Supersymmetric Yang–Mills theories on the lattice

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and more to come with R. G. Jha, A. Joseph, A. Sherletov & T. Wiseman
Overview and plan

Preserve (some) susy in discrete space-time
    → practical lattice investigations

First reproduce perturbative and holographic results,
    then access new domains

Why: Lattice supersymmetry motivation

How: Lattice $\mathcal{N} = 4$ SYM formulation highlights

What: Recent, ongoing & planned work
Overview and plan

Preserve (some) susy in discrete space-time

→ practical lattice investigations

Why: Lattice supersymmetry motivation

How: Lattice $\mathcal{N} = 4$ SYM formulation highlights

What: Recent, ongoing & planned work

Thermodynamics in 1+1 and 2+1 dimensions

(3+1)d static potential and scaling dimensions

Sign problems, supersymmetric QCD, ...
Motivations

Lattice field theory promises first-principles predictions for strongly coupled supersymmetric QFTs.
Lattice field theory in a nutshell

Formally \[ \langle O \rangle = \frac{1}{Z} \int \mathcal{D}\Phi \, O(\Phi) \, e^{-S[\Phi]} \]

Regularize by formulating theory in finite, discrete, euclidean space-time

\[ \overset{\leftarrow}{\text{Gauge invariant, non-perturbative, } d\text{-dimensional}} \]

Spacing between lattice sites ("a") \[ \rightarrow \text{UV cutoff scale } 1/a \]

Remove cutoff: \[ a \rightarrow 0 \ (L/a \rightarrow \infty) \]

Discrete \[ \rightarrow \text{continuous symmetries } \checkmark \]
Numerical lattice field theory calculations

High-performance computing $\longrightarrow$ evaluate up to $\sim$billion-dimensional integrals
(Dirac operator as $\sim 10^9 \times 10^9$ matrix)

Results to be shown, and work in progress, require state-of-the-art resources

Many thanks to USQCD–DOE, DiRAC–STFC–UKRI, and computing centres!
Importance sampling Monte Carlo

Algorithms sample field configurations with probability \( \frac{1}{\mathcal{Z}} e^{-S[\Phi]} \)

\[
\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \; O(\Phi) \; e^{-S[\Phi]} \rightarrow \frac{1}{N} \sum_{i=1}^{N} O(\Phi_i) \text{ with stat. uncertainty } \propto \frac{1}{\sqrt{N}}
\]
Supersymmetry must be broken on the lattice

Supersymmetry is a space-time symmetry, \((I = 1, \cdots, \mathcal{N})\)

adding spinor generators \(Q^I_\alpha\) and \(\overline{Q}^I_{\dot{\alpha}}\) to translations, rotations, boosts

\[
\{Q^I_\alpha, \overline{Q}^J_{\dot{\alpha}}\} = 2\delta^I_J \sigma^\mu_{\alpha\dot{\alpha}} P_\mu
\]

broken in discrete space-time

\[\longrightarrow\] relevant susy-violating operators

Scalar mass  Yukawas  Quartics  Quark mass  Gluino mass
Supersymmetry need not be *completely* broken on the lattice

Preserve susy sub-algebra in discrete lattice space-time

\[ \implies \text{correct continuum limit with little or no fine tuning} \]

Equivalent constructions from ‘topological’ twisting and dim’l deconstruction

Review:
Catterall–Kaplan–Ünsal,

arXiv:0903.4881

Need $2^d$ supersymmetries in $d$ dimensions

\[ d = 4 \implies \mathcal{N} = 4 \text{ super-Yang–Mills (SYM)} \]
Twisting $\mathcal{N} = 4$ SYM

