

# Numerical methods in lattice supersymmetry

David Schaich (University of Liverpool)



Numerical Methods in Theoretical Physics

Asia Pacific Center for Theoretical Physics, Pohang, 18 May 2022

[arXiv:1810.09282](https://arxiv.org/abs/1810.09282)

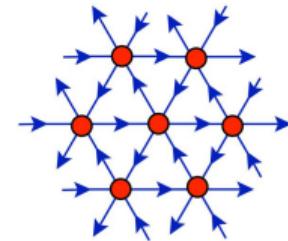
[arXiv:2010.00026](https://arxiv.org/abs/2010.00026)

[arXiv:2201.08626](https://arxiv.org/abs/2201.08626)

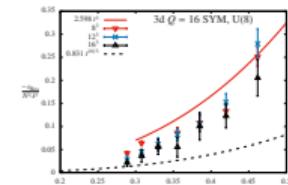
and more to come with R. G. Jha, A. Joseph, A. Sherletov & T. Wiseman

# Overview and plan

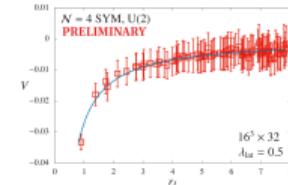
Preserve (some) supersymmetry in discrete space-time  
→ practical numerical investigations



**Why:** Lattice supersymmetry motivation



**How:** Lattice formulation highlights

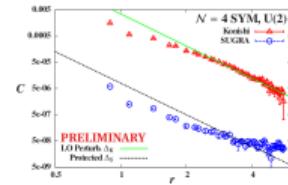


**What:** Recent, ongoing & planned work

Thermodynamics in 1+1 and 2+1 dimensions

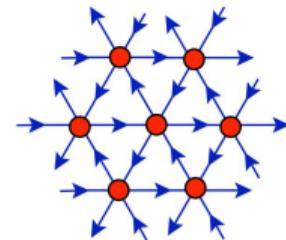
(3+1)d static potential and scaling dimensions

Sign problems, supersymmetric QCD, ...

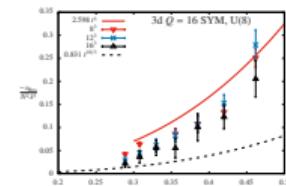


# Overview and plan

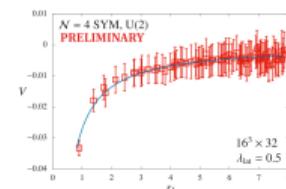
Preserve (some) supersymmetry in discrete space-time  
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**Why:** Lattice supersymmetry motivation



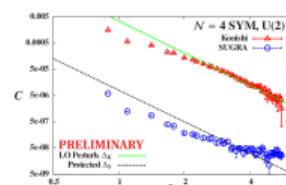
**How:** Lattice formulation highlights



**What:** Recent, ongoing & planned work

These slides: [davidschaich.net/talks/2205APCTP.pdf](http://davidschaich.net/talks/2205APCTP.pdf)

Interaction encouraged — complete coverage unnecessary



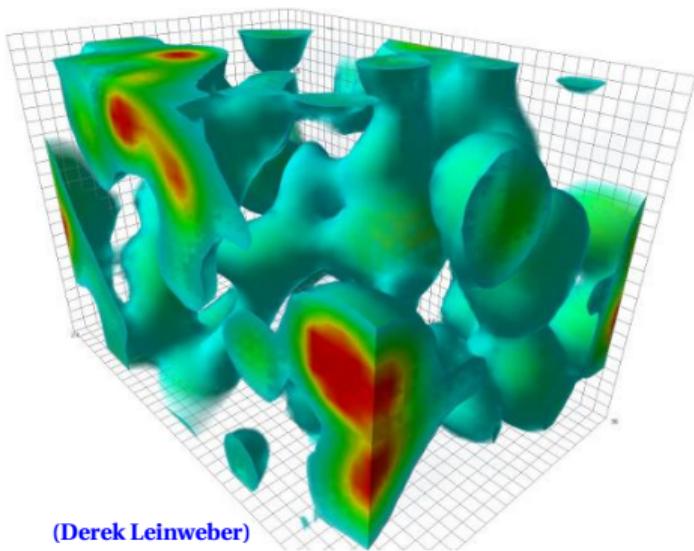
# Motivations

Lattice field theory promises first-principles predictions  
for strongly coupled supersymmetric QFTs

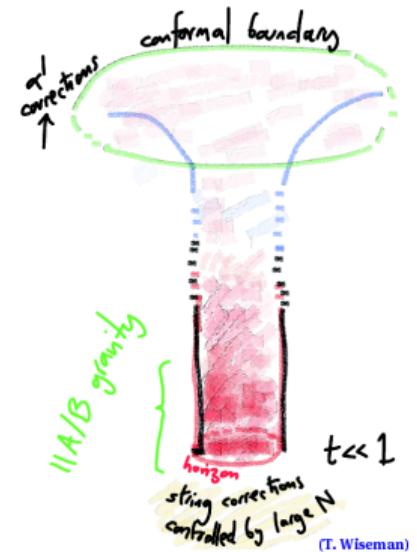
**BSM**



**QFT**



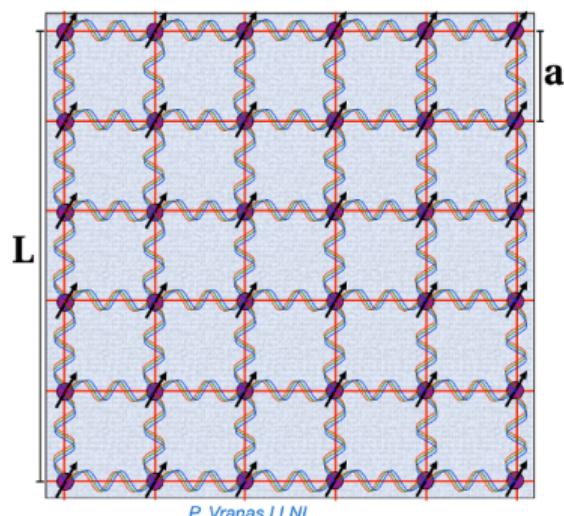
**Holography**



# Lattice regularization of quantum field theories

Formally  $\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \ \mathcal{O}(\Phi) \ e^{-S[\Phi]}$

Regularize by formulating theory in finite, discrete, euclidean space-time  
↗ Gauge invariant, non-perturbative,  $d$ -dimensional



Spacing between lattice sites (" $a$ ")  
→ UV cutoff scale  $1/a$

Remove cutoff:  $a \rightarrow 0$  ( $L/a \rightarrow \infty$ )

Discrete → continuous symmetries ✓

# Numerical lattice field theory calculations

High-performance computing → evaluate up to  $\sim$ billion-dimensional integrals  
(Dirac operator as  $\sim 10^9 \times 10^9$  matrix)

Results to be shown, and work in progress, require state-of-the-art resources

Many thanks to USQCD–DOE, DiRAC–STFC–UKRI, and computing centres!



USQCD @Fermilab



DiRAC @Cambridge



Barkla @Liverpool

# Numerical lattice field theory calculations



USQCD @Fermilab



DiRAC @Cambridge



Barkla @Liverpool

## Importance sampling Monte Carlo

Algorithms sample field configurations with probability  $\frac{1}{\mathcal{Z}} e^{-S[\Phi]}$

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \ \mathcal{O}(\Phi) \ e^{-S[\Phi]} \longrightarrow \frac{1}{N} \sum_{i=1}^N \mathcal{O}(\Phi_i) \text{ with stat. uncertainty } \propto \frac{1}{\sqrt{N}}$$

# Supersymmetry must be broken on the lattice

Supersymmetry is a space-time symmetry,

$(I = 1, \dots, N)$

adding spinor generators  $Q_\alpha^I$  and  $\bar{Q}_{\dot{\alpha}}^I$  to translations, rotations, boosts

$$\{ Q_\alpha^I, \bar{Q}_{\dot{\alpha}}^J \} = 2\delta^{IJ}\sigma_{\alpha\dot{\alpha}}^\mu P_\mu \quad \text{broken in discrete space-time}$$

→ relevant susy-violating operators

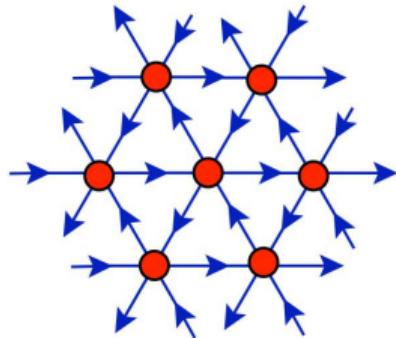


Supersymmetry need not be *completely* broken on the lattice

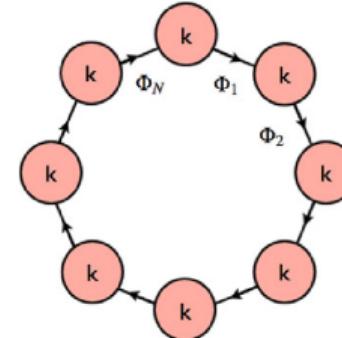
Preserve susy sub-algebra in discrete lattice space-time

⇒ correct continuum limit with little or no fine tuning

Equivalent constructions from ‘topological’ twisting and dim'l deconstruction



Review:  
Catterall–Kaplan–Ünsal,  
[arXiv:0903.4881](https://arxiv.org/abs/0903.4881)



Need  $2^d$  supersymmetries in  $d$  dimensions

$d = 4 \rightarrow \mathcal{N} = 4$  super-Yang–Mills (SYM)

# $\mathcal{N} = 4$ SYM in a nutshell

Arguably simplest non-trivial 4d QFT  $\rightarrow$  dualities, amplitudes, ...

SU( $N$ ) gauge theory with  $\mathcal{N} = 4$  fermions  $\Psi^I$  and 6 scalars  $\Phi^{IJ}$ ,  
all massless and in adjoint rep.

**Symmetries** relate coefficients of kinetic, Yukawa and  $\Phi^4$  terms

Maximal 16 supersymmetries  $Q_\alpha^I$  and  $\bar{Q}_{\dot{\alpha}}^I$   $I = 1, \dots, 4$   
transform under global  $SU(4) \sim SO(6)$  **R symmetry**

Conformal  $\rightarrow$   $\beta$  function is zero for all values of  $\lambda = g^2 N$

# Twisting $\mathcal{N} = 4$ SYM

Intuitive 4d picture — expand  $4 \times 4$  matrix of supersymmetries

$$\begin{pmatrix} Q_\alpha^1 & Q_\alpha^2 & Q_\alpha^3 & Q_\alpha^4 \\ \bar{Q}_{\dot{\alpha}}^1 & \bar{Q}_{\dot{\alpha}}^2 & \bar{Q}_{\dot{\alpha}}^3 & \bar{Q}_{\dot{\alpha}}^4 \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \bar{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \bar{\mathcal{Q}} \gamma_5$$
$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$

with  $a, b = 1, \dots, 5$

Lorentz index  $\times$  R-symmetry index  $\implies$  reps of ‘twisted rotation group’

$$\mathrm{SO}(4)_{\mathrm{tw}} \equiv \mathrm{diag} \left[ \mathrm{SO}(4)_{\mathrm{euc}} \otimes \mathrm{SO}(4)_R \right] \quad \mathrm{SO}(4)_R \subset \mathrm{SO}(6)_R$$

Change of variables  $\longrightarrow$   $\mathcal{Q}$ s transform with integer ‘spin’ under  $\mathrm{SO}(4)_{\mathrm{tw}}$

# Twisting $\mathcal{N} = 4$ SYM

Intuitive 4d picture — expand  $4 \times 4$  matrix of supersymmetries

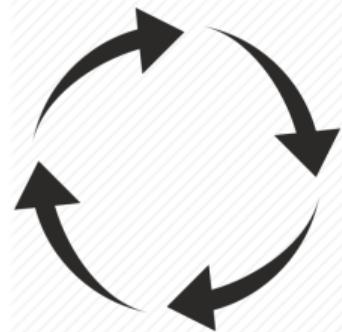
$$\begin{pmatrix} Q_\alpha^1 & Q_\alpha^2 & Q_\alpha^3 & Q_\alpha^4 \\ \bar{Q}_{\dot{\alpha}}^1 & \bar{Q}_{\dot{\alpha}}^2 & \bar{Q}_{\dot{\alpha}}^3 & \bar{Q}_{\dot{\alpha}}^4 \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \bar{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \bar{\mathcal{Q}} \gamma_5$$
$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$

with  $a, b = 1, \dots, 5$

Discrete space-time

Can preserve closed sub-algebra

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$$



# Twisting $\mathcal{N} = 4$ SYM

Intuitive 4d picture — expand  $4 \times 4$  matrix of supersymmetries

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with  $a, b = 1, \dots, 5$

Discrete space-time

Can preserve closed sub-algebra



$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$$

## Completing the twist

Fields also transform with integer spin under  $\text{SO}(4)_{\text{tw}}$  — no spinors

$\psi$  and  $\bar{\psi}$   $\rightarrow \eta, \psi_a$  and  $\chi_{ab}$

$A_\mu$  and  $\phi^I$   $\rightarrow$  complexified gauge field  $\mathcal{A}_a$  and  $\bar{\mathcal{A}}_a$   
 $\rightarrow U(N) = SU(N) \otimes U(1)$  gauge theory

✓  $\mathcal{Q}$  interchanges bosonic  $\longleftrightarrow$  fermionic d.o.f. with  $\mathcal{Q}^2 = 0$

$$\mathcal{Q} A_a = \psi_a$$

$$\mathcal{Q} \psi_a = 0$$

$$\mathcal{Q} \chi_{ab} = -\bar{\mathcal{F}}_{ab}$$

$$\mathcal{Q} \bar{\mathcal{A}}_a = 0$$

$$\mathcal{Q} \eta = d$$

$$\mathcal{Q} d = 0$$

↗ bosonic auxiliary field with e.o.m.  $d = \bar{\mathcal{D}}_a \mathcal{A}_a$

## Lattice $\mathcal{N} = 4$ SYM

Lattice theory looks nearly the same despite breaking  $\mathcal{Q}_a$  and  $\mathcal{Q}_{ab}$

Covariant derivatives  $\rightarrow$  finite difference operators

Complexified gauge fields  $\mathcal{A}_a \rightarrow$  gauge links  $\mathcal{U}_a \in \mathfrak{gl}(N, \mathbb{C})$

$$\mathcal{Q} \mathcal{A}_a \rightarrow \mathcal{Q} \mathcal{U}_a = \psi_a \quad \mathcal{Q} \psi_a = 0$$

$$\mathcal{Q} \chi_{ab} = -\bar{\mathcal{F}}_{ab} \quad \mathcal{Q} \bar{\mathcal{A}}_a \rightarrow \mathcal{Q} \bar{\mathcal{U}}_a = 0$$

$$\mathcal{Q} \eta = d \quad \mathcal{Q} d = 0$$

**Geometry:**  $\eta$  on sites,  $\psi_a$  on links, etc.

Supersymmetric lattice action ( $\mathcal{Q}S = 0$ ) from  $\mathcal{Q}^2 \cdot = 0$  and **Bianchi identity**

