Lattice studies of three-dimensional super-Yang–Mills

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Nonperturbative and Numerical Approaches to Quantum Gravity, String Theory and Holography

International Centre for Theoretical Sciences, Bangalore, 29 August 2022


and more to come with R. G. Jha, A. Joseph, A. Sherletov & T. Wiseman
Overview and plan

Three dimensions is a promising frontier for practical lattice studies of supersymmetric QFTs

Twisted lattice super-Yang–Mills (SYM) brief review

Recent work: $Q = 16$ SYM and dual D2-branes

Ongoing work: $Q = 16$ SYM phase diagram

$Q = 8$ SYM & 2d quiver super-QCD

Interaction encouraged — complete coverage unnecessary
Motivations

Lattice field theory promises first-principles predictions for strongly coupled supersymmetric QFTs

BSM

QFT

Holography

(Derek Leinweber)

(T. Wiseman)
Motivations

Lattice field theory promises first-principles predictions for strongly coupled supersymmetric QFTs.

Three dimensions
Rich field theory and holographic dynamics

(Derek Leinweber)
More manageable computational costs

Results to be shown, and work in progress

use importance sampling to evaluate up to $\sim$crore-dimensional integrals

(\text{Dirac operator as } \sim 10^7 \times 10^7 \text{ matrix})

Significant computational resources required

Many thanks to USQCD–DOE, DiRAC–STFC–UKRI, and computing centres!
Supersymmetry must be broken on the lattice

Supersymmetry is a space-time symmetry, 
adding spinor generators $Q^I_\alpha$ and $\bar{Q}^I_{\dot{\alpha}}$ to translations, rotations, boosts 

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\delta^{IJ}\sigma^\mu_{\alpha\dot{\alpha}} P_\mu$$ broken in discrete space-time

$\longrightarrow$ relevant susy-violating operators

Scalor mass  Yukawas  Quartics  Quark mass  Gluino mass
Supersymmetry need not be *completely* broken on the lattice

Preserve susy sub-algebra in discrete lattice space-time

\[ \implies \text{correct continuum limit with little or no fine tuning} \]

Equivalent constructions from ‘topological’ twisting and dim’l deconstruction

Review:
Catterall–Kaplan–Ünsal,
arXiv:0903.4881

Need \( Q = 2^d \) supersymmetries in \( d \) dimensions

\[ d = 3 \implies \text{super-Yang–Mills (SYM) with } Q = 8 \text{ or (maximal) } Q = 16 \]
May be easiest to grok as dimensional reduction of 4d $\mathcal{N} = 4$ SYM

All fields massless and in adjoint rep. of SU($N$) gauge group

4d: Gauge field $A_\mu$ plus 6 scalars $\Phi^I$

$\mathcal{N} = 4$ four-component fermions $\psi^I \leftrightarrow 16$ supersymmetries $Q^I_\alpha$ and $\bar{Q}^I_{\dot{\alpha}}$

Global $\text{SU}(4) \sim \text{SO}(6)$ R symmetry

3d: Gauge field $A_\mu$ plus 7 scalars $\Phi$

$\mathcal{N} = 8$ two-component fermions $\psi \leftrightarrow 16$ supersymmetries

Global $\text{Spin}(7, \mathbb{C}) \sim \text{SO}(8) \supset \text{SO}(4) \sim \text{SU}(2) \times \text{SU}(2)$ R symmetry

Symmetries relate kinetic, Yukawa and $\Phi^4$ terms $\rightarrow$ single coupling $\lambda = g^2N$
Twisting 3d maximal SYM

May be easiest to grok as dim’l reduction of 4d twisted $\mathcal{N} = 4$ SYM [2002.10517]

\[
\begin{align*}
\{Q, Q_a, Q_{ab}\} &\rightarrow \{Q, Q_0, Q_\mu, Q_{0\mu}, Q_{\mu\nu}\} \\
\{\eta, \psi_a, \chi_{ab}\} &\rightarrow \{\eta, \eta_0, \psi_\mu, \psi_{0\mu}, \chi_{\mu\nu}\} \\
\{U_a, U_a\} &\rightarrow \{\phi, \phi, U_\mu, U_\mu\}
\end{align*}
\]

with $\mu, \nu = 1, \ldots, 4$

Twisted rotation group now

\[
SO(3)_{tw} \equiv \text{diag} \left[ SO(3)_{\text{euc}} \otimes SO(3)_R \right]
\]