Intuitive 4d picture — expand $4 \times 4$ matrix of supersymmetries

\[
\begin{pmatrix}
Q_1^{\alpha} & Q_2^{\alpha} & Q_3^{\alpha} & Q_4^{\alpha} \\
\bar{Q}_1^{\dot{\alpha}} & \bar{Q}_2^{\dot{\alpha}} & \bar{Q}_3^{\dot{\alpha}} & \bar{Q}_4^{\dot{\alpha}}
\end{pmatrix} = Q + Q_\mu \gamma_\mu + \bar{Q}_\mu \gamma_\mu \gamma_5 + \bar{Q} \gamma_5 \\
\rightarrow Q + Q_a \gamma_a + Q_{ab} \gamma_a \gamma_b \text{ with } a, b = 1, \cdots, 5
\]

Lorentz index $\times$ R-symmetry index $\rightarrow$ reps of ‘twisted rotation group’

\[
\text{SO(4)}_{tw} \equiv \text{diag} \left[ \text{SO(4)}_{\text{euc}} \otimes \text{SO(4)}_R \right]
\]

$\text{SO(4)}_R \subset \text{SO(6)}_R$

Change of variables $\rightarrow$ $Q$s transform with integer ‘spin’ under $\text{SO(4)}_{tw}$
Twisting $\mathcal{N} = 4$ SYM

Intuitive 4d picture — expand $4 \times 4$ matrix of supersymmetries

\[
\begin{pmatrix}
Q_1^\alpha & Q_2^\alpha & Q_3^\alpha & Q_4^\alpha \\
Q_1^{\dot{\alpha}} & Q_2^{\dot{\alpha}} & Q_3^{\dot{\alpha}} & Q_4^{\dot{\alpha}}
\end{pmatrix}
= Q + Q_\mu \gamma_\mu + Q_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{Q}_\mu \gamma_\mu \gamma_5 + \overline{Q} \gamma_5
\]

\[
\longrightarrow Q + Q_a \gamma_a + Q_{ab} \gamma_a \gamma_b
\]

with $a, b = 1, \ldots, 5$

Discrete space-time

Can preserve closed sub-algebra

\[
\{ Q, Q \} = 2Q^2 = 0
\]
Intuitive 4d picture — expand $4 \times 4$ matrix of supersymmetries

$$\begin{pmatrix}
Q_1^\alpha & Q_2^\alpha & Q_3^\alpha & Q_4^\alpha \\
\bar{Q}_1^{\dot{\alpha}} & \bar{Q}_2^{\dot{\alpha}} & \bar{Q}_3^{\dot{\alpha}} & \bar{Q}_4^{\dot{\alpha}}
\end{pmatrix} = Q + Q_{\mu} \gamma_{\mu} + Q_{\mu\nu} \gamma_{\mu} \gamma_{\nu} + \bar{Q}_{\mu} \gamma_{\mu} \gamma_{5} + \bar{Q} \gamma_{5}$$

$$\rightarrow Q + Q_{a} \gamma_{a} + Q_{ab} \gamma_{a} \gamma_{b}$$

with $a, b = 1, \cdots, 5$

Discrete space-time

Can preserve closed sub-algebra

$$\{Q, Q\} = 2Q^2 = 0$$
Completing the twist

Fields also transform with integer spin under $\text{SO}(4)_{\text{tw}}$ — no spinors

\[ \psi \text{ and } \bar{\psi} \rightarrow \eta, \psi_a \text{ and } \chi_{ab} \]

\[ A_\mu \text{ and } \phi^I \rightarrow \text{complexified gauge field } A_a \text{ and } \overline{A}_a \]

\[ \rightarrow \text{U}(N) = \text{SU}(N) \otimes \text{U}(1) \text{ gauge theory} \]

✓ $Q$ interchanges bosonic $\leftrightarrow$ fermionic d.o.f. with $Q^2 = 0$

\[ Q A_a = \psi_a \]

\[ Q \psi_a = 0 \]

\[ Q \chi_{ab} = -\tilde{F}_{ab} \]

\[ Q \overline{A}_a = 0 \]

\[ Q \eta = d \]

\[ Q d = 0 \]

\[ \text{bosonic auxiliary field with e.o.m. } d = \overline{D}_a A_a \]
Lattice $\mathcal{N} = 4$ SYM