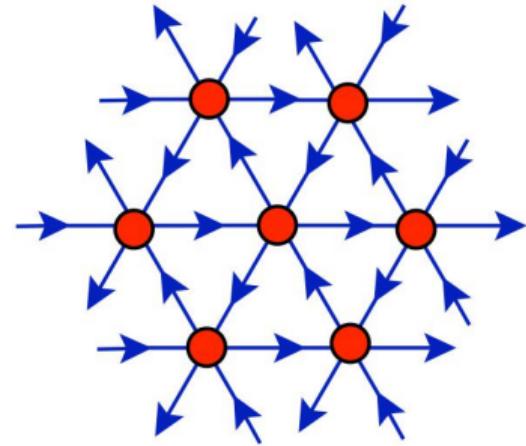
$$S = \frac{N}{4\lambda_{\text{lat}}} \text{Tr} \left[ \mathcal{Q} \left( \chi_{ab} \mathcal{F}_{ab} + \eta \bar{\mathcal{D}}_a \mathcal{U}_a - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon^{abcde} \chi_{ab} \bar{\mathcal{D}}_c \chi_{de} \right]$$

Five links in four dimensions  $\rightarrow A_4^*$  lattice

$A_4^*$   $\sim$  4d analog of 2d triangular lattice

Basis vectors linearly dependent and non-orthogonal

Large  $S_5$  point group symmetry



$S_5$  irreps precisely match onto irreps of twisted  $\text{SO}(4)_{tw}$

$$\psi_a \rightarrow \psi_\mu, \bar{\eta} \quad \text{is} \quad \mathbf{5} \rightarrow \mathbf{4} \oplus \mathbf{1}$$

$$\chi_{ab} \rightarrow \chi_{\mu\nu}, \bar{\psi}_\mu \quad \text{is} \quad \mathbf{10} \rightarrow \mathbf{6} \oplus \mathbf{4}$$

$S_5 \rightarrow \text{SO}(4)_{tw}$  in continuum limit restores  $\mathcal{Q}_a$  and  $\mathcal{Q}_{ab}$

## Checkpoint

Analytic results for twisted  $\mathcal{N} = 4$  SYM on  $A_4^*$  lattice

$U(N)$  gauge invariance +  $\mathcal{Q}$  +  $S_5$  lattice symmetries

→ Moduli space preserved to all orders

→ One-loop lattice  $\beta$  function vanishes

→ Only one log. tuning to recover continuum  $\mathcal{Q}_a$  and  $\mathcal{Q}_{ab}$

[arXiv:1102.1725, arXiv:1306.3891, arXiv:1408.7067]

Not yet suitable for numerical calculations

Must regulate zero modes and flat directions, especially in  $U(1)$  sector

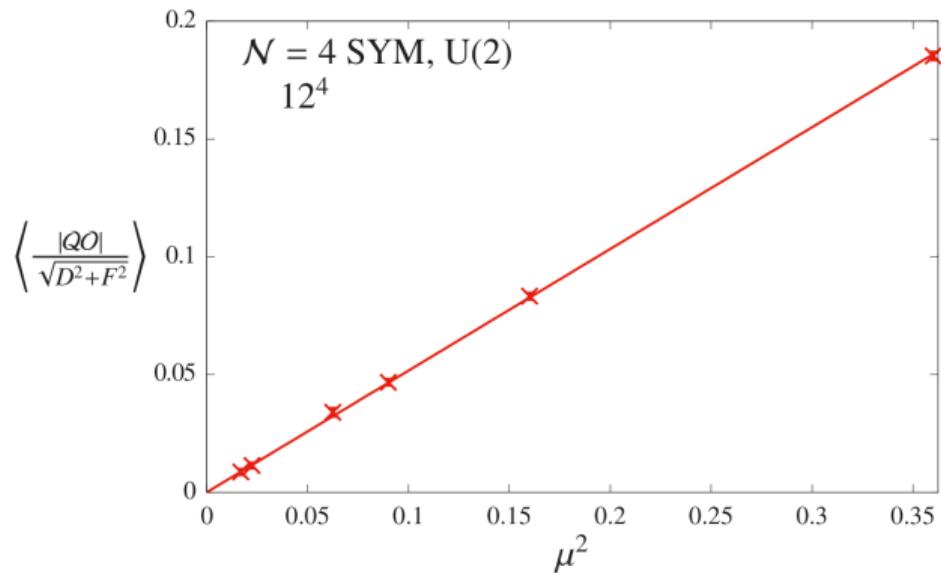
## Two deformations stabilize lattice calculations

1) Add SU( $N$ ) scalar potential  $\propto \mu^2 \sum_a (\text{Tr} [\mathcal{U}_a \bar{\mathcal{U}}_a] - N)^2$

**Softly** breaks susy  $\rightarrow \mathcal{Q}$ -violating operators vanish  $\propto \mu^2 \rightarrow 0$

Test via Ward identity violations

$$\mathcal{Q} [\eta \mathcal{U}_a \bar{\mathcal{U}}_a] \neq 0$$



# Two deformations stabilize lattice calculations

2) Constrain U(1) plaquette determinant  $\sim G \sum_{a < b} (\det \mathcal{P}_{ab} - 1)$

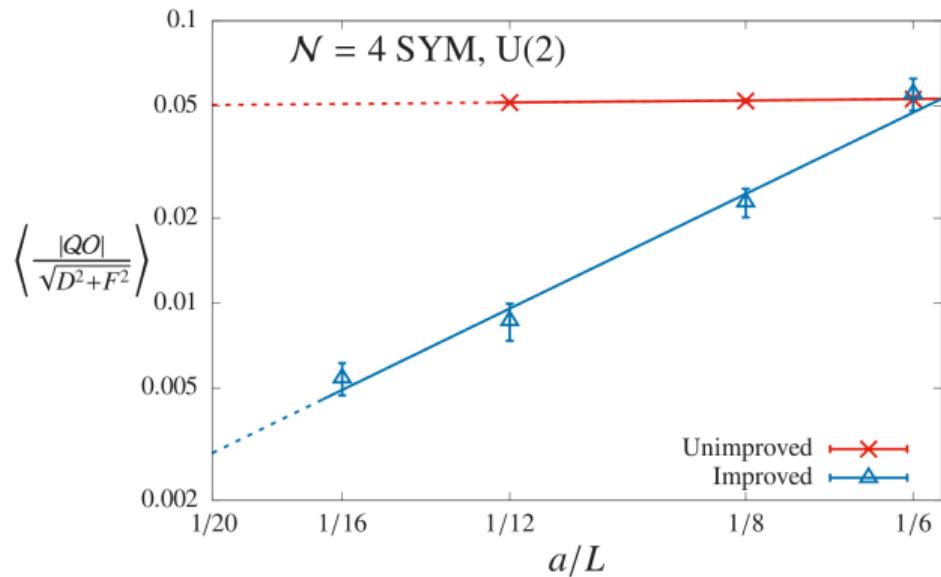
Implemented supersymmetrically by modifying auxiliary field equations of motion

Test via Ward identity violations

$$Q [\eta \mathcal{U}_a \bar{\mathcal{U}}_a] \neq 0$$

Log-log axes

$$\rightarrow \text{violations} \propto (a/L)^2$$



# Public code for supersymmetric lattice field theories

so that the full improved action becomes

$$S_{\text{imp}} = S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \quad (18)$$

$$\begin{aligned} S'_{\text{exact}} &= \frac{N}{4\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ -\bar{\mathcal{F}}_{ab}(n) \mathcal{F}_{ab}(n) - \chi_{ab}(n) \mathcal{D}_{[a}^{(+)} \psi_{b]}(n) - \eta(n) \bar{\mathcal{D}}_a^{(-)} \psi_a(n) \right. \\ &\quad \left. + \frac{1}{2} \left( \bar{\mathcal{D}}_a^{(-)} \mathcal{U}_a(n) + G \sum_{a \neq b} (\det \mathcal{P}_{ab}(n) - 1) \mathbb{I}_N \right)^2 \right] - S_{\text{det}} \end{aligned}$$

$$S_{\text{det}} = \frac{N}{4\lambda_{\text{lat}}} G \sum_n \text{Tr} [\eta(n)] \sum_{a \neq b} [\det \mathcal{P}_{ab}(n)] \text{Tr} [\mathcal{U}_b^{-1}(n) \psi_b(n) + \mathcal{U}_a^{-1}(n + \hat{\mu}_b) \psi_a(n + \hat{\mu}_b)]$$

$$S_{\text{closed}} = -\frac{N}{16\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ \epsilon_{abcde} \chi_{de}(n + \hat{\mu}_a + \hat{\mu}_b + \hat{\mu}_c) \bar{\mathcal{D}}_c^{(-)} \chi_{ab}(n) \right],$$

$$S'_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}} \mu^2 \sum_n \sum_a \left( \frac{1}{N} \text{Tr} [\mathcal{U}_a(n) \bar{\mathcal{U}}_a(n)] - 1 \right)^2$$

$\gtrsim 100$  inter-node data transfers in the fermion operator — non-trivial...

Public parallel code to reduce barriers to entry: [github.com/daschaich/susy](https://github.com/daschaich/susy)

Evolved from MILC QCD code, user guide in [arXiv:1410.6971](https://arxiv.org/abs/1410.6971)

Naive dimensional reduction  $\rightarrow$  skewed tori

$r_L \times r_\beta$  with  $r_\beta = \sqrt{\lambda}/T$  and four scalar  $\mathcal{Q}$

$r_1 \times r_2 \times r_\beta$  with  $r_\beta = \lambda/T$  and two scalar  $\mathcal{Q}$

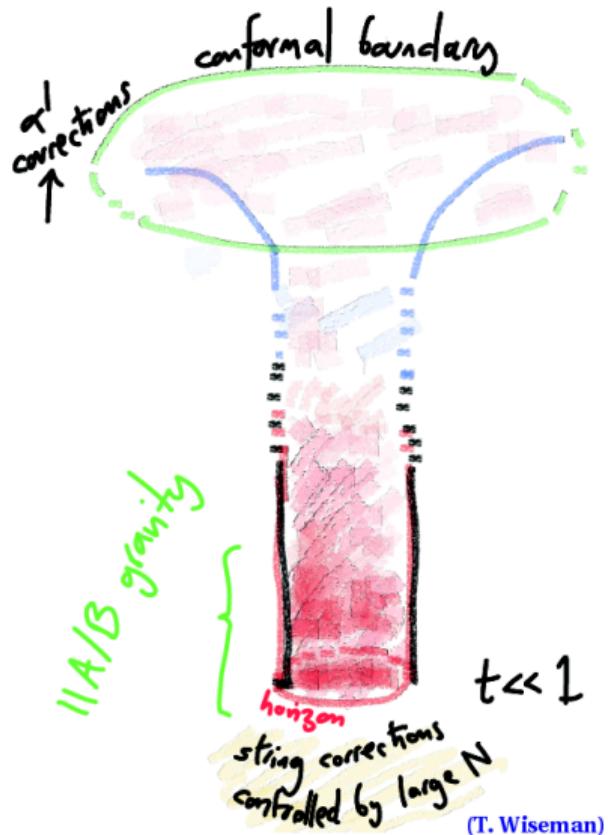
Thermal boundary conditions

$\rightarrow$  dimensionless temperature  $t = 1/r_\beta$

Low temperatures  $t$  at large  $N$



Black branes in dual supergravity



# 2d $\mathcal{N} = (8, 8)$ SYM phase diagram

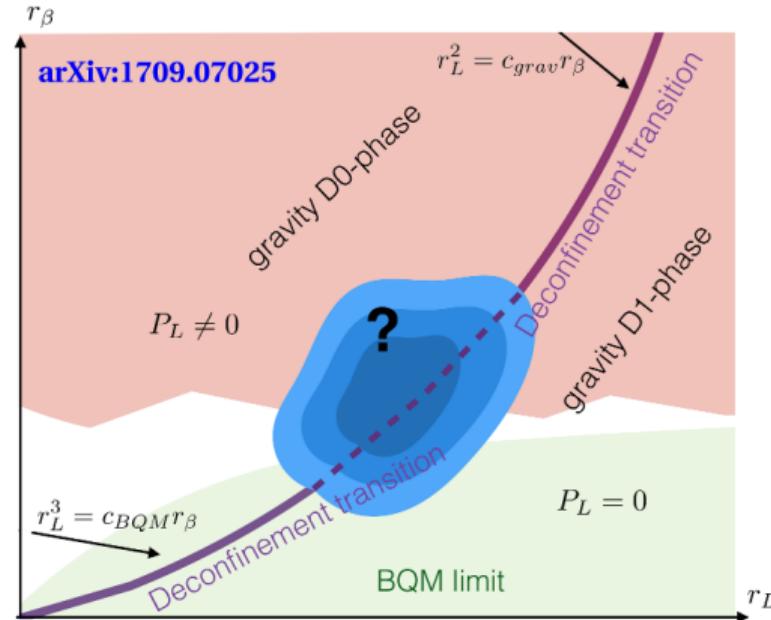
First-order transitions predicted from bosonic QM at high  $t$  ( $r_\beta \ll 1$ )  
from holography at low  $t$  ( $r_\beta \gg 1$ )

For decreasing  $r_L$  at large  $N$

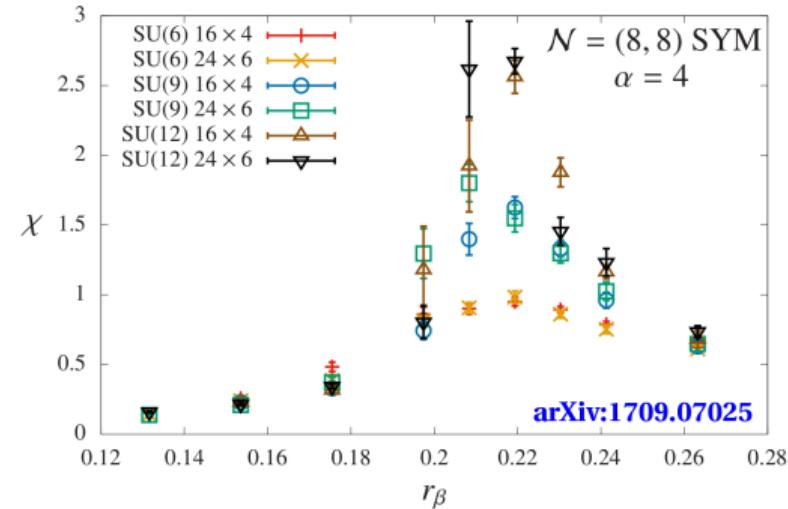
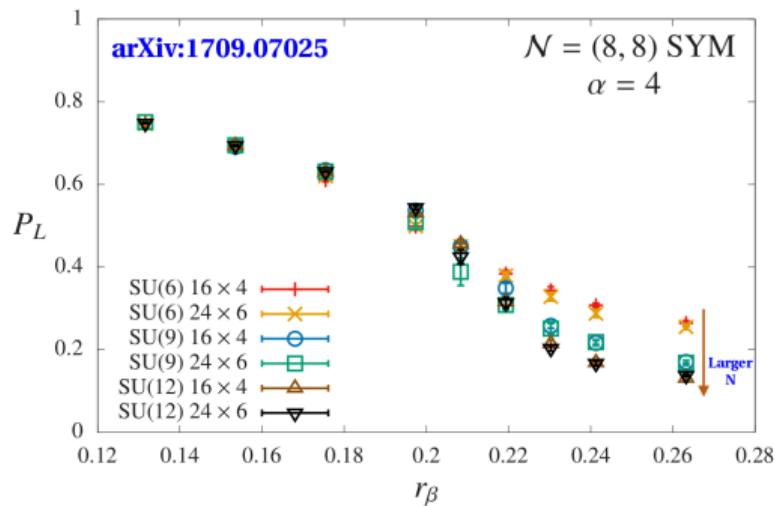
homogeneous black string (D1)  
→ localized black hole (D0)



