columns so not really matrix)

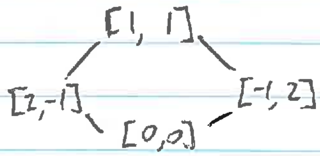
Roots for SU(3) and G_2, Reconstruct G_2 from simple roots 3/22/11

For any positive root $\phi = \sum_j k_j \vec{\alpha}_j$, $\quad l_i = \sum_j k_j \, 2\alpha_i \cdot \alpha_j / \alpha_i^2 = \sum_j k_j A_{ji}$

The factors of 2 along the diagonal appear because each simple root is the highest weight of an adjoint rep of an SU(2) subalgebra

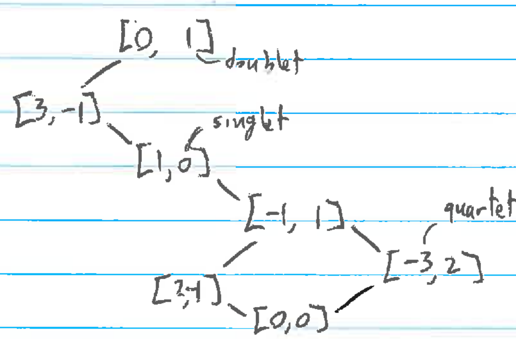
For SU(3), $A_{ji} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ (can see from O–O: same length, 120°)

Still have two [0,0] vectors, two simple roots [2,-1] and [-1,2]



(and mirror image for negative roots)

For G_2,



(structure constants)

Not hard to explicitly reconstruct f_{abc} from this notation

Exploit SU(2) subalgebras of simple roots to calculate all correlators

For G_2, define raising operators $E_1^+ = E_{\alpha_1}$ $\quad E_2^+ = \frac{1}{\sqrt{3}} E_{\alpha_2}$ $\quad$!

$|E_{\alpha_2}\rangle = |\frac{3}{2}, -\frac{3}{2}; 1\rangle$ so $E_1^+ |E_{\alpha_2}\rangle = \sqrt{\frac{3}{2}} |\frac{3}{2}, -\frac{1}{2}; 1\rangle = \sqrt{\frac{3}{2}} |E_{\alpha_1+\alpha_2}\rangle$

with respect to α_1 SU(2) subalgebra

$E_1^+ |[E_{\alpha_1}, E_{\alpha_2}]\rangle = |[E_{\alpha_1}, [E_{\alpha_1}, E_{\alpha_2}]]\rangle = 2\sqrt{\frac{3}{2}} |E_{2\alpha_1+\alpha_2}\rangle$

So $f_{\alpha_1, \alpha_2, \alpha_1+\alpha_2} = \sqrt{\frac{3}{2}}$ $\quad f_{\alpha_1, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2} = \sqrt{6}\sqrt{\frac{2}{3}}$ $\quad f_{\alpha_1, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2} = 3/\sqrt{6}$

$|[E_{\alpha_1}, [E_{\alpha_1}, [E_{\alpha_1}, E_{\alpha_2}]]]\rangle = 3|E_{3\alpha_1+\alpha_2}\rangle = 3|\frac{1}{2}, -\frac{1}{2}; 2\rangle$

$|[E_{\alpha_2}, [E_{\alpha_1}, [E_{\alpha_1}, [E_{\alpha_1}, E_{\alpha_2}]]]]\rangle = 3\sqrt{3}\left(\frac{1}{\sqrt{2}}\right)|\frac{1}{2}, -\frac{1}{2}; 2\rangle = \frac{9}{\sqrt{6}} |E_{3\alpha_1+2\alpha_2}\rangle$

This gives us all possible commutators of positive roots

Commutators of negative roots are just ~~the~~ hermitian conjugates

Only need to worry about commutators of positive and negative roots

$[E_{-\alpha_1}, E_{\alpha_1+\alpha_2}] = [E_{-\alpha_1}, [E_{\alpha_1}, E_{\alpha_2}]]\sqrt{\frac{2}{3}}$

$[E_{-\alpha_1}, E_{\alpha_1}] = -\alpha_1 \cdot H$ $\quad [E_{-\alpha_1}, E_{\alpha_2}] = 0$ since α_2 is simple root

Reconstructing G_2, Roots for C_3, Generalizing irreps: highest weights   3/24/11 3/22/11

Use Jacobi identity on double commutator:

$$\sqrt{\tfrac{2}{3}}\,[E_{-\alpha_1}, [E_{\alpha_1}, E_{\alpha_2}]] = -\sqrt{\tfrac{2}{3}}\left([E_{\alpha_{2}}, \overbrace{[E_{\alpha_2}, E_{-\alpha_1}]}^{0}] + [E_{\alpha_2}, \overbrace{[E_{-\alpha_1}, E_{\alpha_1}]}^{-\alpha_1 \cdot H}]\right)$$
$$= \sqrt{\tfrac{2}{3}}\,\alpha_1\cdot\alpha_2\, E_{\alpha_2} = -\sqrt{\tfrac{3}{2}}\, E_{\alpha_2}$$

Conclusion: can obtain Full algebra (structure constants) From just the Dynkin diagram.