Lattice theory looks nearly the same despite breaking $Q_a$ and $Q_{ab}$

Covariant derivatives $\longrightarrow$ finite difference operators

Complexified gauge fields $A_a \longrightarrow$ gauge links $U_a \in \mathfrak{gl}(N, \mathbb{C})$

\[ Q A_a \longrightarrow Q U_a = \psi_a \]
\[ Q \chi_{ab} = -\overline{F}_{ab} \]
\[ Q \eta = d \]

\[ Q \psi_a = 0 \]
\[ Q \overline{A}_a \longrightarrow Q \overline{U}_a = 0 \]
\[ Q d = 0 \]

**Geometry:** $\eta$ on sites, $\psi_a$ on links, etc.

Supersymmetric lattice action ($QS = 0$) from $Q^2 \cdot = 0$ and Bianchi identity

\[ S = \frac{N}{4\lambda_{\text{lat}}} \text{Tr} \left[ Q \left( \chi_{ab} F_{ab} + \eta \overline{D}_a U_a - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \overline{D}_c \chi_{de} \right] \]
Five links in four dimensions \( \longrightarrow \) \( A_4^* \) lattice

\[ A_4^* \sim 4\text{d analog of 2d triangular lattice} \]

Basis vectors linearly dependent and non-orthogonal

Large \( S_5 \) point group symmetry

\( S_5 \) irreps precisely match onto irreps of twisted \( SO(4)_{tw} \)

\[
\begin{align*}
\psi_a &\longrightarrow \psi_\mu, \quad \bar{\eta} &\quad \text{is} & \quad 5 \longrightarrow 4 \oplus 1 \\
\chi_{ab} &\longrightarrow \chi_{\mu\nu}, \quad \bar{\psi}_\mu &\quad \text{is} & \quad 10 \longrightarrow 6 \oplus 4
\end{align*}
\]

\( S_5 \longrightarrow SO(4)_{tw} \) in continuum limit restores \( Q_a \) and \( Q_{ab} \)
**Checkpoint**

**Analytic results for twisted $\mathcal{N} = 4$ SYM on $A_4^*$ lattice**

U($N$) gauge invariance + $Q$ + $S_5$ lattice symmetries

$\rightarrow$ Moduli space preserved to all orders

$\rightarrow$ One-loop lattice $\beta$ function vanishes

$\rightarrow$ Only one log. tuning to recover continuum $Q_a$ and $Q_{ab}$


**Not yet suitable for numerical calculations**

Must regulate zero modes and flat directions, especially in U(1) sector
Two deformations stabilize lattice calculations

1) Add SU($N$) scalar potential $\propto \mu^2 \sum_a (\text{Tr} \left[ U_a \bar{U}_a \right] - N)^2$

Softly breaks susy $\rightarrow$ $Q$-violating operators vanish $\propto \mu^2 \rightarrow 0$

Test via Ward identity violations

$Q \left[ \eta U_a \bar{U}_a \right] \neq 0$
Two deformations stabilize lattice calculations

2) Constrain U(1) plaquette determinant \( \sim G \sum_{a<b} (\det P_{ab} - 1) \)

Implemented supersymmetrically by modifying auxiliary field equations of motion

Test via Ward identity violations
\[ Q \left[ \eta U_a \bar{U}_a \right] \neq 0 \]