“spatial deconfinement”  
signalled by Wilson line  $P_L$



# Spatial deconfinement transition signals — high- $t$ example



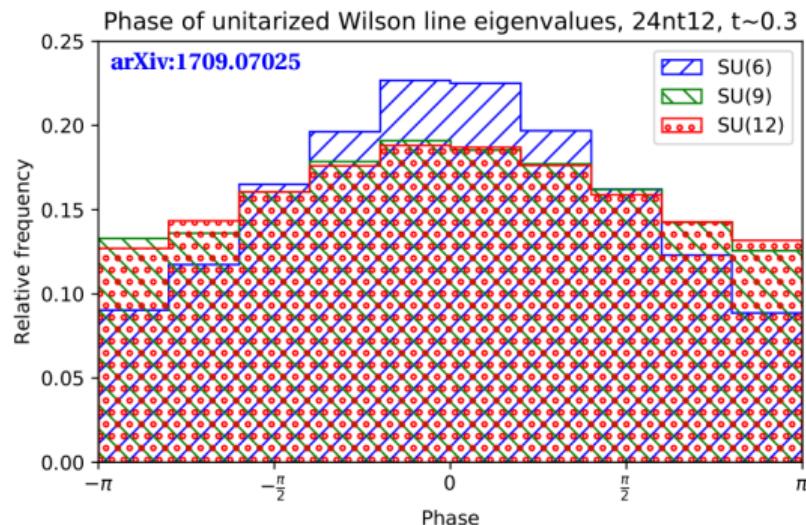
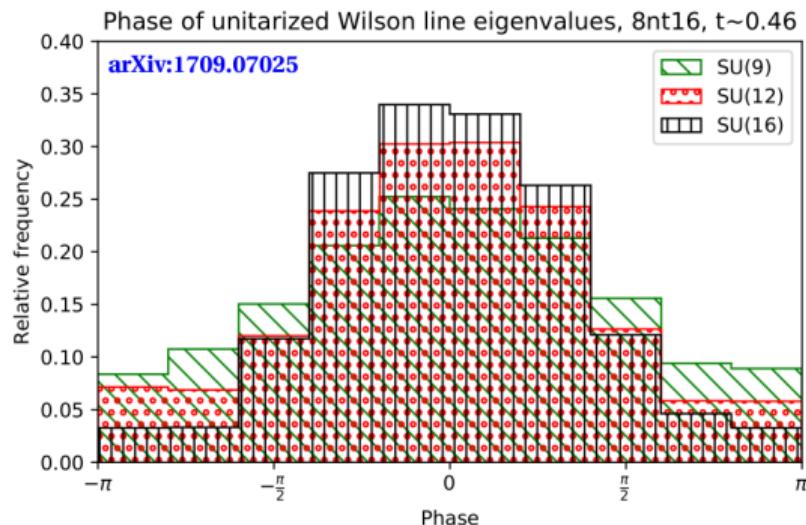
Fix aspect ratio  $\alpha = r_L/r_\beta = 4$

Check  $16 \times 4$  vs.  $24 \times 6$  lattices agree

Peaks in  $\text{Tr } P_L$  susceptibility match change in its magnitude,  
grow with size of  $U(N)$  gauge group, comparing  $N = 6, 9, 12$

# Wilson line eigenvalues for low $t$

Large- $N$  eigenvalue phase distribution also signals spatial deconfinement



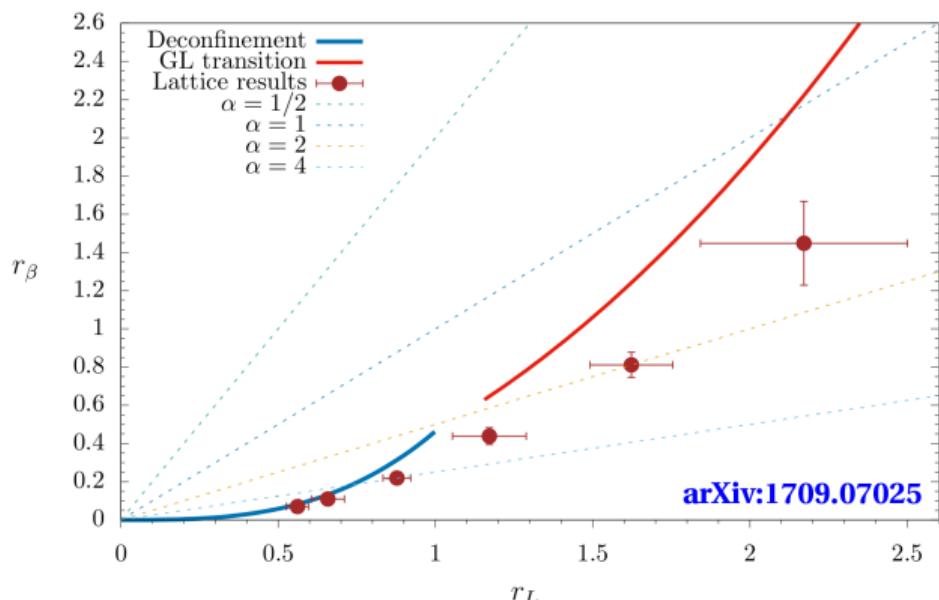
**Left:**  $\alpha = 1/2$  distributions more localized as  $N$  increases  $\rightarrow$  D0 black hole

**Right:**  $\alpha = 2$  distributions more uniform as  $N$  increases  $\rightarrow$  D1 black string

# Lattice results for 2d $\mathcal{N} = (8, 8)$ SYM phase diagram

Good agreement with bosonic QM at high temperatures ( $\alpha \gtrsim 4$ )

Harder to control low-temperature uncertainties (larger  $N > 16$  should help)



Overall consistent with holography

Comparing multiple lattice sizes  
and  $6 \leq N \leq 16$

Controlled extrapolations  
not yet attempted in 2d

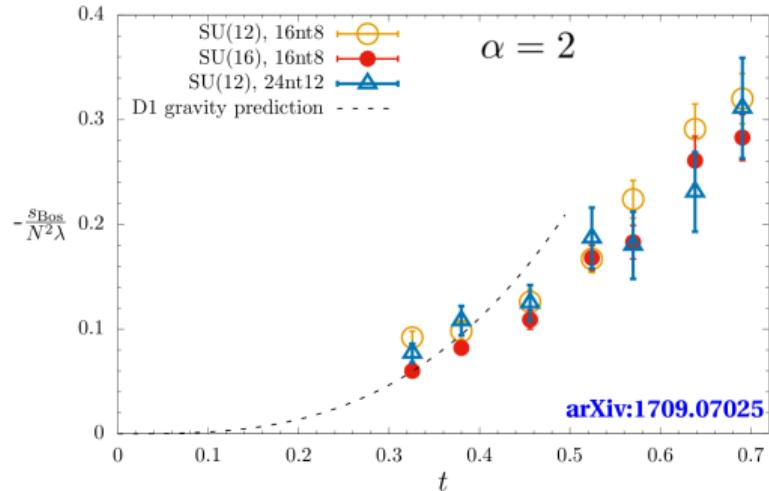
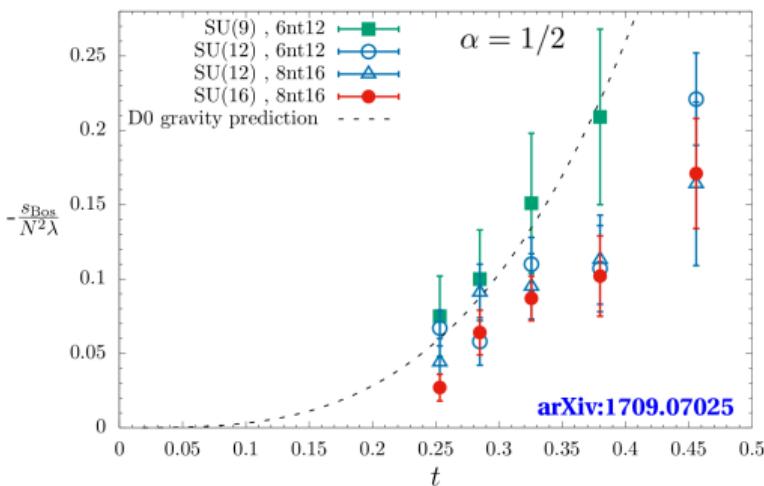
# Holographic black hole energies

Lattice results consistent with leading expectation for sufficiently low  $t \lesssim 0.4$

Similar behavior  $\rightarrow$  difficult to distinguish phases

$\propto t^{3.2}$  for small- $r_L$  D0 phase

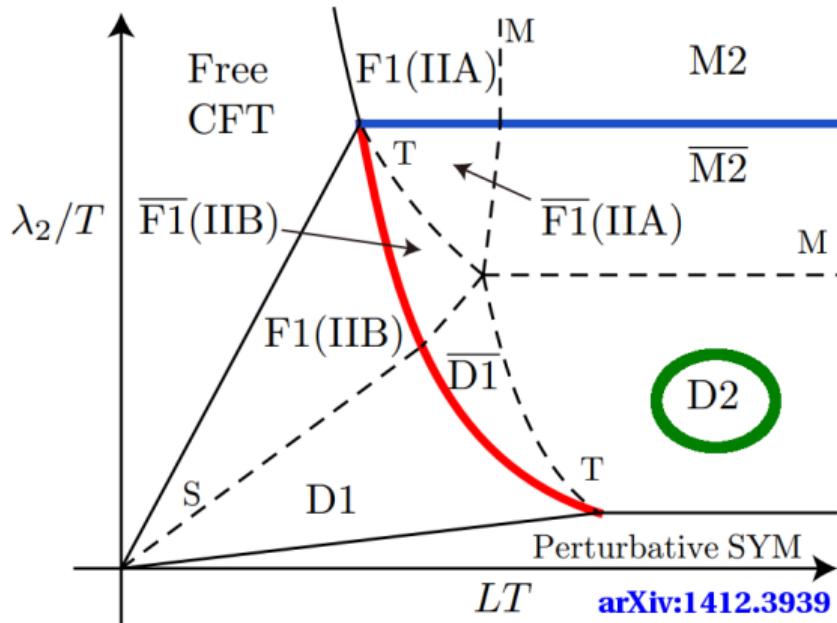
$\propto t^3$  for large- $r_L$  D1 phase



## 3d maximal SYM

Holography  $\rightarrow$  much richer low- $t$  phase diagram than for 2d  $\mathcal{N} = (8, 8)$  SYM

For now consider simplest homogeneous black D2-branes  $\rightarrow r_1 = r_2 = r_\beta$



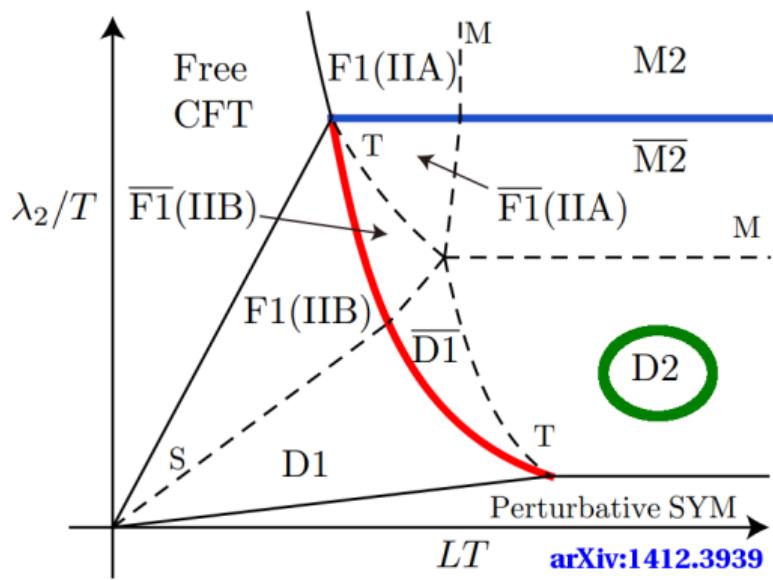
arXiv:1412.3939

# Homogeneous D2 phase

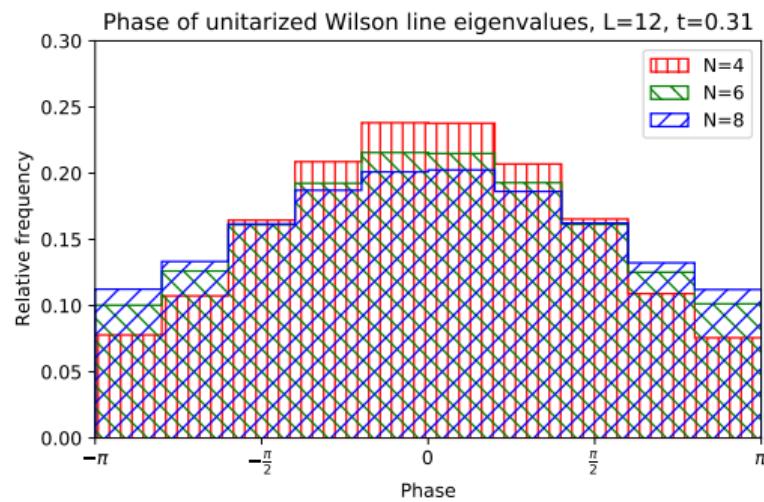
arXiv:2010.00026

Lattice volume  $(L/a)^3 \rightarrow$  continuum limit  $L/a \rightarrow \infty$  with fixed  $t = 1/r_\beta = L/\lambda$

