Lattice supersymmetry in a nutshell
David Schaich, 28 May 2015

Goals of this informal and pedagogical introduction
• Write down supersymmetric lattice system
  (Four-dimensional $\mathcal{N} = 4$ SYM — analogous lattice systems in 2d & 3d)
• Point out connections to staggered fermions
• Raise issue of potential sign problem,
  possible connections to complex Langevin and Lefschetz thimble methods
• Main reference: arXiv:0903.4881
• Skip motivations: Take it for granted that we care about lattice susy

Supersymmetry and naive lattice obstacle
• Supersymmetries extend Poincaré spacetime symmetry
• Lorentz: Work in euclidean space $\rightarrow SO(d)_{\text{euc}}$ rotations $\Lambda_{\mu\nu}$
  “Time” arbitrary – transfer matrix can be defined along any lattice vector
• Poincaré: Add spacetime translations $P_{\mu}$ to (euclidean) Lorentz
  $[P_{\mu}, P_{\nu}] = 0$ $[P_{\mu}, \Lambda_{\rho\sigma}] \propto \delta_{\mu\rho} P_{\sigma} - \delta_{\mu\sigma} P_{\rho}$
  $[\Lambda_{\mu\nu}, \Lambda_{\rho\sigma}] \propto \delta_{\mu\rho} \Lambda_{\nu\sigma} + \delta_{\nu\sigma} \Lambda_{\mu\rho} - \delta_{\mu\sigma} \Lambda_{\nu\rho} - \delta_{\nu\rho} \Lambda_{\mu\sigma}$

• Supercharges: Spinorial generators $Q^I_{\alpha}$ and $\overline{Q}^I_{\dot{\alpha}}$ with $I = 1, \cdots, \mathcal{N}$
  Transform under global SU(\mathcal{N})_R flavor symmetry (“R symmetry”)
  $\{Q^I_{\alpha}, Q^J_{\beta}\} = 0$ $\left\{Q^I_{\alpha}, \overline{Q}^J_{\dot{\alpha}}\right\} = 2\delta^{IJ} \sigma^\mu_{\alpha\dot{\alpha}} P_\mu$
  $[Q^I_{\alpha}, P_\mu] = 0$ $[Q^I_{\alpha}, \Lambda_{\mu\nu}] \propto \frac{1}{4} [\gamma_\mu, \gamma_\nu]_{\alpha}^\beta Q^I_{\beta}$
• Lattice: $P_\mu$ generates infinitesimal spacetime translations
  $P_\mu$ does not exist in discrete spacetime $\implies$ no supersymmetry protection,
  have to fine tune (typically many) relevant or marginal operators
• Aside: Banks & Windey ’82 tried using hamiltonian formulation
  to preserve $\left\{Q^I_{\alpha}, \overline{Q}^J_{\dot{\alpha}}\right\} = 2\delta^{IJ} \sigma^0_{\alpha\dot{\alpha}} H$
  Now fine-tuning required to recover Lorentz symmetry in continuum limit
Kähler–Dirac selects $\mathcal{N} = 4$ SYM in 4d

- **Consider 4d:** Expand $4 \times 4$ matrix of 16 supercharges in basis of $\gamma$ matrices

\[
\begin{pmatrix}
Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\
\overline{Q}^1_{\dot{\alpha}} & \overline{Q}^2_{\dot{\alpha}} & \overline{Q}^3_{\dot{\alpha}} & \overline{Q}^4_{\dot{\alpha}}
\end{pmatrix} = Q + Q_\mu \gamma_\mu + Q_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{Q}_\mu \gamma_\mu \gamma_5 + \overline{Q} \gamma_5
\]

- **Observation:** Simple change of variables (in flat spacetime) that replaces spinors with anti-symmetric tensors

- **Key:** Susy subalgebra $\{Q, \overline{Q}\} = 2Q^2 = 0$ can be preserved on lattice

- **Restriction:** Need at least $2^d$ supercharges for expansion
  Only one possibility in 4d: $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM)

- **$\mathcal{N} = 4$ SYM:** Restricting to spins $\leq 1$ allows only single super-multiplet
  Contains the gauge field $A_\mu$, four fermions $\Psi^I$ and six scalars $\Phi^{IJ}$
  all in adjoint rep. of gauge group — typically $SU(N)$