Log–log axes
\[ \rightarrow \text{violations } \propto (a/L)^2 \]
Public code for supersymmetric lattice field theories

so that the full improved action becomes

\begin{align}
S_{\text{imp}} &= S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \\
S'_{\text{exact}} &= \frac{N}{4\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ -\mathcal{F}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}_{a}^{(+)}\psi_{b}(n) - \eta(n)\mathcal{D}_{a}^{(-)}\psi_{a}(n) + \frac{1}{2} \left( \mathcal{D}_{a}^{(-)} U_{a}(n) + G \sum_{a \neq b} (\text{det} P_{ab}(n) - 1) I_N \right)^2 \right] - S_{\text{det}} \\
S_{\text{det}} &= \frac{N}{4\lambda_{\text{lat}}} G \sum_n \text{Tr} [\eta(n)] \sum_{a \neq b} [\text{det} P_{ab}(n)] \text{Tr} [U_{b}^{-1}(n)\psi_{b}(n) + U_{a}^{-1}(n + \hat{\mu}_{b})\psi_{a}(n + \hat{\mu}_{b})] \\
S_{\text{closed}} &= -\frac{N}{16\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ \epsilon_{abcdef} \chi_{dc}(n + \hat{\mu}_{a} + \hat{\mu}_{b} + \hat{\mu}_{c})\mathcal{D}_{c}^{(-)}\chi_{ab}(n) \right] , \\
S'_{\text{soft}} &= \frac{N}{4\lambda_{\text{lat}}} \mu^{2} \sum_n \sum_{a} \left( \frac{1}{N} \text{Tr} [U_{a}(n)\overline{U}_{a}(n)] - 1 \right)^{2}
\end{align}

\geq 100 \text{ inter-node data transfers in the fermion operator} \quad \text{— non-trivial. . .}

Public parallel code to reduce barriers to entry: github.com/daschaich/susy

Evolved from MILC QCD code, user guide in arXiv:1410.6971
Naive dimensional reduction $\rightarrow$ skewed tori

$r_L \times r_\beta$ with $r_\beta = \sqrt{\lambda}/T$ and four scalar $Q$

$r_1 \times r_2 \times r_\beta$ with $r_\beta = \lambda/T$ and two scalar $Q$

Thermal boundary conditions $\rightarrow$ dimensionless temperature $t = 1/r_\beta$

Low temperatures $t$ at large $N$

Black branes in dual supergravity
2d $\mathcal{N} = (8, 8)$ SYM phase diagram

First-order transitions predicted from bosonic QM at high $t$ ($r_\beta \ll 1$) from holography at low $t$ ($r_\beta \gg 1$)

For decreasing $r_L$ at large $N$

homogeneous black string (D1) \[\rightarrow\] localized black hole (D0)

“spatial deconfinement” signalled by Wilson line $P_L$
Spatial deconfinement transition signals — high-$t$ example

Fix aspect ratio $\alpha = r_L/r_\beta = 4$  
Check $16 \times 4$ vs. $24 \times 6$ lattices agree

Peaks in $\text{Tr} P_L$ susceptibility match change in its magnitude,  
grow with size of SU($N$) gauge group, comparing $N = 6, 9, 12$
Wilson line eigenvalues for low $t$

Large-$N$ eigenvalue phase distribution also signals spatial deconfinement

**Left:** $\alpha = 1/2$ distributions more localized as $N$ increases $\rightarrow$ D0 black hole

**Right:** $\alpha = 2$ distributions more uniform as $N$ increases $\rightarrow$ D1 black string
Lattice results for 2d $\mathcal{N} = (8, 8)$ SYM phase diagram

Good agreement with bosonic QM at high temperatures ($\alpha \gtrsim 4$)

Harder to control low-temperature uncertainties (larger $N > 16$ should help)

Overall consistent with holography

Comparing multiple lattice sizes and $6 \leq N \leq 16$

Controlled extrapolations not yet attempted in 2d
Holographic black hole energies

Lattice results consistent with leading expectation for sufficiently low $t \lesssim 0.4$

Similar behavior $\rightarrow$ difficult to distinguish phases

$\propto t^{3.2}$ for small-$r_L$ D0 phase

$\propto t^3$ for large-$r_L$ D1 phase

\[ \text{arXiv:1709.07025} \]
Compare with $\mathcal{N} = (2, 2)$ SYM

Much simpler twisted formulation: $Q = 4$ supercharges $\{Q, Q_a, Q_{ab}\} \rightarrow$ site / link / plaquette fermions $\{\eta, \psi_a, \chi_{ab}\}$ on square lattice $(a, b = 1, 2)$