Homogeneous D2-branes  $\longleftrightarrow$  uniform Wilson line eigenvalue phases at large  $N$



arXiv:1412.3939

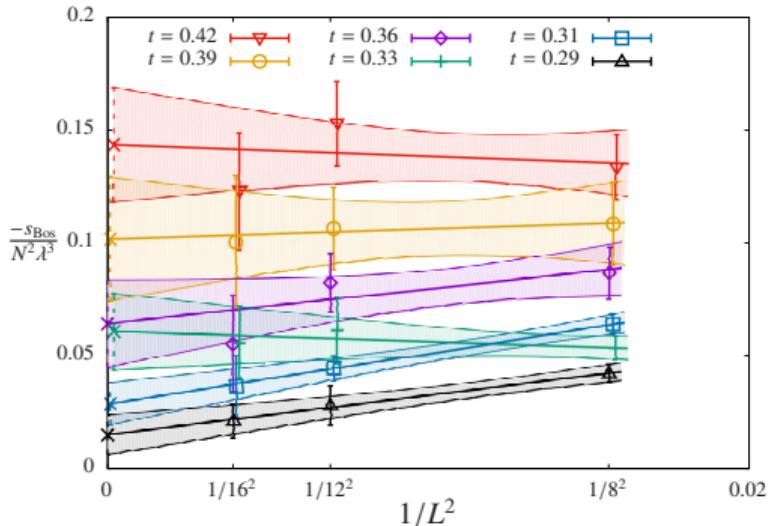
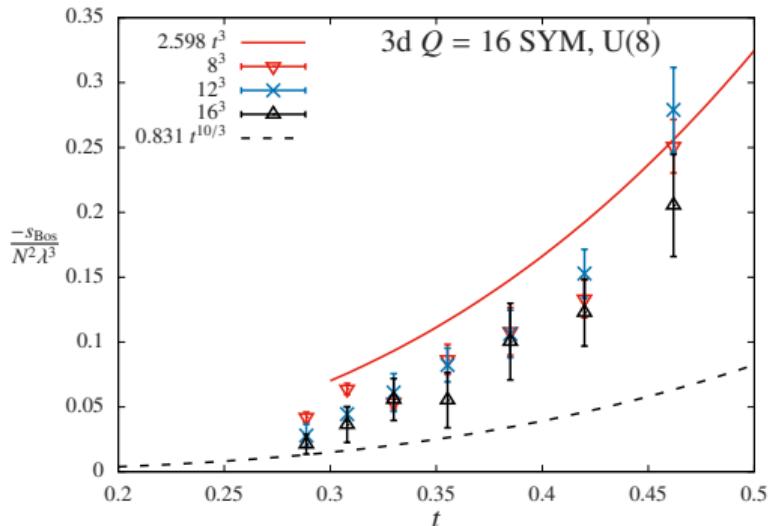


# Holographic black brane energies and continuum extrapolation

Lattice volume  $(L/a)^3$  with fixed  $N = 8$

→ results approach leading holographic expectation  $\propto t^{10/3}$  for low  $t \lesssim 0.4$

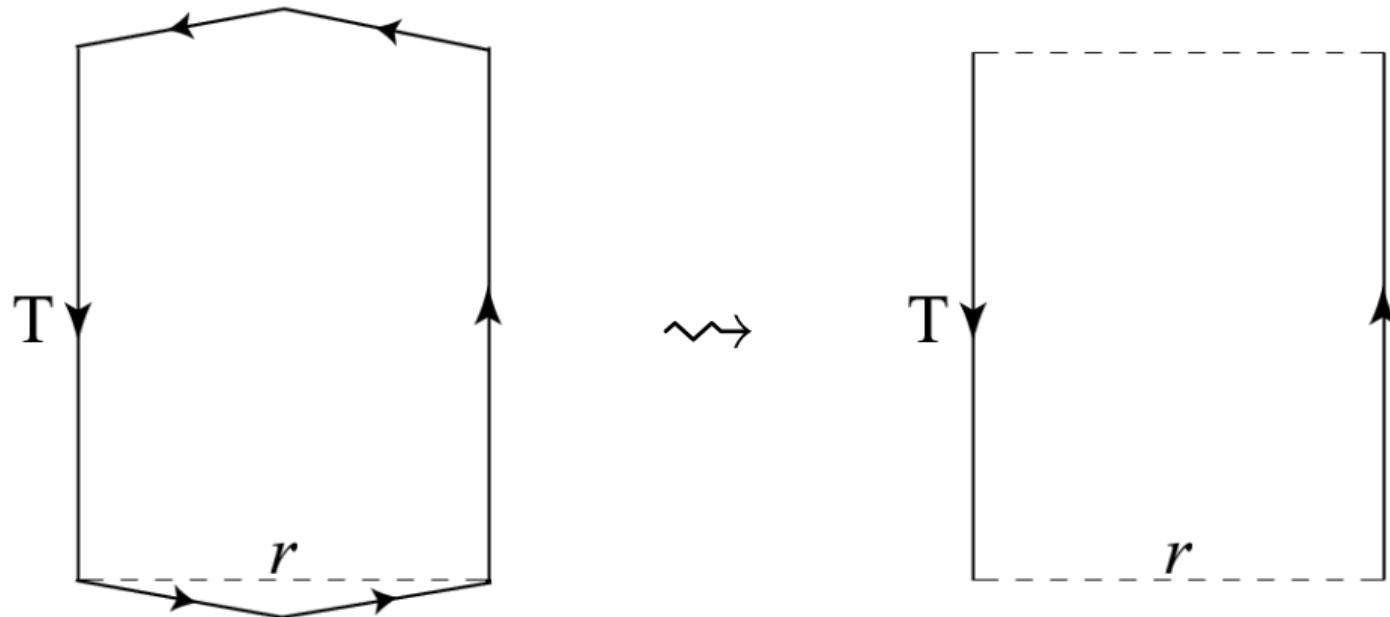
Carry out first  $L/a \rightarrow \infty$  continuum extrapolations



## 4d $\mathcal{N} = 4$ SYM static potential $V(r)$

Static probes  $\rightarrow r \times T$  Wilson loops  $W(r, T) \propto e^{-V(r) T}$

Coulomb gauge trick reduces  $A_4^*$  lattice complications



Static potential is Coulombic at all  $\lambda$

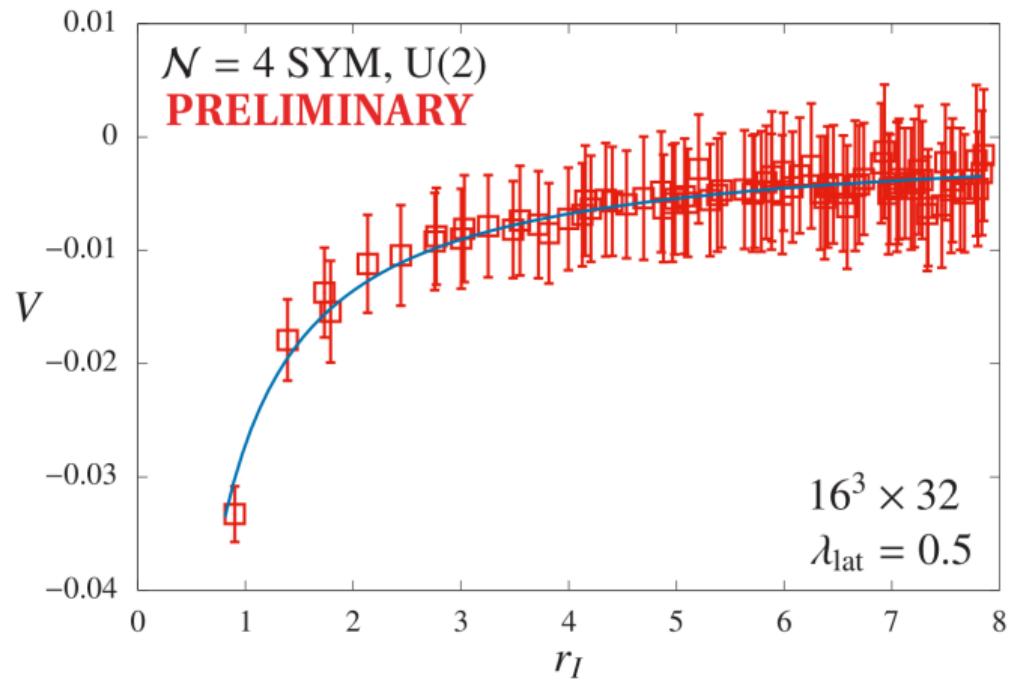
Fits to confining  $V(r) = A - C/r + \sigma r$   $\longrightarrow$  vanishing string tension  $\sigma$

Therefore fit

$$V(r) = A - C/r$$

to find Coulomb coefficient  $C(\lambda)$

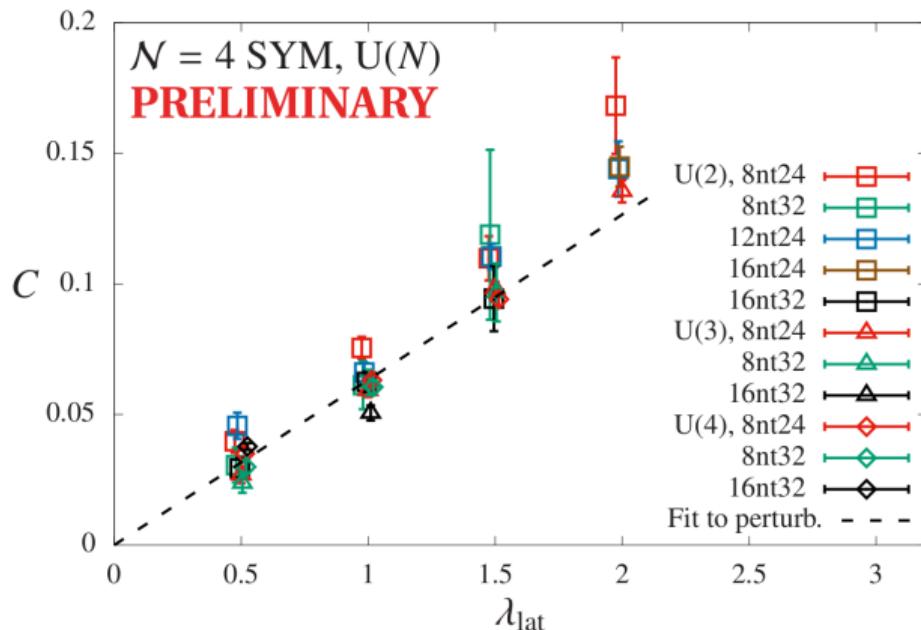
Discretization artifacts reduced  
by tree-level improved analysis



# Coupling dependence of Coulomb coefficient

Continuum perturbation theory  $\rightarrow C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

Holography  $\rightarrow C(\lambda) \propto \sqrt{\lambda}$  for  $N \rightarrow \infty$  and  $\lambda \rightarrow \infty$  with  $\lambda \ll N$



Again comparing different volumes  
and  $N = 2, 3, 4$

For  $\lambda_{\text{lat}} \leq 2$ , consistent with  
leading-order perturbation theory

# Konishi operator scaling dimension $\Delta_K$

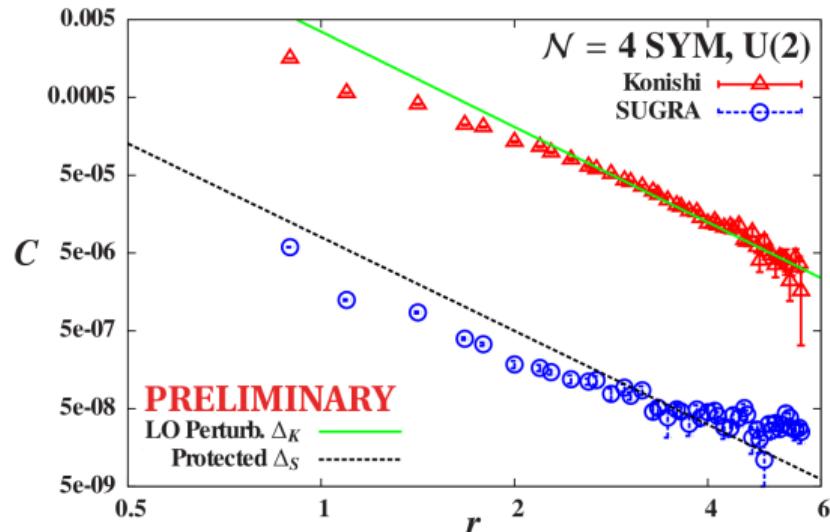
$\mathcal{O}_K(x) = \sum_I \text{Tr} [\Phi^I(x) \Phi^I(x)]$  is simplest conformal primary operator

Scaling dimension  $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$  investigated through  
perturbation theory (& S duality), holography, conformal bootstrap

$$C_K(r) \equiv \mathcal{O}_K(x+r) \mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

'SUGRA' is 20' op.,  $\Delta_S = 2$

Work in progress to compare:  
Direct power-law decay  
Finite-size scaling  
Monte Carlo RG



# Konishi operator scaling dimension $\Delta_K$

Lattice scalars  $\varphi(n)$  from polar decomposition  $\mathcal{U}_a(n) = e^{\varphi_a(n)} U_a(n)$

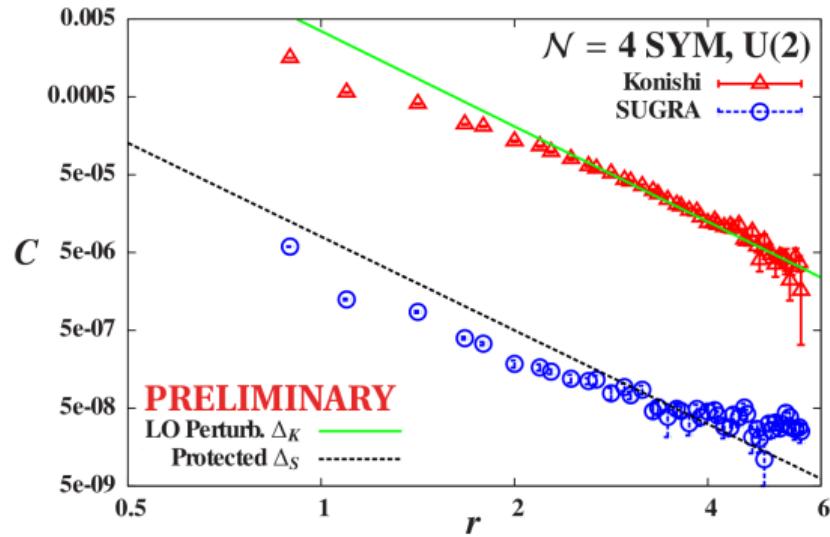
$$\mathcal{O}_K^{\text{lat}}(n) = \sum_a \text{Tr} [\varphi_a(n) \varphi_a(n)] - \text{vev}$$

$$\mathcal{O}_S^{\text{lat}}(n) \sim \text{Tr} [\varphi_a(n) \varphi_b(n)]$$

$$C_K(r) \equiv \mathcal{O}_K(x+r) \mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

'SUGRA' is 20' op.,  $\Delta_S = 2$

Work in progress to compare:  
Direct power-law decay  
Finite-size scaling  
Monte Carlo RG