Less trivial example: 

Rank: 3

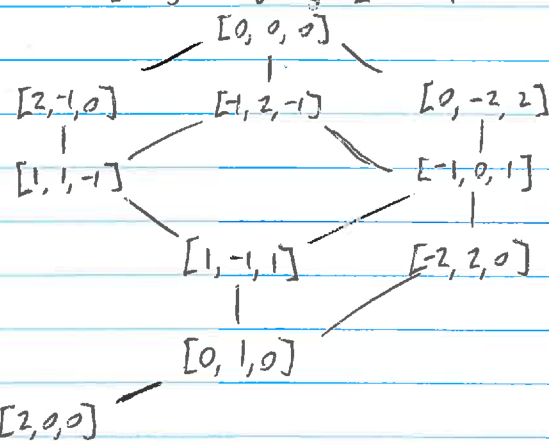
Two possibilities for lengths $1:1:\sqrt{2}$ or $\sqrt{2}:\sqrt{2}:1$

C_3: $\vec{\alpha}_1^{\,2} = \vec{\alpha}_2^{\,2} = 1$, $\vec{\alpha}_3^{\,2} = 2$

$\vec{\alpha}_1\cdot\vec{\alpha}_2 = \frac{-1}{2}$ $\vec{\alpha}_1\cdot\vec{\alpha}_3 = 0$ $\vec{\alpha}_2\cdot\vec{\alpha}_3 = -\sqrt{\alpha_3^2}\,\frac{\sqrt{2}}{2} = -1$

$$A_{ji} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

Upside-down root diagram:



In general, could be multiple states with the same weight, however the <u>roots</u> of the adjoint rep uniquely correspond to generators. Conclude $\dim(C_3) = 21$.

3/22/11

3/24/11

Need irreps for general algebras – use highest weight procedure

If $\vec{\mu}$ is a highest weight, then $\vec{\mu}+\vec{\phi}$ is not a weight for any positive ~~weight~~ root $\vec{\phi} = \sum_i k_i \vec{\alpha}_i \Rightarrow E_{\alpha_i}|\mu\rangle = 0 \;\forall$ simple roots α_i

This will turn out to be sufficient to identify a highest weight

We will also prove that an irrep has exactly one highest weight vector

So for the highest weight $p=0$, $2\alpha_i\cdot\mu/\alpha_i^2 = q_i$ non-negative integer

Since α_i are complete, q_i (Dynkin indices of μ) specify μ

So can write highest weight as linear combination of Fund. weights!

Fundamental reps, SU(3), Uniqueness of highest weight 3/24/11

Highest weights are linear combinations of fundamental weights (m of them) with non-negative integer coefficients

Simplest case: Fund. weights themselves $[1,0,\cdots 0]$, $[0,1,0,\cdots 0]$, $\ldots$ $[0,0,\ldots 0,1]$

These are the m fundamental reps

Then can take tensor products to find more reps

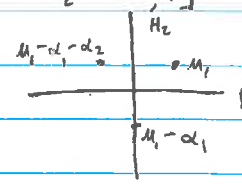
Example: SU(3) $\circ\!\!-\!\!\circ$

Choose $\alpha_1 = \frac{1}{2}(1, \sqrt{3})$ $\alpha_2 = \frac{1}{2}(1, -\sqrt{3})$ (fixes basis)

Fund. weights: $\mu_1 = \frac{1}{2}(1, \frac{1}{\sqrt{3}})$ $\mu_2 = \frac{1}{2}(1, \frac{-1}{\sqrt{3}})$

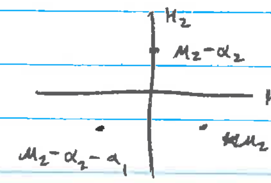
$\mu_1 = [1, 0]$ $\mu_2 = [0, 1]$ by definition

Fund. reps: $[1, 0] \xrightarrow{\alpha_1} [-1, 1] \xrightarrow{\alpha_2} [0, -1]$



For μ_1 "3"

$[0, 1] \to [1, -1] \to [-1, 0]$



For μ_2 "$\bar{3}$"

Can an irrep have more than one highest weight?

Any state in the irrep can be written $E_{\phi_1} \cdots E_{\phi_n}|\mu\rangle$ where all ϕ_i are negative. $\vec{\phi}_i = \sum_j k_{ji}(-\alpha_j)$, so all states can be written $E_{-\alpha_1} \cdots E_{-\alpha_n}|\mu\rangle$

Suppose $|\mu\rangle$ and $|\mu'\rangle$ are distinct ($\langle\mu|\mu'\rangle = 0$) highest weights

Then $\langle\mu| E_{-\alpha_1} \cdots E_{-\alpha_n}|\mu'\rangle = 0$ since $|\mu\rangle$ can't be raised

Implies that $|\mu\rangle$ and $|\mu'\rangle$ generate different invariant subspaces so that the group would not be simple

By the same argument, any state obtainable by lowering from the highest weight in a unique way is unique

$\therefore$ both fundamental irreps of SU(3) are three-dimensional

Irreps inherit symmetry from SU(2) subalgebras associated with simple roots...