Work by Navdeep Singh Dhindsa

Prelim. $\mu^2 \rightarrow 0$ extrapolations
for $r_L = r_\beta \leftrightarrow \alpha = 1$

Energy independent of $t \lesssim 0.33$
vs. $\sim t^3$ for $\mathcal{N} = (8, 8)$ SYM
3d maximal SYM

Holography $\rightarrow$ much richer low-$t$ phase diagram than for 2d $\mathcal{N} = (8, 8)$ SYM

For now consider simplest homogeneous black D2-branes $\rightarrow r_1 = r_2 = r_\beta$

\[\text{arXiv:1412.3939}\]
Homogeneous D2 phase

Lattice volume \((L/a)^3\) \(\longrightarrow\) continuum limit \(L/a \rightarrow \infty\) with fixed \(t = 1/r_\beta = L/\lambda\)

Homogeneous D2-branes \(\leftrightarrow\) uniform Wilson line eigenvalue phases at large \(N\)

Phase of unitarized Wilson line eigenvalues, \(L=12, t=0.31\)
Holographic black brane energies and continuum extrapolation

Lattice volume \((L/a)^3\) with fixed \(N = 8\)

\[ \rightarrow \] results approach leading holographic expectation \(\propto t^{10/3}\) for low \(t \lesssim 0.4\)

Carry out first \(L/a \to \infty\) continuum extrapolations
4d $\mathcal{N} = 4$ SYM static potential $V(r)$

Static probes $\rightarrow r \times T$ Wilson loops $W(r, T) \propto e^{-V(r)T}$

Coulomb gauge trick reduces $A_4^*$ lattice complications
Static potential is Coulombic at all $\lambda$

Fits to confining $V(r) = A - C/r + \sigma r \rightarrow$ vanishing string tension $\sigma$

Therefore fit

$$V(r) = A - C/r$$

to find Coulomb coefficient $C(\lambda)$

Discretization artifacts reduced by tree-level improved analysis
Coupling dependence of Coulomb coefficient

Continuum perturbation theory $\rightarrow$ \( C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2) \)

Holography $\rightarrow$ \( C(\lambda) \propto \sqrt{\lambda} \) for \( N \to \infty \) and \( \lambda \to \infty \) with \( \lambda \ll N \)

Again comparing different volumes and \( N = 2, 3, 4 \)

For \( \lambda_{\text{lat}} \leq 2 \), consistent with leading-order perturbation theory
Konishi operator scaling dimension $\Delta_K$

$\mathcal{O}_K(x) = \sum I \text{Tr} [\Phi^I(x)\Phi^I(x)]$ is simplest conformal primary operator

Scaling dimension $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$ investigated through perturbation theory (& S duality), holography, conformal bootstrap

$C_K(r) \equiv \mathcal{O}_K(x + r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$

‘SUGRA’ is 20’ op., $\Delta_S = 2$

Work in progress to compare:
- Direct power-law decay
- Finite-size scaling
- Monte Carlo RG
Konishi operator scaling dimension $\Delta_K$

Lattice scalars $\varphi(n)$ from polar decomposition $U_a(n) = e^{\varphi_a(n)} U_a(n)$

$$O_{\text{lat}}^K(n) = \sum_a \text{Tr} [\varphi_a(n) \varphi_a(n)] - \text{vev}$$

$$O_{\text{lat}}^S(n) \sim \text{Tr} [\varphi_a(n) \varphi_b(n)]$$

$$C_K(r) \equiv O_K(x + r) O_K(x) \propto r^{-2\Delta_K}$$

‘SUGRA’ is 20’ op., $\Delta_S = 2$

Work in progress to compare:
- Direct power-law decay
- Finite-size scaling
- Monte Carlo RG
Preliminary $\Delta_K$ results from Monte Carlo RG