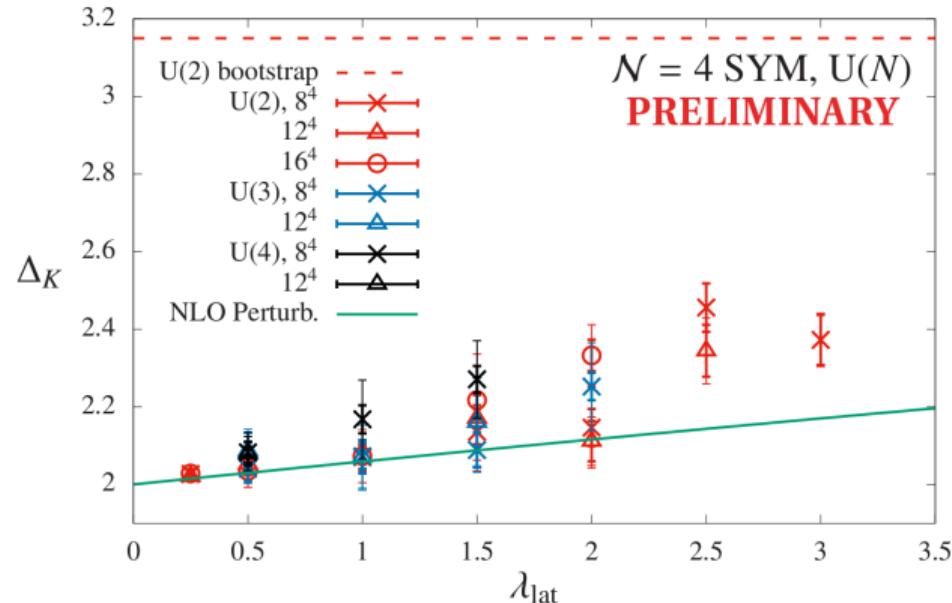


# Preliminary $\Delta_K$ results from Monte Carlo RG

Analyzing both  $\mathcal{O}_K^{\text{lat}}$  and  $\mathcal{O}_S^{\text{lat}}$

Imposing protected  $\Delta_S = 2$   
→  $\Delta_K(\lambda)$  looks perturbative

Systematic uncertainties from  
different amounts of smearing



Complication from twisting  $SO(4)_R \subset SO(6)_R$

$\mathcal{O}_K^{\text{lat}}$  mixes with  $SO(4)_R$ -singlet part of  $SO(6)_R$ -nonsinglet  $\mathcal{O}_S$   
→ disentangle via variational analyses

## Future: Pushing $\mathcal{N} = 4$ SYM to stronger coupling

Sign problem seems to obstruct access to  $\lambda_{\text{lat}} \gtrsim 3$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int [d\mathcal{U}] [d\bar{\mathcal{U}}] \mathcal{O} e^{-S_B[\mathcal{U}, \bar{\mathcal{U}}]} \text{pf } \mathcal{D}[\mathcal{U}, \bar{\mathcal{U}}]$$

Complex pfaffian  $\text{pf } \mathcal{D} = |\text{pf } \mathcal{D}| e^{i\alpha}$  complicates importance sampling

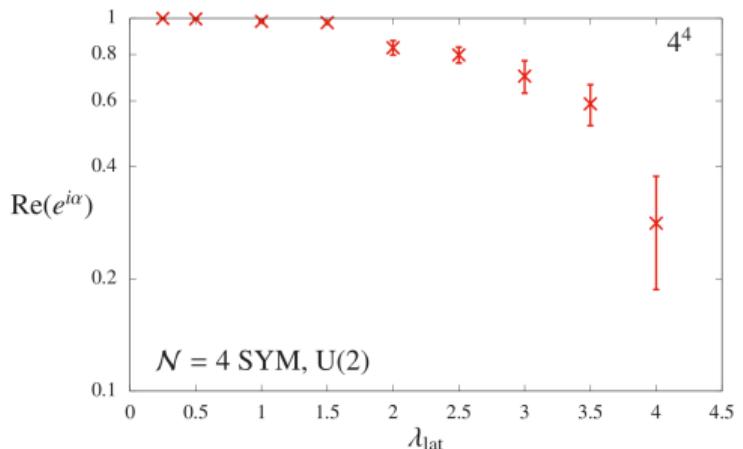
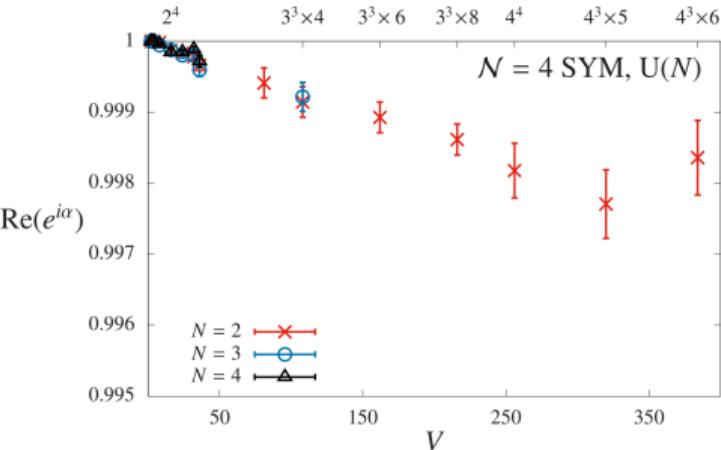
We phase quench,  $\text{pf } \mathcal{D} \rightarrow |\text{pf } \mathcal{D}|$ , need to reweight  $\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}}$

$$\implies \langle e^{i\alpha} \rangle_{pq} = \frac{Z}{Z_{pq}} \text{ quantifies severity of sign problem}$$

# $\mathcal{N} = 4$ SYM sign problem

Fix  $\lambda_{\text{lat}} = g_{\text{lat}}^2 N = 0.5$

Pfaffian nearly real positive  
for all accessible volumes



Fix  $4^4$  volume  
Fluctuations increase with coupling  
Signal-to-noise  
becomes obstruction for  $\lambda_{\text{lat}} \gtrsim 4$

## Future: Supersymmetric QCD

Add matter multiplets → investigate electric–magnetic dualities,  
dynamical supersymmetry breaking and more



Quiver construction based on twisted SYM

[arXiv:1505.00467]

preserves susy sub-algebra in  $(d - 1)$  dims. to reduce fine-tuning

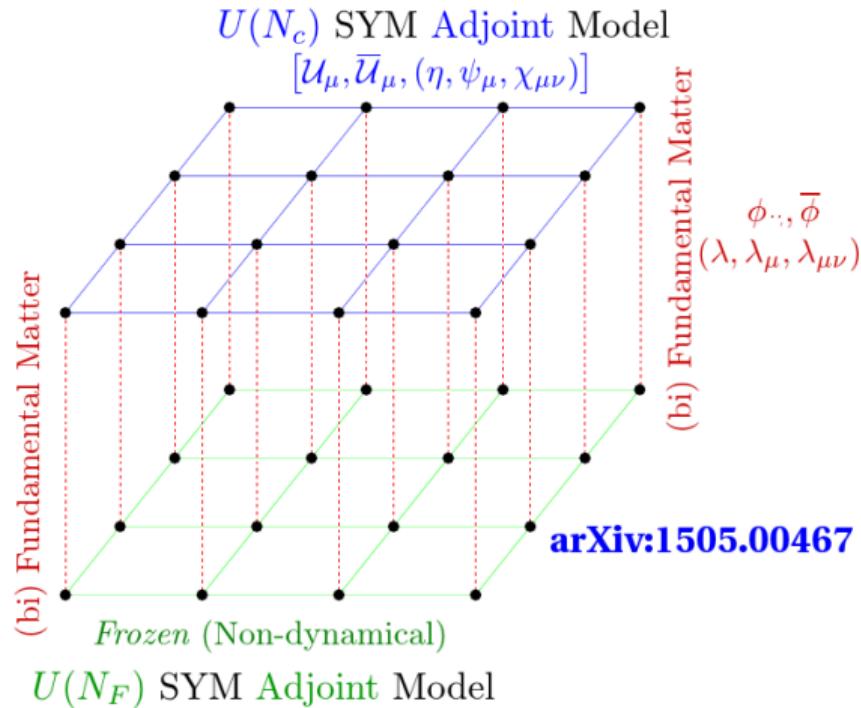
# Quiver superQCD from twisted SYM

2-slice lattice SYM  
with  $U(N) \times U(F)$  gauge group

Adj. fields on each slice

Bi-fundamental in between

Decouple  $U(F)$  slice  
 $\rightarrow U(N)$  SQCD in  $(d - 1)$  dims.  
with  $F$  fund. hypermultiplets

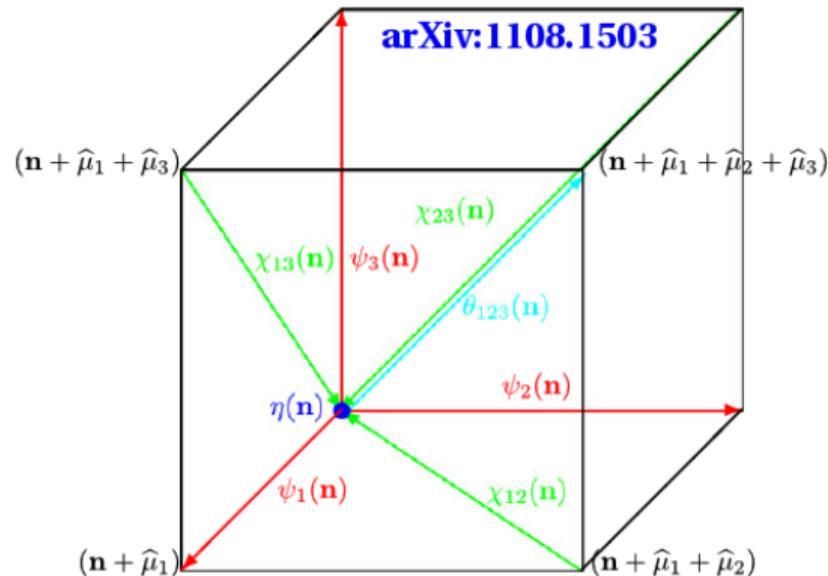


First check 3d SYM  $\rightarrow$  2d superQCD      then new 4d SYM  $\rightarrow$  3d superQCD

## Simpler twisted formulation

$Q = 8$  supercharges  $\{\mathcal{Q}, \mathcal{Q}_a, \mathcal{Q}_{ab}, \mathcal{Q}_{abc}\}$  with  $a, b = 1, \dots, 3$

→ site / link / plaquette / cube fermions  $\{\eta, \psi_a, \chi_{ab}, \theta_{abc}\}$  on simple cubic lattice



## Simpler twisted formulation

$Q = 8$  supercharges  $\{\mathcal{Q}, \mathcal{Q}_a, \mathcal{Q}_{ab}, \mathcal{Q}_{abc}\}$  with  $a, b = 1, \dots, 3$

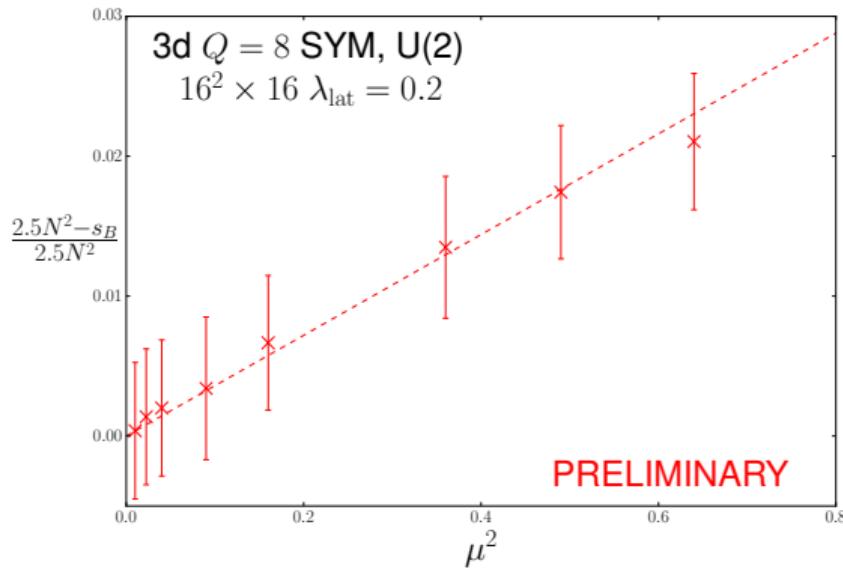
→ site / link / plaquette / cube fermions  $\{\eta, \psi_a, \chi_{ab}, \theta_{abc}\}$  on simple cubic lattice

Work by Angel Sherletov

Parallel code developed

3d SYM tests passed

→ 2d quiver superQCD



# Recap: An exciting time for numerical lattice supersymmetry

✓ Preserve (some) susy in discrete space-time

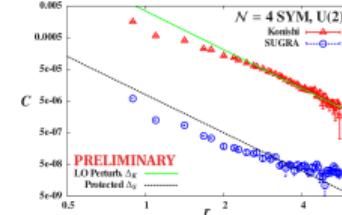
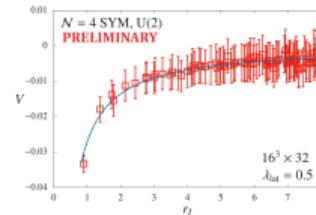
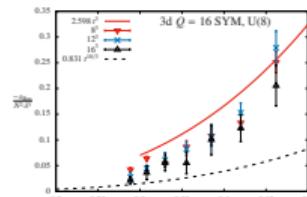
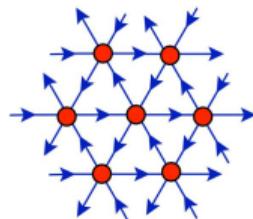
→ practical lattice  $\mathcal{N} = 4$  SYM, public code available

Reproduce reliable analytic results

✓ 2d and 3d thermodynamics consistent with holography

✓ Perturbative  $\mathcal{N} = 4$  SYM static potential Coulomb coefficient  $C(\lambda)$   
and Konishi operator scaling dimension  $\Delta_K(\lambda)$

Access new domains → sign problems, supersymmetric QCD and more...



Thanks for your attention!

Any further questions?