<table>
<thead>
<tr>
<th>State</th>
<th>Helicity</th>
<th>Flavor SU(4)$_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\Omega_1\rangle$</td>
<td>1</td>
</tr>
<tr>
<td>$Q^I_\alpha</td>
<td>\Omega_1\rangle$</td>
<td>1/2</td>
</tr>
<tr>
<td>$Q^J_\beta Q^I_\alpha</td>
<td>\Omega_1\rangle$</td>
<td>0</td>
</tr>
<tr>
<td>$Q^K_\gamma Q^K_\beta Q^I_\alpha</td>
<td>\Omega_1\rangle$</td>
<td>$-1/2$</td>
</tr>
<tr>
<td>$Q^K_\delta Q^K_\gamma Q^I_\alpha</td>
<td>\Omega_1\rangle$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

- **Conformal:** $\beta = 0$ for all couplings (line of fixed points)

**Connection to (reduced) staggered fermions**

- **Four fermions:** Majorana $\Psi^I$ expand just like supercharges

\[
\begin{pmatrix}
\Psi^1 & \Psi^2 & \Psi^3 & \Psi^4
\end{pmatrix} \rightarrow (\eta, \psi_\mu, \chi_{\mu\nu}, \overline{\psi}_\mu, \overline{\eta})
\]

- **Observation:** Expansion mixes flavor symmetry (horizontal in matrix) and spacetime symmetry (vertical in matrix)

  Equivalent to (reduced) staggered fermions: Banks, Dothan & Horn ’82

  More formal Kähler–Dirac foundation in Rabin ’82, Becher & Joos ’82
Topological twisting

- **More formally:** Matrix gives $SO(4)_{\text{euc}} \otimes SO(4)_R$
  
  We’re expanding in reps of “twisted rotation group”
  
  \[
  SO(4)_{tw} \equiv \text{diag} \left[ SO(4)_{\text{euc}} \otimes SO(4)_R \right]
  \]

- **Complication:** Only have $SO(4)_R \subset SU(4)_R \simeq SO(6)_R$
  
  $\implies$ Scalar fields $\Phi^{IJ} \rightarrow (B_\mu, \phi, \bar{\phi})$

- **Solution:** Combine $4 + 6$ bosons in complexified $A_a = (A_\mu, \phi) + i(B_\mu, \bar{\phi})$
  
  Similarly combine $\psi_a = (\psi_\mu, \eta)$ and $\chi_{ab} = (\chi_{\mu\nu}, \bar{\psi}_\mu)$

- **Question:** Could complexified gauge field be related to [hep-lat/0301028](https://arxiv.org/abs/hep-lat/0301028) complex Langevin or Lefschetz thimble approaches to sign problem?

$A_4^*$ lattice and its $S_5$ point group symmetry

- $A_4^*$: Contains five links symmetrically spanning four dimensions
  
  Four-dimensional analog of 2d triangular lattice
  
  Can obtain from dimensional reduction with symmetric constraint $\sum_a \partial_a = 0$

- **$S_5$ point group symmetry:** $S_5$ irreps match those of $SO(4)_{tw}$
  
  Extracted by orthogonal matrix
  
  \[
P = \begin{pmatrix}
  \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
  \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & 0 & 0 \\
  \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} & -\frac{3}{\sqrt{12}} & 0 \\
  \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{12}} & \frac{\sqrt{20}}{\sqrt{12}} & -\frac{4}{\sqrt{20}} \\
  \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}}
\end{pmatrix}
\]

\[
\implies \begin{pmatrix}
  \psi_\mu \\
  \bar{\eta}
\end{pmatrix} = P \begin{pmatrix}
  \psi_a \\
  \chi_{\mu\nu}
\end{pmatrix}
\]

- **Scalar $Q$:** Nilpotent ($Q^2 = 0$), exchanges bosons $\leftrightarrow$ fermions
  
  $Q \, U_a = \psi_a$

  $Q \, \chi_{ab} = -\bar{\mathcal{F}}_{ab}$

  $Q \, \eta = d$

  $Q \, \bar{U}_a = 0$

  $Q \, \psi_a = 0$

  $Q \, \bar{U}_a = 0$

  $Q \, d = 0$

  $d$ is bosonic auxiliary field, with standard e.o.m. $d = \bar{D}_a U_a$
Sign problem

- **Phase reweighting:** Allows importance sampling Monte Carlo using real non-negative Boltzmann factor $|\text{pf } \mathcal{D}| e^{-S_B}$

$$
\langle \mathcal{O} \rangle = \frac{1}{Z} \int [d\mathcal{U}_a][d\overline{\mathcal{U}}_a][d\Psi] \mathcal{O} e^{-S_B[\mathcal{U}_a,\overline{\mathcal{U}}_a] - \Psi^T \mathcal{D}[\mathcal{U}_a,\overline{\mathcal{U}}_a] \Psi}
= \frac{1}{Z} \int [d\mathcal{U}_a][d\overline{\mathcal{U}}_a] \mathcal{O} e^{i\alpha} |\text{pf } \mathcal{D}| e^{-S_B}
= \frac{\langle \mathcal{O} e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}}
$$