$\text{SO}(3)_R \subset \text{SO}(4)_R$

Two closed supersymmetry sub-algebras in discrete space-time

\[
\begin{align*}
\{Q, Q\} &= 2Q^2 = 0 \\
\{Q_0, Q_0\} &= 2Q_0^2 = 0
\end{align*}
\]
Four links in three dimensions $\longrightarrow A_3^*$ lattice

$A_3^*$ (body-centered cubic) lattice
from dimensional reduction of 4d $A_4^*$ lattice

Basis vectors linearly dependent and non-orthogonal

Large $S_4$ point group symmetry
$\longrightarrow$ continuum limit without fine-tuning

Numerical calculations require regulating zero modes and flat directions and stabilizing dimensional reduction

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ICTS Bangalore, 29 August 2022
Deformations to stabilize lattice calculations

Recall soft $Q$-breaking SU($N$) scalar potential $\propto \mu^2 \sum_a \text{Tr} \left[ (U_a \bar{U}_a - I_N)^2 \right]

and supersymmetric U(1) constraint $\sim G \sum_{a<b} (\det P_{ab} - 1)$

Monitor $Q$ restoration via Ward identity violations $\langle \text{Tr} \, Q \left[ \eta U_a \bar{U}_a \right] \rangle \neq 0$
Deformations to stabilize lattice calculations

Enable naive dimensional reduction (4d code with $N_x = 1$)

Potential $\propto \mu^2 \text{Tr} \left[ (\varphi - \mathbb{I}_N)^\dagger (\varphi - \mathbb{I}_N) \right]$ to break center symmetry in reduced dir

$\rightarrow$ Kaluza–Klein reduction rather than Eguchi–Kawai
so that the full improved action becomes

\[ S_{\text{imp}} = S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \]  

\[ S'_{\text{exact}} = \frac{N}{4\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ -\mathcal{F}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}^{(+)}_{a} \psi_{b}(n) - \eta(n)\mathcal{D}^{(-)}_{a} \psi_{a}(n) \right] \]

\[ + \frac{1}{2} \left( \mathcal{D}^{(-)}_{a} \mathcal{U}_{a}(n) + G \sum_{a \neq b} (\det \mathcal{P}_{ab}(n) - 1) I_{N} \right)^{2} - S_{\text{det}} \]

\[ S_{\text{det}} = \frac{N}{4\lambda_{\text{lat}}} G \sum_n \text{Tr} [\eta(n)] \sum_{a \neq b} [\det \mathcal{P}_{ab}(n)] \text{Tr} [\mathcal{U}_{b}^{-1}(n)\psi_{b}(n) + \mathcal{U}_{a}^{-1}(n + \mu_{b})\psi_{a}(n + \mu_{b})] \]

\[ S_{\text{closed}} = -\frac{N}{16\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ \epsilon_{abcde} \chi_{de}(n + \mu_{a} + \mu_{b} + \mu_{c})\mathcal{D}^{(-)}_{c} \chi_{ab}(n) \right] \]

\[ S'_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}} \mu^{2} \sum_{n} \sum_{a} \left( \frac{1}{N} \text{Tr} [\mathcal{U}_{a}(n)\overline{\mathcal{U}}_{a}(n)] - 1 \right)^{2} \]

\[ \geq 100 \text{ inter-node data transfers in the fermion operator} \quad \text{— non-trivial} \]
3d maximal SYM thermodynamics

Formulate on $r_1 \times r_2 \times r_\beta$ (skewed) 3-torus

Thermal boundary conditions

$\rightarrow$ dimensionless temperature $t = \frac{T}{\lambda} = \frac{1}{r_\beta}$

Low temperatures $t$ at large $N$

Black branes in dual supergravity

(T. Wiseman)
Holographic expectations for 3d maximal SYM

Rich holographic phase diagram, especially when $r_1 \neq r_2$

Left: $r_1 = r_2$

Right: $r_1 = \infty$, $r_2 = L\lambda$

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Holographic expectations for 3d maximal SYM

Rich holographic phase diagram, especially when $r_1 \neq r_2$

First consider simplest homogeneous black D2-branes $\rightarrow r_1 = r_2 = r_\beta$

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3d lattice SYM
arXiv:1412.3939
Homogeneous D2 phase