Analyzing both $\mathcal{O}_K^{\text{lat}}$ and $\mathcal{O}_S^{\text{lat}}$

Imposing protected $\Delta_S = 2$

$\rightarrow \Delta_K(\lambda)$ looks perturbative

Systematic uncertainties from different amounts of smearing

Complication from twisting $\text{SO}(4)_R \subset \text{SO}(6)_R$

$\mathcal{O}_K^{\text{lat}}$ mixes with $\text{SO}(4)_R$-singlet part of $\text{SO}(6)_R$-nonsinglet $\mathcal{O}_S$

$\rightarrow$ disentangle via variational analyses
Supplement: Pushing $\mathcal{N} = 4$ SYM to stronger coupling

✓ Reproduce reliable 4d results in perturbative regime

→ Check holographic predictions and access new domains

Sign problem seems to become obstruction

\[ \langle \mathcal{O} \rangle = \frac{1}{Z} \int [dU][d\bar{U}] \mathcal{O} e^{-S_{B[U,\bar{U}]} \text{ pf } D[U,\bar{U}]} \]

Complex pfaffian $\text{ pf } D = |\text{ pf } D| e^{i\alpha}$ complicates importance sampling

We phase quench, $\text{ pf } D \longrightarrow |\text{ pf } D|$, need to reweight
\[ \langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}} \]
Supplement: Pushing $\mathcal{N} = 4$ SYM to stronger coupling

Sign problem seems to become obstruction

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int [dU][d\overline{U}] \mathcal{O} \ e^{-S_B[U,\overline{U}]} \ pf \mathcal{D}[U,\overline{U}]$$

Complex pfaffian $pf \mathcal{D} = |pf \mathcal{D}| e^{i\alpha}$ complicates importance sampling

We phase quench, $pf \mathcal{D} \rightarrow |pf \mathcal{D}|$, need to reweight

$$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}}$$

$$\rightarrow \langle e^{i\alpha} \rangle_{pq} = \frac{Z}{Z_{pq}} \text{ quantifies severity of sign problem}$$
$\mathcal{N} = 4$ SYM sign problem

Fix $\lambda_{\text{lat}} = g_{\text{lat}}^2, N = 0.5$

Pfaffian nearly real positive
for all accessible volumes

Fix $4^4$ volume
Fluctuations increase with coupling
Signal-to-noise
becomes obstruction for $\lambda_{\text{lat}} \gtrsim 4$
Supplement: Supersymmetric QCD

Add matter multiplets → investigate electric–magnetic dualities, dynamical supersymmetry breaking and more

Quiver construction based on twisted SYM
preserves susy sub-algebra in $(d - 1)$ dims. to reduce fine-tuning

[arXiv:1505.00467]
Quiver superQCD from twisted SYM

2-slice lattice SYM
with $U(N) \times U(F)$ gauge group
Adj. fields on each slice
Bi-fundamental in between

Decouple $U(F)$ slice
\[ \rightarrow U(N) \text{ SQCD in } (d - 1) \text{ dims.} \]
with $F$ fund. hypermultiplets

First check 3d SYM \[ \rightarrow \text{ 2d superQCD} \]
then new 4d SYM \[ \rightarrow \text{ 3d superQCD} \]
First step: 8-supercharge SYM in 3d

Simpler twisted formulation

\[ Q = 8 \text{ supercharges } \{Q, Q_a, Q_{ab}, Q_{abc}\} \text{ with } a, b = 1, \cdots, 3 \]

\[ \rightarrow \text{ site / link / plaquette / cube fermions } \{\eta, \psi_a, \chi_{ab}, \theta_{abc}\} \text{ on simple cubic lattice} \]
First step: 8-supercharge SYM in 3d