- **Sign problem:** When the phase $\alpha$ fluctuates so much that $\langle e^{i\alpha} \rangle_{pq}$ is consistent with zero

- **Numerical results:** Phase fluctuations strangely sensitive to temporal BCs $e^{i\alpha} \approx 1$ with anti-periodic BCs, $\langle e^{i\alpha} \rangle_{pq} \approx 0$ with periodic BCs
Even more strangely, other observables change little for different BCs

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Lattice action

- **Twisted action:** $S$ is manifestly $\mathcal{Q}$-supersymmetric

$$
S = \frac{N}{2\lambda_{\text{lat}}} \mathcal{Q} \left( \chi_{ab} \mathcal{F}_{ab} + \eta \mathcal{D}_a \mathcal{A}_a - \frac{1}{2} \eta d \right) - \frac{N}{8\lambda_{\text{lat}}} \epsilon_{abce} \chi_{ab} \mathcal{D}_c \chi_{de}
$$

$\mathcal{Q}S = 0$ follows from $\mathcal{Q}^2 = 0$ and Bianchi identity $\epsilon_{abce} \mathcal{D}_c \mathcal{F}_{de} = 0$

- **Expand:** Apply $\mathcal{Q}$ and integrate out auxiliary field $d$:

$$
S = \frac{N}{2\lambda_{\text{lat}}} \left[ \mathcal{F}_{ab} \mathcal{F}_{ab} + \frac{1}{2} (\mathcal{D}_a \mathcal{U}_a)^2 - \chi_{ab} \mathcal{D}_{[a} \psi_{b]} - \eta \mathcal{D}_a \psi_a \right]
- \frac{N}{8\lambda_{\text{lat}}} \epsilon_{abce} \chi_{ab} \mathcal{D}_c \chi_{de}
$$
Numerical complications

- **Non-compact links live in algebra:** \( Q \mathcal{U}_a = \psi_a \implies \mathcal{U}_a \in \mathfrak{gl}(N, \mathbb{C}) \)
  Flat measure in path integral is gauge invariant due to complexification
  Need \( \mathcal{U}_a = \frac{1}{a} \mathbb{I}_N + \mathcal{A}_a + \mathcal{O}(a^2) \) in continuum limit,
  stabilized by scalar potential \( \sum_a \left( \frac{1}{N} \text{Tr} \left[ \mathcal{U}_a \bar{\mathcal{U}}_a \right] - 1 \right)^2 \)

- **U(N) gauge invariance:** Due to complexified links
  \( U(N) = \text{SU}(N) \otimes \text{U}(1) \) but \( \text{U}(1) \) only decouples in continuum

- **Flat directions:** Those in \( \text{U}(1) \) sector seem especially problematic
  Include all constant \( \text{U}(1) \) shifts of \( x \)-independent fields, even if \( S \neq 0 \)
  \( \text{SU}(N) \) flat directions restricted to supersymmetric vacua with \( S = 0 \)

- **Lifting:** Scalar potential lifts \( \text{SU}(N) \) flat directions but softly breaks susy
  Plaquette determinant lifts \( \text{U}(1) \) flat directions,
  can be implemented supersymmetrically \( (1505.03135) \)
  Modify e.o.m. for auxiliary field \( d = \overline{D}_a \mathcal{U}_a + 2 \text{GRe} \sum_{a < b} (\det \mathcal{P}_{ab} - 1) \mathbb{I}_N \)

PS: Motivations / context for lattice supersymmetry

- **BSM:** Supersymmetry most familiar as ingredient in new physics models
  Relies on (dynamical) spontaneous supersymmetry breaking \( \rightarrow \) lattice

- **Symmetries:** Simplify analytic calculations, allowing insight into
  confinement, dynamical symmetry breaking, conformality, \( \ldots \)
  Lattice is new non-perturbative method to confirm / refine / extend insights

- **Dualities:** As in spin systems (e.g., Kramers & Wannier on 2d Ising),
  theories with different fields & interactions produce identical physics
  Relate “electric” & “magnetic” gauge theories — Seiberg duality
  Relate gauge & gravity theories — AdS/CFT duality or “holography”
  Method: Conjecture & check (exploiting susy), may be extended by lattice

- **Modelling:** Attempts to study everything from QCD at finite density
  to non-Fermi liquids based on AdS/CFT holography
  Lattice could provide new input to these efforts — validate or refine