Homogeneous D2-branes $\leftrightarrow$ uniform Wilson line eigenvalue phases at large $N$

Phase of unitarized Wilson line eigenvalues, $L=12$, $t=0.31$

- $N=4$
- $N=6$
- $N=8$

Relative frequency

$P_L \neq 0$

$P_L = 0$

D0 gravity

D2 gravity

2d pure glue

3d lattice SYM
Holographic black brane energies and continuum extrapolation

Lattice volume $L^3$ with gauge group $U(8)$

\[ \rightarrow \text{results approach leading holographic expectation } \propto t^{10/3} \text{ for low } t \lesssim 0.4 \]

Carry out first 3d continuum extrapolations, $L \to \infty$ with fixed $t = 1/(L\lambda_{\text{lat}})$
Aside: No sign problem for 3d maximal SYM

Continuum limit $L \to \infty$ with fixed $t = 1/(L\lambda_{\text{lat}}) \implies \lambda_{\text{lat}} \to 0$

Pfaffian nearly real positive for $\lambda_{\text{lat}} \leq 1$ on small volumes $\implies$ no sign problem

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**Preliminary**

3d $Q = 16$ SYM, U(4)

- $\lambda_{\text{lat}} = 0.25$
- $\lambda_{\text{lat}} = 0.5$
- $\lambda_{\text{lat}} = 1.0$

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Next step: Exploring the 3d $Q = 16$ SYM phase diagram

Work in progress to investigate D2–D0 transition with $r_L = r_1 = r_2$ → scan in $r_\beta$ fixing aspect ratio $\alpha = r_L/r_\beta$

For decreasing $r_L$ at large $N$

homogeneous black D2 brane → localized D0 black hole

“spatial deconfinement” signalled by Wilson line $P_L$
Preliminary U(8) results for $8^3$ vs. $12^3$ vs. $16^3$ lattices (aspect ratio $\alpha = 1$)

Still locating peaks in Wilson line susceptibility and checking hysteresis
Work in progress: 3d $Q = 8$ SYM

**Simpler [Blau–Thompson] twisted formulation**

$Q = 8$ supercharges $\{Q, Q_a, Q_{ab}, Q_{abc}\}$ with $a, b = 1, \cdots, 3$

$\rightarrow$ site / link / plaquette / cube fermions $\{\eta, \psi_a, \chi_{ab}, \theta_{abc}\}$ on simple cubic lattice

Diagram showing fermions on a simple cubic lattice with labels $\eta(n), \psi_1(n), \psi_2(n), \psi_3(n), \chi_{12}(n), \chi_{13}(n), \chi_{23}(n), \theta_{123}(n)$.
Work in progress: 3d $Q = 8$ SYM

Simpler [Blau–Thompson] twisted formulation

$Q = 8$ supercharges $\{Q, Q_a, Q_{ab}, Q_{abc}\}$ with $a, b = 1, \ldots, 3$

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Parallel code developed

( Angel Sherletov )

Tests passed

$\rightarrow$ larger-scale calculations [mirror symmetry?]

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[mirror symmetry?]
2-slice lattice SYM
with $U(N) \times U(F)$ gauge group
Adj. fields on each slice
Bi-fundamental in between

Decouple $U(F)$ slice
$\rightarrow U(N)$ SQCD in $d-1$ dims.
with $F$ fund. hypermultiplets
Recap: An exciting time for lattice supersymmetry

Three dimensions is a promising frontier for practical lattice studies of supersymmetric QFTs.

Preserving susy sub-algebra enables lattice calculations, public code available.

3d $Q = 16$ SYM thermodynamics consistent with holography, work in progress on phase diagram.

Work in progress on 3d $Q = 8$ SYM $\rightarrow$ 2d superQCD and much more for the future.
Thanks for your attention! Any further questions?