Simpler twisted formulation

\[ Q = 8 \text{ supercharges } \{ Q, Q_a, Q_{ab}, Q_{abc} \} \text{ with } a, b = 1, \ldots, 3 \]

\[ \rightarrow \text{ site / link / plaquette / cube fermions } \{ \eta, \psi_a, \chi_{ab}, \theta_{abc} \} \text{ on simple cubic lattice} \]

Work by Angel Sherletov

Parallel code developed

Initial tests passed

\[ \rightarrow \text{ larger-scale calculations} \]
Recap: An exciting time for lattice supersymmetry

✓ Preserve (some) susy in discrete space-time

→ practical lattice $\mathcal{N} = 4$ SYM, public code available

Reproduce reliable analytic results

✓ 2d and 3d thermodynamics consistent with holography

✓ Perturbative $\mathcal{N} = 4$ SYM static potential Coulomb coefficient $C(\lambda)$
  and Konishi operator scaling dimension $\Delta_K(\lambda)$

Access new domains → sign problems, supersymmetric QCD and more...
Thanks for your attention! Any further questions?

Collaborators
Raghav Jha, Anosh Joseph, Angel Sherletov, Toby Wiseman
also Georg Bergner, Simon Catterall, Poul Damgaard, Joel Giedt

Funding and computing resources
Backup: Breakdown of Leibniz rule on the lattice

\[ \{ Q_\alpha, \bar{Q}_{\dot{\alpha}} \} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = 2i\sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \] is problematic

\[ \implies \text{try finite difference} \quad \partial_\phi(x) \rightarrow \Delta_\phi(x) = \frac{1}{a} [\phi(x + a) - \phi(x)] \]

Crucial difference between \( \partial \) and \( \Delta \)

\[ \Delta [\phi \eta] = a^{-1} [\phi(x + a) \eta(x + a) - \phi(x) \eta(x)] \]

\[ = [\Delta \phi] \eta + \phi \Delta \eta + a [\Delta \phi] \Delta \eta \]

Full supersymmetry requires Leibniz rule \( \partial [\phi \eta] = [\partial \phi] \eta + \phi \partial \eta \) only recovered in \( a \to 0 \) continuum limit for any local finite difference
Backup: Breakdown of Leibniz rule on the lattice

Full supersymmetry requires Leibniz rule $\partial [\phi \eta] = [\partial \phi] \eta + \phi \partial \eta$
only recovered in $a \to 0$ continuum limit for any local finite difference

Supersymmetry vs. locality ‘no-go’ theorems

Complicated constructions to balance locality vs. supersymmetry
Non-ultralocal product operator $\longrightarrow$ lattice Leibniz rule but not gauge invariance

Cyclic Leibniz rule $\longrightarrow$ partial lattice supersymmetry but only (0+1)d QM so far
Backup: $\mathcal{N} = 4$ SYM in a nutshell

Arguably simplest non-trivial 4d QFT $\rightarrow$ dualities, amplitudes, ... 

SU($N$) gauge theory with $\mathcal{N} = 4$ fermions $\Psi^I$ and 6 scalars $\Phi^{IJ}$, all massless and in adjoint rep.

**Symmetries** relate coefficients of kinetic, Yukawa and $\Phi^4$ terms

Maximal 16 supersymmetries $Q_I^\alpha$ and $\overline{Q}^I_{\dot{\alpha}}$, $I = 1, \cdots, 4$

transform under global $SU(4) \sim SO(6)$ $R$ symmetry

**Conformal** $\rightarrow$ $\beta$ function is zero for all values of $\lambda = g^2 N$
Backup: Complexified gauge field from twisting

Combining $A_\mu$ and $\Phi^I \rightarrow A_a$ and $\bar{A}_a$

produces $U(N) = SU(N) \otimes U(1)$ gauge theory

Complicates lattice action but needed so that $Q A_a = \psi_a$

Further motivation: Under $SO(d)_{\text{tw}} = \text{diag}[SO(d)_{\text{euc}} \otimes SO(d)_R]$