## Collaborators

Raghav Jha, Anosh Joseph, Angel Sherletov, Toby Wiseman  
also Georg Bergner, Simon Catterall, Poul Damgaard, Joel Giedt

## Funding and computing resources

UK Research  
and Innovation



## Backup: Breakdown of Leibniz rule on the lattice

$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu = 2i\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu$  is problematic

$\implies$  try finite difference  $\partial\phi(x) \longrightarrow \Delta\phi(x) = \frac{1}{a} [\phi(x + a) - \phi(x)]$

Crucial difference between  $\partial$  and  $\Delta$

$$\begin{aligned}\Delta[\phi\eta] &= a^{-1} [\phi(x + a)\eta(x + a) - \phi(x)\eta(x)] \\ &= [\Delta\phi]\eta + \phi\Delta\eta + a[\Delta\phi]\Delta\eta\end{aligned}$$

Full supersymmetry requires Leibniz rule  $\partial[\phi\eta] = [\partial\phi]\eta + \phi\partial\eta$

only recovered in  $a \rightarrow 0$  continuum limit for any local finite difference

## Backup: Breakdown of Leibniz rule on the lattice

Full supersymmetry requires Leibniz rule  $\partial[\phi\eta] = [\partial\phi]\eta + \phi\partial\eta$

only recoverd in  $a \rightarrow 0$  continuum limit for any local finite difference

Supersymmetry vs. locality ‘no-go’ theorems

by Kato–Sakamoto–So [[arXiv:0803.3121](#)] and Bergner [[arXiv:0909.4791](#)]

Complicated constructions to balance locality vs. supersymmetry

Non-ultralocal product operator  $\rightarrow$  lattice Leibniz rule but not gauge invariance

D’Adda–Kawamoto–Saito, [arXiv:1706.02615](#)

Cyclic Leibniz rule  $\rightarrow$  partial lattice supersymmetry but only (0+1)d QM so far

Kadoh–Kamei–So, [arXiv:1904.09275](#)

## Backup: Complexified gauge field from twisting

Combining  $A_\mu$  and  $\Phi^I \rightarrow \mathcal{A}_a$  and  $\bar{\mathcal{A}}_a$

produces  $U(N) = SU(N) \otimes U(1)$  gauge theory

Complicates lattice action but needed so that  $\mathcal{Q} \mathcal{A}_a = \psi_a$

Further motivation: Under  $SO(d)_{\text{tw}} = \text{diag}[SO(d)_{\text{euc}} \otimes SO(d)_R]$

$A_\mu \sim \text{vector} \otimes \text{scalar} = \text{vector}$

$\Phi^I \sim \text{scalar} \otimes \text{vector} = \text{vector}$

Easiest to see in 5d, then dimensionally reduce

$$\mathcal{A}_a = A_a + i\Phi_a \rightarrow (A_\mu, \phi) + i(\Phi_\mu, \bar{\phi})$$

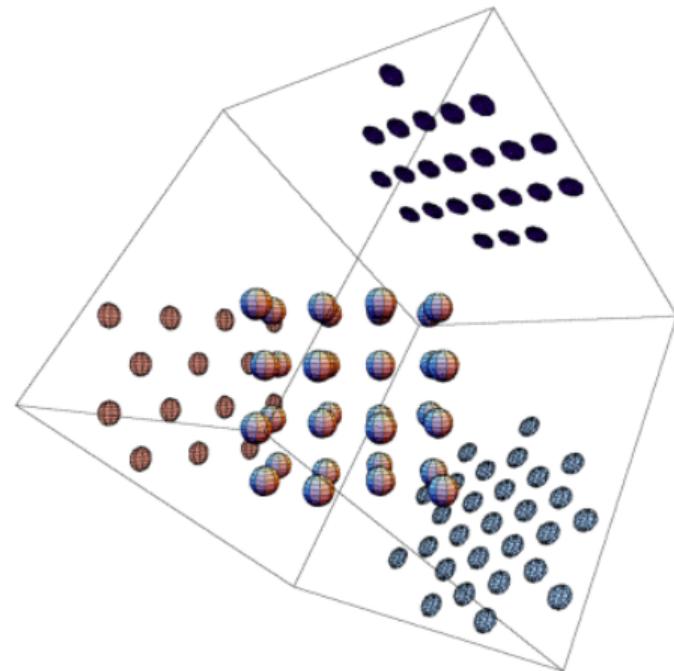
## Backup: $A_4^*$ lattice from five dimensions

Again dimensionally reduce, treating all five gauge links symmetrically

Start with hypercubic lattice  
in 5d momentum space

**Symmetric** constraint  $\sum_a \partial_a = 0$   
projects to 4d momentum space

Result is  $A_4$  lattice  
→ dual  $A_4^*$  lattice in position space



# Backup: Restoration of $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supersymmetries

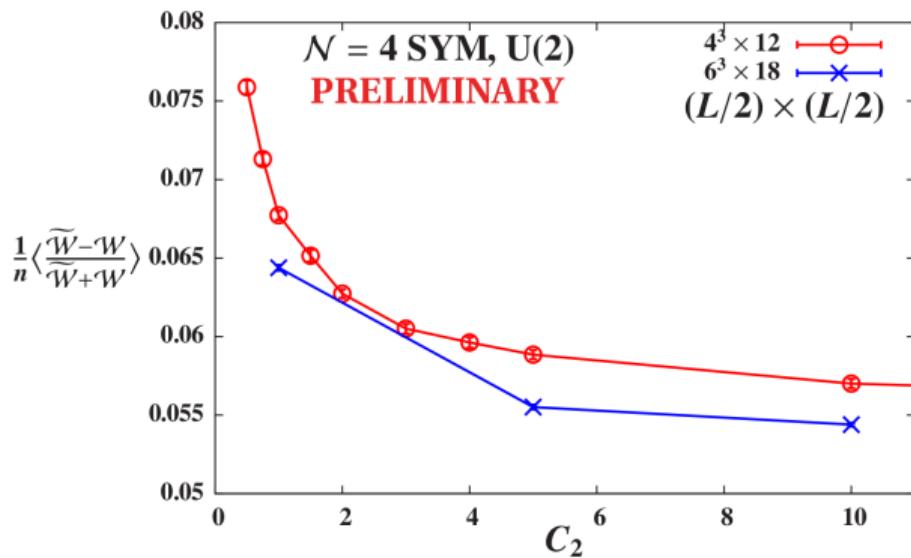
“ $\mathcal{Q}$  + discrete  $R_a \subset \text{SO}(4)_{\text{tw}}$  =  $\mathcal{Q}_a$  and  $\mathcal{Q}_{ab}$ ”

[arXiv:1306.3891]

Test  $R_a$  on Wilson loops

$$\widetilde{\mathcal{W}}_{ab} \equiv R_a \mathcal{W}_{ab}$$

Tune coeff.  $c_2$  of  $d^2$  term in action  
for fastest restoration  
towards continuum limit



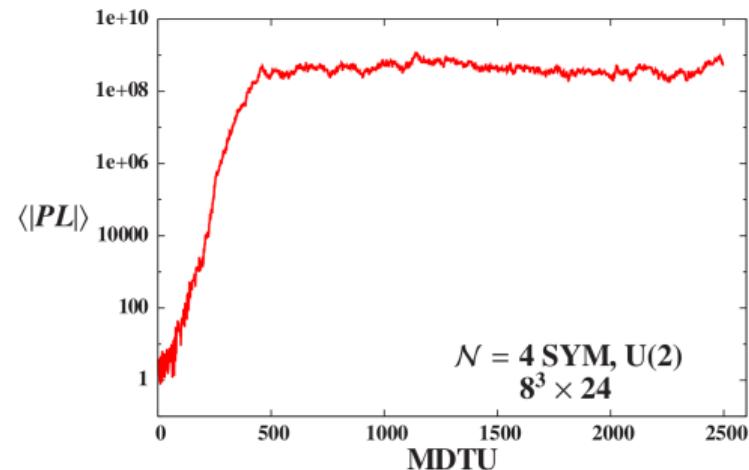
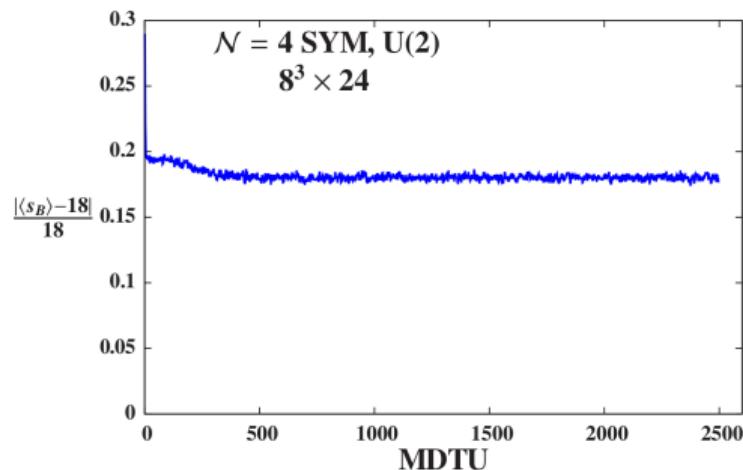
## Backup: Problem with $SU(N)$ flat directions

$\mu^2/\lambda_{\text{lat}}$  too small  $\rightarrow \mathcal{U}_a$  can move far from continuum form  $\mathbb{I}_N + \mathcal{A}_a$

Example:  $\mu = 0.2$  and  $\lambda_{\text{lat}} = 2.5$  on  $8^3 \times 24$  volume

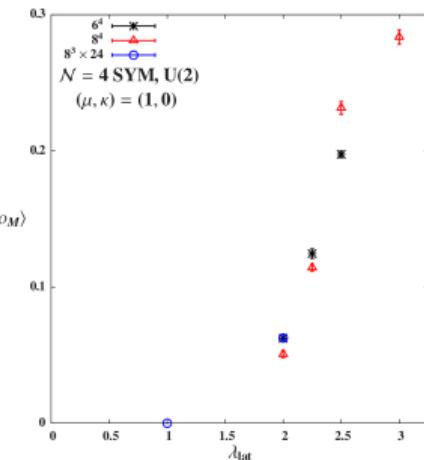
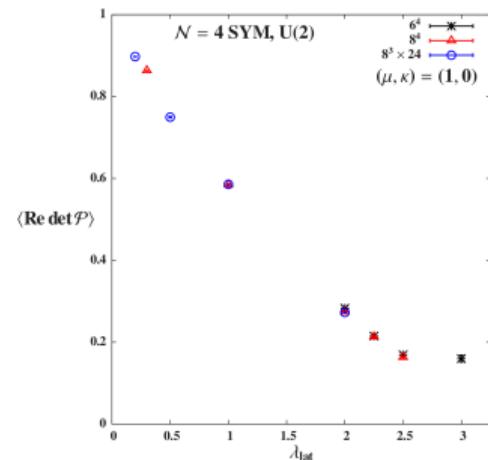
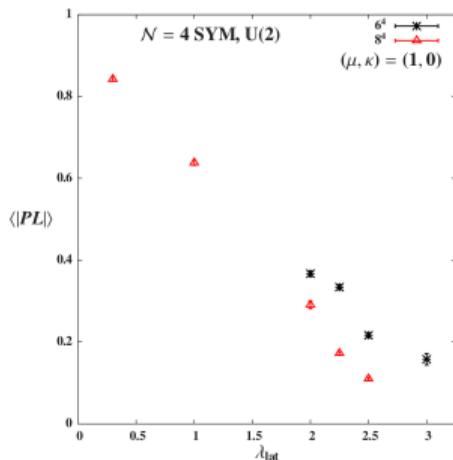
**Left:** Bosonic action stable  $\sim 18\%$  off its supersymmetric value

**Right:** (Complexified) Polyakov loop wanders off to  $\sim 10^9$



# Backup: Problem with U(1) flat directions

Monopole condensation  $\rightarrow$  confined lattice phase not present in continuum



Around the same  $2\lambda_{\text{lat}} \approx 2\dots$

**Left:** Polyakov loop falls towards zero

**Center:** Plaquette determinant falls towards zero

**Right:** Density of U(1) monopole world lines becomes non-zero

## Backup: Naively regulating U(1) flat directions

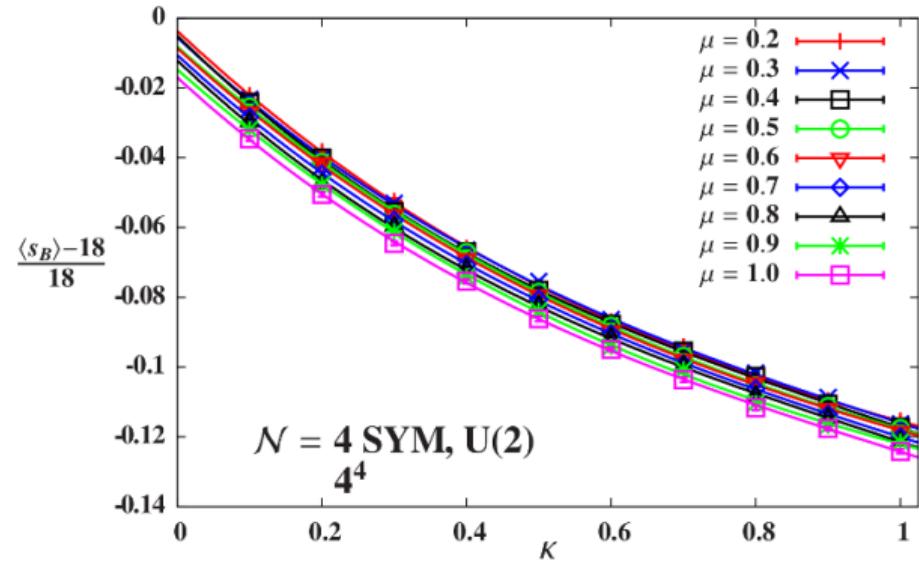
In earlier work we added another soft  $\mathcal{Q}$ -breaking term

$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}} \mu^2 \sum_a \left( \frac{1}{N} \text{Tr} [\mathcal{U}_a \bar{\mathcal{U}}_a] - 1 \right)^2 + \kappa \sum_{a < b} |\det \mathcal{P}_{ab} - 1|^2$$

More sensitivity to  $\kappa$  than to  $\mu^2$

Showing  $\mathcal{Q}$  Ward identity  
from bosonic action

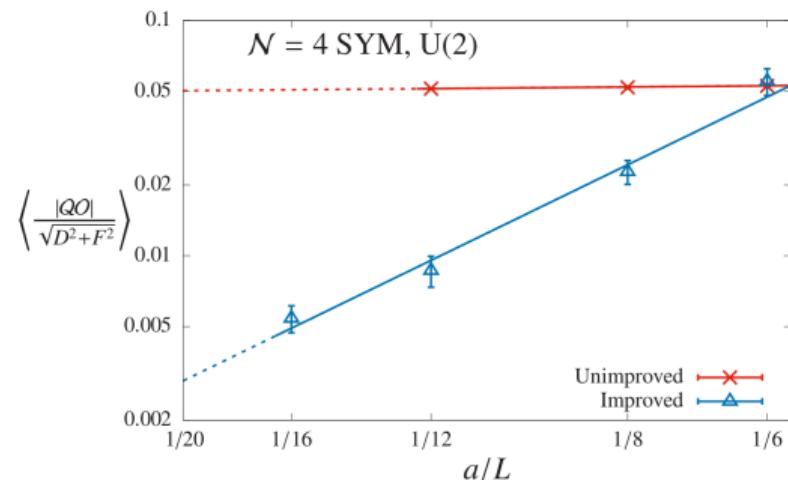
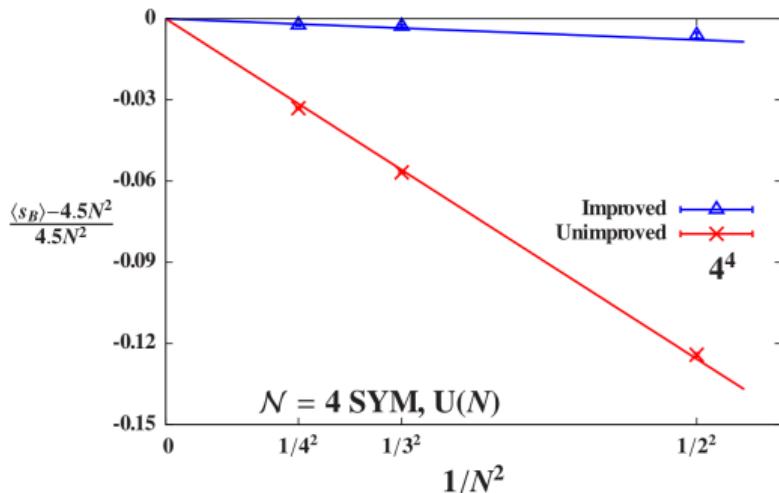
$$\langle s_B \rangle = 9N^2/2$$



# Backup: Better regulating U(1) flat directions

$$S = \frac{N}{4\lambda_{\text{lat}}} \left[ \mathcal{Q} \left( \chi_{ab} \mathcal{F}_{ab} + \eta \left\{ \bar{\mathcal{D}}_a \mathcal{U}_a + G \sum_{a < b} [\det \mathcal{P}_{ab} - 1] \mathbb{I}_N \right\} - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \bar{\mathcal{D}}_c \chi_{de} + \mu^2 V \right]$$