Collaborators
Simon Catterall, Joel Giedt, Raghav Jha,
Anosh Joseph, Angel Sherletov, Toby Wiseman

Funding and computing resources

UK Research and Innovation

USQCD
Lattice theory looks nearly the same despite breaking $Q_a$ and $Q_{ab}$

Covariant derivatives $\rightarrow$ finite difference operators

Complexified gauge fields $\{A_a, \bar{A}_a\} \rightarrow$ gauge links $\{U_a, \bar{U}_a\} \in gl(N, \mathbb{C})$

with gauge-invariant flat measure $DU D\bar{U}$

Need $U_a \rightarrow I_N + A_a$ to recover continuum covariant derivative

$\checkmark$ $Q$ interchanges bosonic $\leftrightarrow$ fermionic d.o.f. with $Q^2 = 0$

$Q A_a \rightarrow Q U_a = \psi_a$

$Q \chi_{ab} = -\bar{F}_{ab}$

$Q \eta = d$

$Q \bar{A}_a \rightarrow Q \bar{U}_a = 0$

$Q \psi_a = 0$

$Q d = 0$
Backup: Sign problems

Recall typical algorithms sample field configurations $\Phi$ with probability $\frac{1}{Z} e^{-S[\Phi]}$ → “sign problem” if action $S[\Phi]$ can be negative or complex

Lattice SYM has complex pfaffian $\text{pf } D = |\text{pf } D| e^{i\alpha}$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int [dU][d\bar{U}] \mathcal{O} \ e^{-S_{B[U,\bar{U}]} \ \text{pf } D[U,\bar{U}]}$$

We phase quench $\text{pf } D \longrightarrow |\text{pf } D|$, need to reweight $\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}}$

$$\longrightarrow \langle e^{i\alpha} \rangle_{pq} = \frac{Z}{Z_{pq}} \quad \text{quantifies severity of sign problem}$$
Dimensionally reduce to 2d $\mathcal{N} = (8, 8)$ SYM on $(r_L \times r_\beta)$ torus with four scalar $Q$.

Low temperatures $t = 1/r_\beta \longleftrightarrow$ black holes in dual supergravity.

For decreasing $r_L$ at large $N$

homogeneous black string (D1) $\longrightarrow$ localized black hole (D0)

“spatial deconfinement” signalled by Wilson line $P_L$
Backup: Dimensional reduction to 2d $\mathcal{N} = (8, 8)$ SYM

Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_x = N_y = 1$

$A_4^* \longrightarrow A_2^*$ (triangular) lattice

Torus **skewed** depending on $\alpha = L/N_t$

Modular transformation into fundamental domain

$\longrightarrow$ some skewed tori actually rectangular

Again need to stabilize compactified links
to ensure broken center symmetries
Peaks in Wilson line susceptibility match change in its magnitude $|PL|$, grow with size of SU($N$) gauge group, comparing $N = 6, 9, 12$

Agreement for $16 \times 4$ vs. $24 \times 6$ lattices (aspect ratio $\alpha = r_L/r_\beta = 4$)
Backup: 2d Wilson line eigenvalues

Large-$N$ eigenvalue phase distribution also signals spatial deconfinement

**Left:** $\alpha = 1/2$ distributions more localized as $N$ increases $\rightarrow$ D0 black hole

**Right:** $\alpha = 2$ distributions more uniform as $N$ increases $\rightarrow$ D1 black string
Backup: Lattice results for 2d $\mathcal{N} = (8, 8)$ SYM phase diagram

Good agreement with bosonic QM at high temperatures ($\alpha \gtrsim 4$)

Harder to control low-temperature uncertainties (larger $N > 16$ should help)

Overall consistent with holography

Comparing multiple lattice sizes and $6 \leq N \leq 16$

Controlled extrapolations are work in progress

arXiv:1709.07025
Backup: 2d holographic black hole energies

Lattice results consistent with leading expectation for sufficiently low $t \lesssim 0.4$

Similar behavior $\rightarrow$ difficult to distinguish phases

$\propto t^{3.2}$ for small-$r_L$ D0 phase

$\propto t^3$ for large-$r_L$ D1 phase
Much simpler twisted formulation: $Q = 4$ supercharges $\{Q, Q_a, Q_{ab}\}$

$\rightarrow$ site / link / plaquette fermions $\{\eta, \psi_a, \chi_{ab}\}$ on square lattice $(a, b = 1, 2)$

Work by Navdeep Singh Dhindsa

Prelim. $\mu^2 \rightarrow 0$ extrapolations

for $r_L = r_\beta \leftrightarrow \alpha = 1$

Energy independent of $t \lesssim 0.33$

vs. $\sim t^3$ for $\mathcal{N} = (8, 8)$ SYM
Backup: High-temperature \((t \gtrsim 1)\) 3d maximal SYM

Wilson line eigenvalue phases localized rather than uniform (left)

Thermodynamics consistent with weak-coupling expectation \(\propto t^3\) (right)