$\mathcal{Q}$  Ward identity violations scale  $\propto 1/N^2$  (**left**) and  $\propto (a/L)^2$  (**right**)  
 ~ effective ‘ $\mathcal{O}(a)$  improvement’ since  $\mathcal{Q}$  forbids all dim-5 operators



# Backup: Supersymmetric moduli space modification

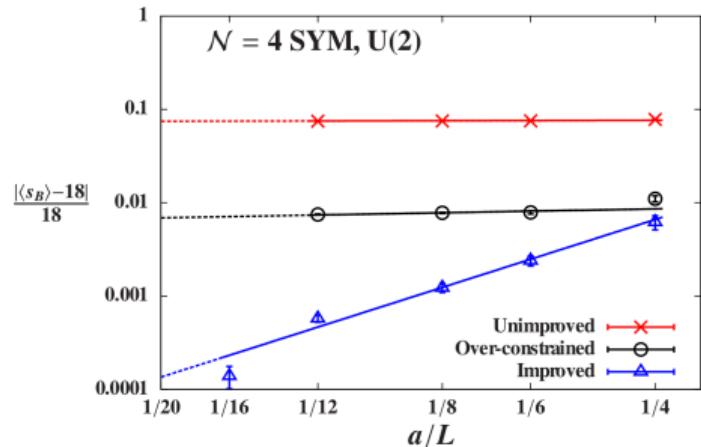
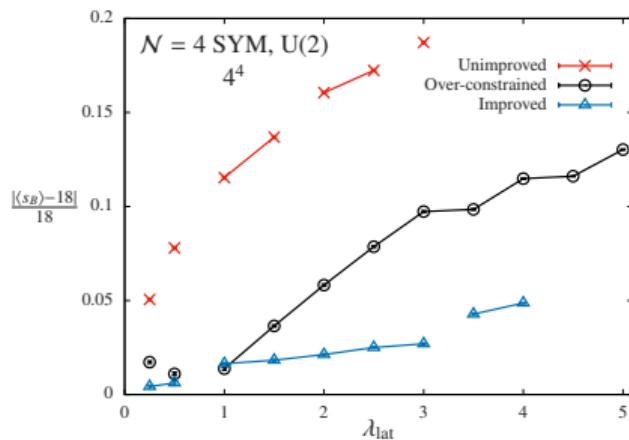
[arXiv:1505.03135]

Method to impose  $\mathcal{Q}$ -invariant constraints on generic site operator  $\mathcal{O}(n)$

Modify auxiliary field equations of motion  $\rightarrow$  moduli space

$$d(n) = \bar{\mathcal{D}}_a^{(-)} \mathcal{U}_a(n) \quad \rightarrow \quad d(n) = \bar{\mathcal{D}}_a^{(-)} \mathcal{U}_a(n) + G\mathcal{O}(n)\mathbb{I}_N$$

Including both  $U(1)$  and  $SU(N) \in \mathcal{O}(n)$  over-constraints system



# Backup: Dimensional reduction to 2d $\mathcal{N} = (8, 8)$ SYM

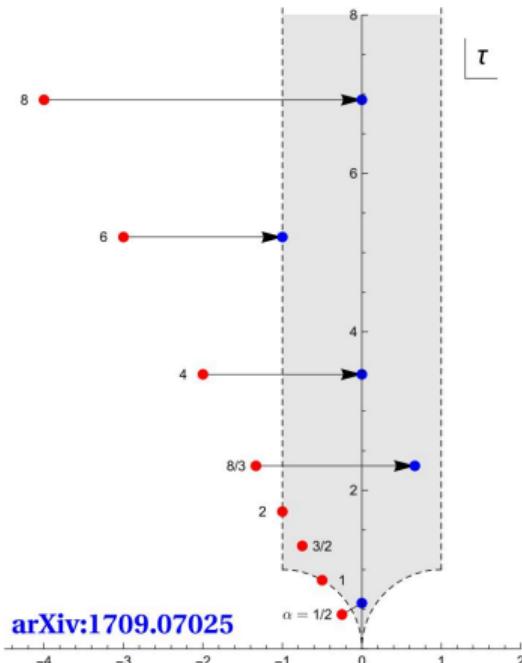
Naive for now: 4d  $\mathcal{N} = 4$  SYM code with  $N_x = N_y = 1$

$A_4^* \longrightarrow A_2^*$  (triangular) lattice

Torus **skewed** depending on  $\alpha = L/N_t$

Modular transformation into fundamental domain  
→ some skewed tori actually rectangular

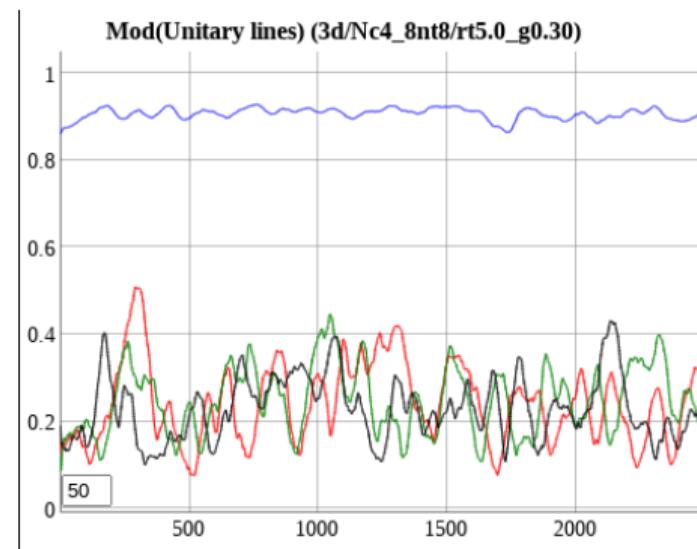
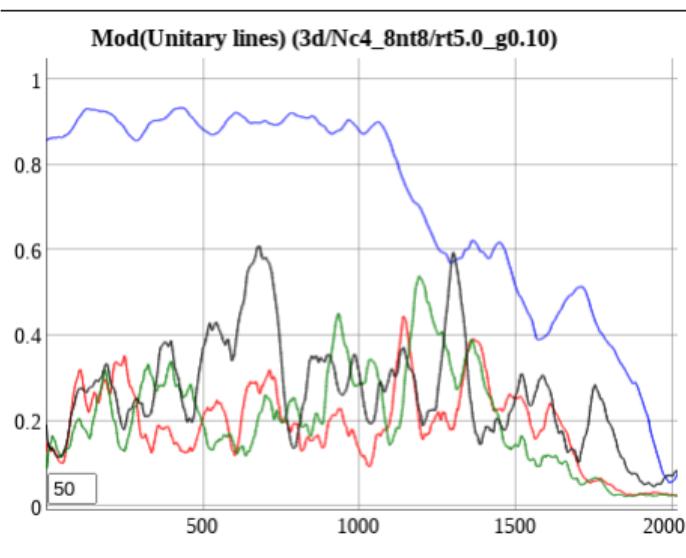
Also need to stabilize compactified links  
to ensure broken center symmetries



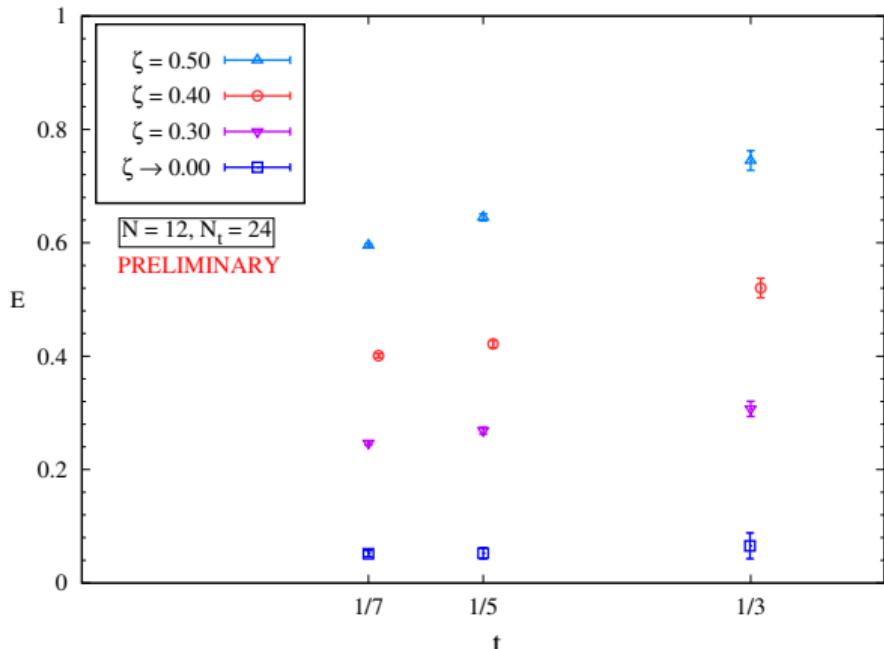
arXiv:1709.07025

## Backup: Stabilizing compactified links

Add potential  $\propto \text{Tr} \left[ (\varphi - \mathbb{I}_N)^\dagger (\varphi - \mathbb{I}_N) \right]$  to break center symmetry in reduced dir(s)  
(~Kaluza–Klein rather than Eguchi–Kawai reduction)



Much simpler twisted formulation:  $Q = 4$  supercharges  $\{\mathcal{Q}, \mathcal{Q}_a, \mathcal{Q}_{ab}\}$   
 → site / link / plaquette fermions  $\{\eta, \psi_a, \chi_{ab}\}$  on square lattice ( $a, b = 1, 2$ )



Work by Navdeep Singh Dhindsa

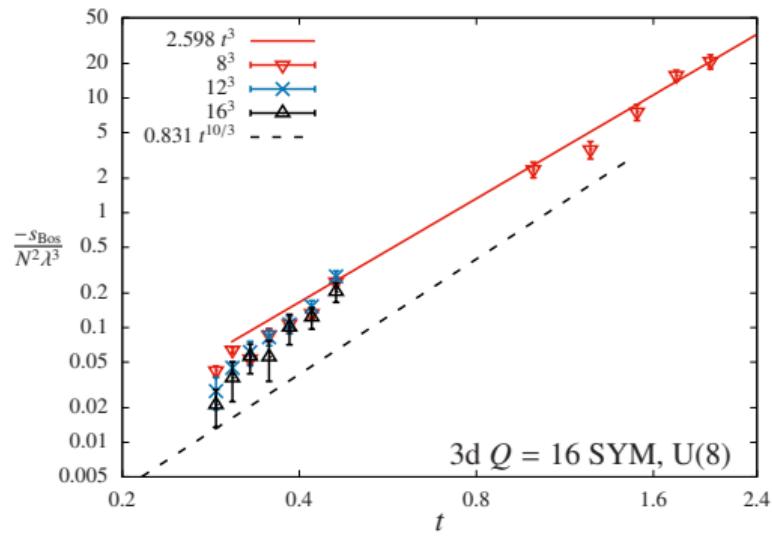
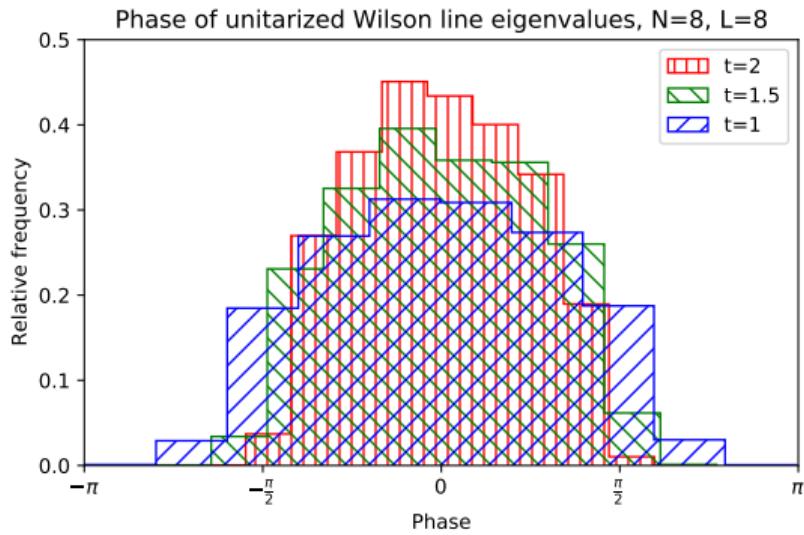
Prelim.  $\mu^2 \rightarrow 0$  extrapolations  
 for  $r_L = r_\beta \longleftrightarrow \alpha = 1$

Energy independent of  $t \lesssim 0.33$   
 vs.  $\sim t^3$  for  $\mathcal{N} = (8, 8)$  SYM

# Backup: High-temperature ( $t \gtrsim 1$ ) 3d maximal SYM

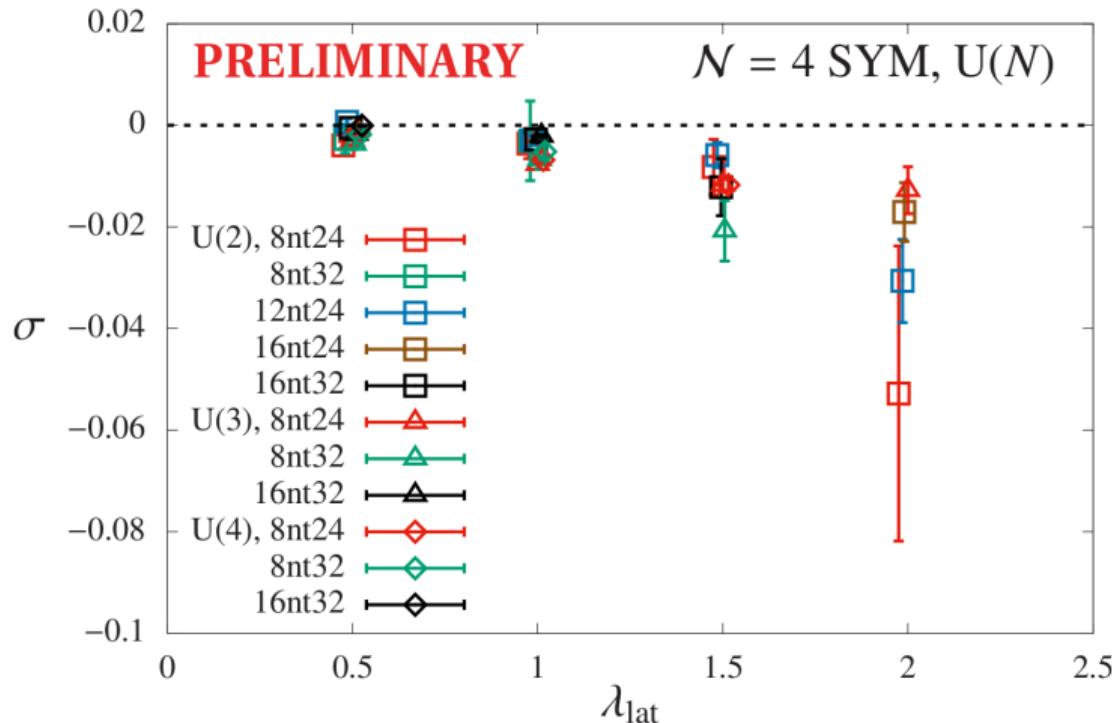
Wilson line eigenvalue phases localized rather than uniform (**left**)

Thermodynamics consistent with weak-coupling expectation  $\propto t^3$  (**right**)



## Backup: Static potential is Coulombic at all $\lambda$

String tension  $\sigma$  from fits to confining form  $V(r) = A - C/r + \sigma r$



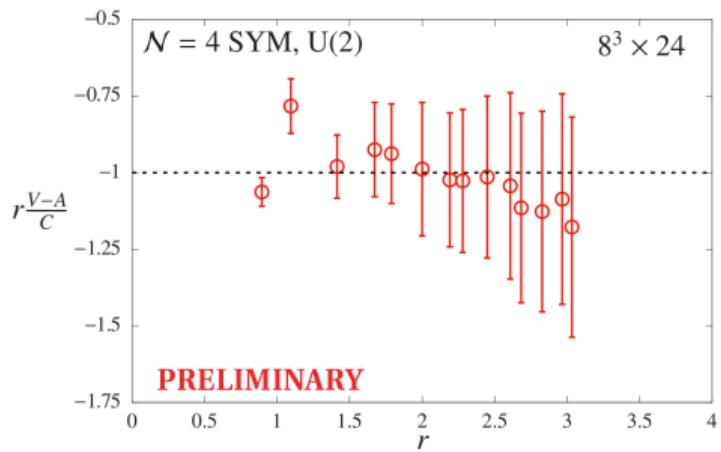
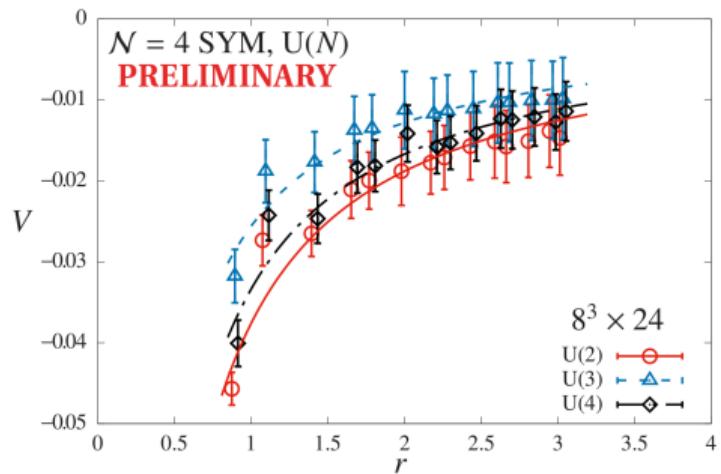
Slightly negative values  
flatten  $V(r_l)$  for  $r_l \lesssim L/2$

$\sigma \rightarrow 0$  as accessible  
range of  $r_l$  increases  
on larger volumes

# Backup: Discretization artifacts in static potential

Discretization artifacts visible at short distances

where Coulomb term in  $V(r) = A - C/r$  is most significant



Danger of distorting Coulomb coefficient  $C$

## Backup: Tree-level improvement

Classic trick to reduce discretization artifacts in static potential

Associate  $V(r_\nu)$  data with ' $r_l$ ' from Fourier transform of gluon propagator

Recall  $\frac{1}{4\pi^2 r^2} = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{e^{ir_\nu k_\nu}}{k^2}$  where  $\frac{1}{k^2} = G(k_\nu)$  in continuum

$$A_4^* \text{ lattice} \rightarrow \frac{1}{r_l^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \frac{\cos(i r_\nu \hat{k}_\nu)}{4 \sum_{\mu=1}^4 \sin^2(\hat{k} \cdot \hat{e}_\mu / 2)}$$

Tree-level lattice propagator from [arXiv:1102.1725](https://arxiv.org/abs/1102.1725)

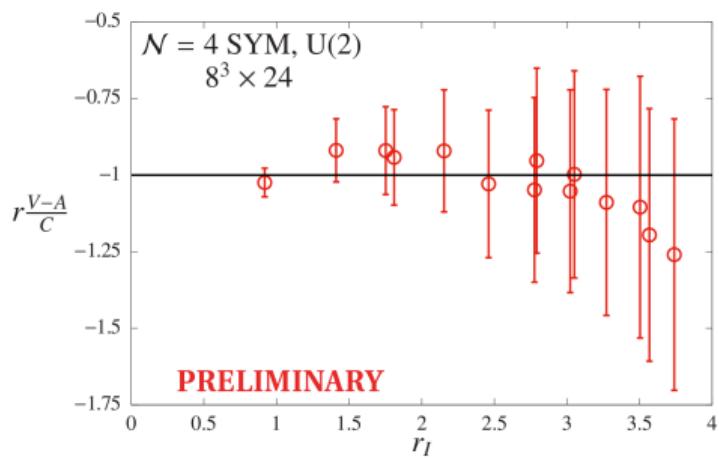
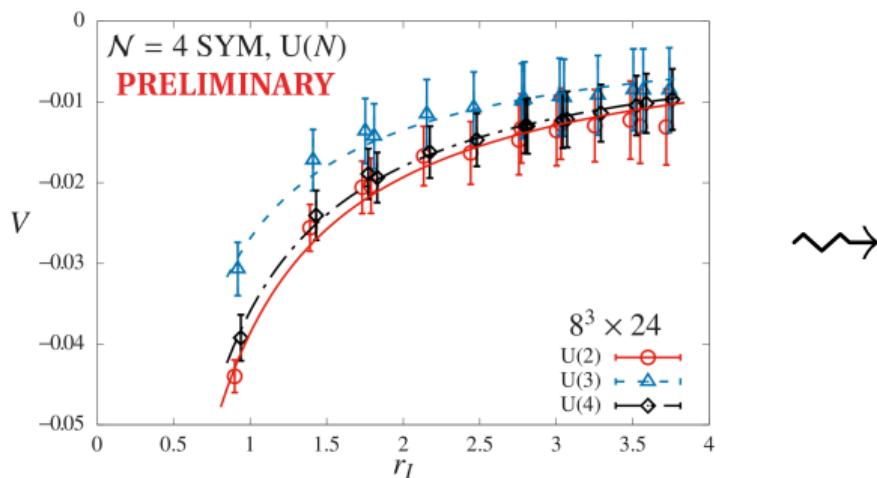
$\hat{e}_\mu$  are  $A_4^*$  lattice basis vectors;

momenta  $\hat{k} = \frac{2\pi}{L} \sum_{\mu=1}^4 n_\mu \hat{g}_\mu$  depend on dual basis vectors

# Backup: Tree-level-improved static potential

$$\frac{1}{r_I^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \frac{\cos(i r_\nu \hat{k}_\nu)}{4 \sum_{\mu=1}^4 \sin^2(\hat{k} \cdot \hat{e}_\mu / 2)}$$

→ significantly reduced discretization artifacts



## Backup: Scaling dimensions from MCRG stability matrix

Lattice system:  $H = \sum_i c_i \mathcal{O}_i$  (infinite sum)

Couplings flow under RG blocking  $\rightarrow H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Conformal fixed point  $\rightarrow H^* = R_b H^*$  with couplings  $c_i^*$

Linear expansion around fixed point  $\rightarrow$  **stability matrix**  $T_{ik}^*$

$$c_i^{(n)} - c_i^* = \sum_k \left. \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \right|_{H^*} (c_k^{(n-1)} - c_k^*) \equiv \sum_k T_{ik}^* (c_k^{(n-1)} - c_k^*)$$

Correlators of  $\mathcal{O}_i, \mathcal{O}_k \rightarrow$  elements of stability matrix

[Swendsen, 1979]

Eigenvalues of  $T_{ik}^*$   $\rightarrow$  scaling dimensions of corresponding operators

## Backup: Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve  $\mathcal{Q}$  and  $S_5$  symmetries  $\longleftrightarrow$  geometric structure

Simple transformation constructed in [arXiv:1408.7067](#)

$$\begin{aligned}\mathcal{U}'_a(n') &= \xi \mathcal{U}_a(n) \mathcal{U}_a(n + \hat{\mu}_a) & \eta'(n') &= \eta(n) \\ \psi'_a(n') &= \xi [\psi_a(n) \mathcal{U}_a(n + \hat{\mu}_a) + \mathcal{U}_a(n) \psi_a(n + \hat{\mu}_a)] & \text{etc.}\end{aligned}$$

Doubles lattice spacing  $a \rightarrow a' = 2a$ , with tunable rescaling factor  $\xi$

Scalar fields from polar decomposition  $\mathcal{U}(n) = e^{\varphi(n)} U(n)$

$\Rightarrow$  shift  $\varphi \rightarrow \varphi + \log \xi$  to keep blocked  $U$  unitary

$\mathcal{Q}$ -preserving RG transformation needed

to show only one log. tuning to recover continuum  $\mathcal{Q}_a$  and  $\mathcal{Q}_{ab}$

# Backup: Smearing for Konishi analyses

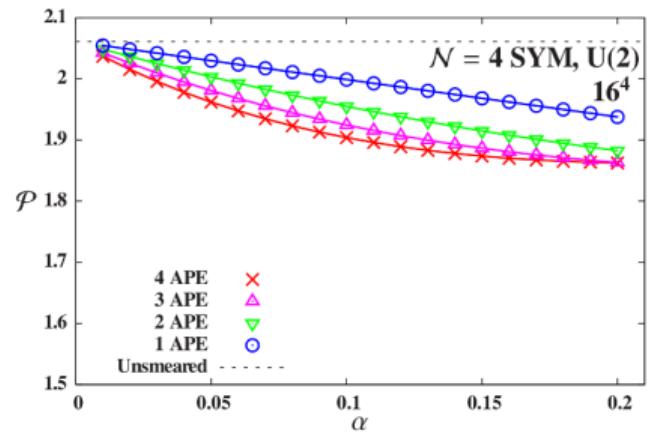
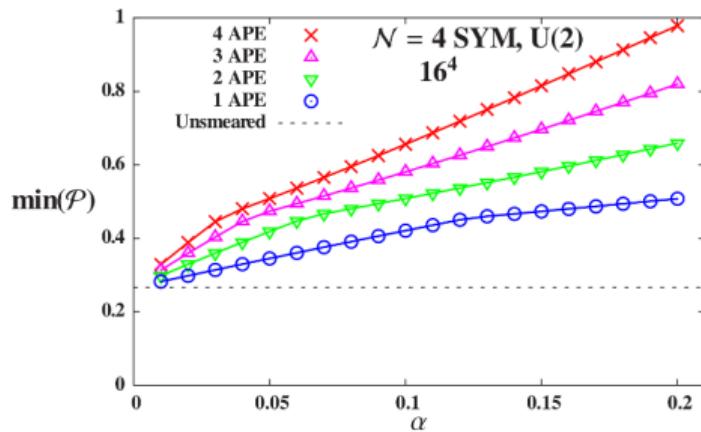
Smear to enlarge (MCRG or variational) operator basis

APE-like smearing:  $\underline{\quad}$   $\rightarrow$   $(1 - \alpha)\underline{\quad} + \frac{\alpha}{8} \sum \square,$

staples built from unitary parts of links but no final unitarization

Average plaquette stable upon smearing (**right**),

minimum plaquette steadily increases (**left**)

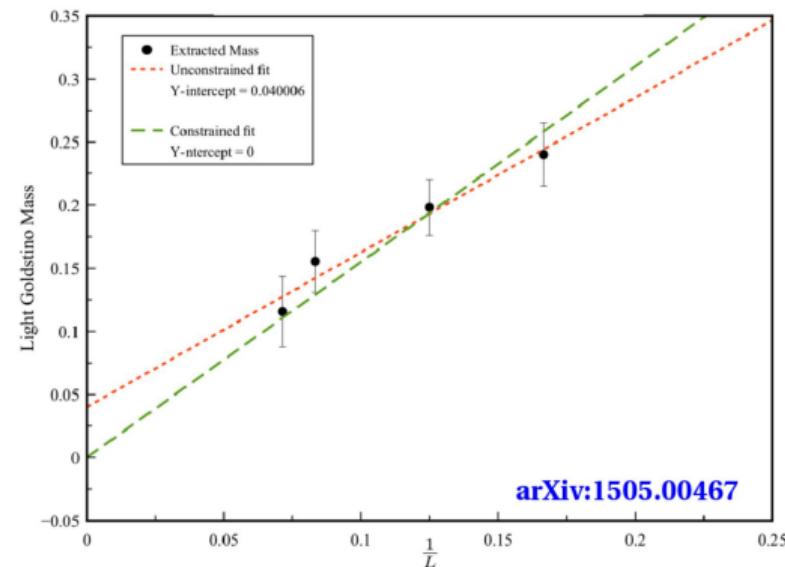
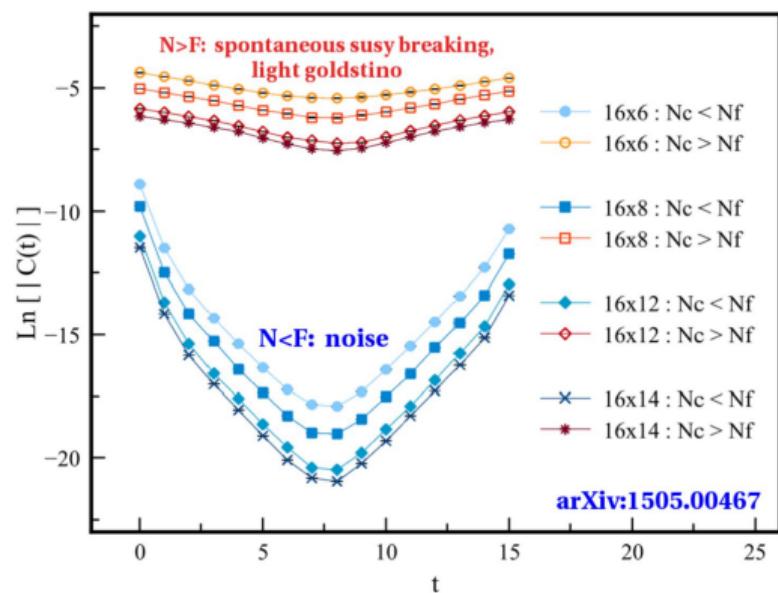


# Backup: Dynamical susy breaking in 2d lattice superQCD

$U(N)$  superQCD with  $F$  fundamental hypermultiplets

Observe spontaneous susy breaking only for  $N > F$ , as expected

Catterall–Veernala, arXiv:1505.00467



## Backup: More on dynamical susy breaking

Spontaneous susy breaking means  $\langle 0 | H | 0 \rangle > 0$  or equivalently  $\langle Q\mathcal{O} \rangle \neq 0$

Twisted superQCD auxiliary field e.o.m.  $\longleftrightarrow$  Fayet–Iliopoulos  $D$ -term potential

$$d = \bar{\mathcal{D}}_a \mathcal{U}_a + \sum_{i=1}^F \phi_i \bar{\phi}_i - r \mathbb{I}_N \quad \longleftrightarrow \quad \text{Tr} \left[ \left( \sum_i \phi_i \bar{\phi}_i - r \mathbb{I}_N \right)^2 \right] \in H$$

Have  $F \times N$  scalar vevs to zero out  $N \times N$  matrix

$\rightarrow N > F$  suggests susy breaking,  $\langle 0 | H | 0 \rangle > 0 \longleftrightarrow \langle Q\eta \rangle = \langle d \rangle \neq 0$