$A_\mu \sim \text{vector} \otimes \text{scalar} = \text{vector}$
$\Phi^I \sim \text{scalar} \otimes \text{vector} = \text{vector}$

Easiest to see in 5d, then dimensionally reduce

$A_a = A_a + i\Phi_a \rightarrow (A_\mu, \phi) + i(\Phi_\mu, \bar{\phi})$
Backup: $A_4^*$ lattice from five dimensions

Again dimensionally reduce, treating all five gauge links symmetrically

Start with hypercubic lattice in 5d momentum space

**Symmetric** constraint $\sum_a \partial_a = 0$

projects to 4d momentum space

Result is $A_4$ lattice

$\rightarrow$ dual $A_4^*$ lattice in position space
Backup: Restoration of $Q_a$ and $Q_{ab}$ supersymmetries

"$Q + \text{discrete } R_a \subset \text{SO}(4)_{\text{tw}} = Q_a \text{ and } Q_{ab}$"

[arXiv:1306.3891]

Test $R_a$ on Wilson loops

$\tilde{\mathcal{W}}_{ab} \equiv R_a \mathcal{W}_{ab}$

Tune coeff. $c_2$ of $d^2$ term in action for fastest restoration towards continuum limit
Backup: Problem with SU($N$) flat directions

$\mu^2/\lambda_{\text{lat}}$ too small $\rightarrow U_a$ can move far from continuum form $\mathbb{I}_N + \mathcal{A}_a$

Example: $\mu = 0.2$ and $\lambda_{\text{lat}} = 2.5$ on $8^3 \times 24$ volume

**Left:** Bosonic action stable $\sim 18\%$ off its supersymmetric value

**Right:** (Complexified) Polyakov loop wanders off to $\sim 10^9$
Backup: Problem with U(1) flat directions

Monopole condensation $\rightarrow$ confined lattice phase not present in continuum

Around the same $2\lambda_{\text{lat}} \approx 2.0$

- **Left:** Polyakov loop falls towards zero
- **Center:** Plaquette determinant falls towards zero
- **Right:** Density of U(1) monopole world lines becomes non-zero
Backup: Naively regulating U(1) flat directions

In earlier work we added another soft $Q$-breaking term

$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}} \mu^2 \sum_a \left( \frac{1}{N} \text{Tr} [U \overline{U}_a] - 1 \right)^2 + \kappa \sum_{a<b} |\det \mathcal{P}_{ab} - 1|^2$$

More sensitivity to $\kappa$ than to $\mu^2$

Showing $Q$ Ward identity from bosonic action

$$\langle s_B \rangle = \frac{9N^2}{2}$$
Backup: Better regulating U(1) flat directions

\[
S = \frac{N}{4\lambda_{\text{lat}}} \left[ Q \left( \chi_{ab} F_{ab} + \eta \left\{ \overline{D}_a U_a + G \sum_{a<b} [\det P_{ab} - 1] \mathbb{I}_N \right\} - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \overline{D}_c \chi_{de} + \mu^2 V \right]
\]

- Ward identity violations scale \( \propto 1/N^2 \) (left) and \( \propto (a/L)^2 \) (right)
- \( \sim \) effective ‘\( \mathcal{O}(a) \) improvement’ since \( Q \) forbids all dim-5 operators
Method to impose $Q$-invariant constraints on generic site operator $\mathcal{O}(n)$

Modify auxiliary field equations of motion $\rightarrow$ moduli space

\[ d(n) = \overline{D}_a^{(-)} U_a(n) \quad \rightarrow \quad d(n) = \overline{D}_a^{(-)} U_a(n) + G\mathcal{O}(n)\mathbb{I}_N \]

Including both $U(1)$ and $SU(N) \in \mathcal{O}(n)$ over-constrains system
Backup: Dimensional reduction to 2d $\mathcal{N} = (8, 8)$ SYM

Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_x = N_y = 1$

$A_4^* \rightarrow A_2^*$ (triangular) lattice

Torus **skewed** depending on $\alpha = L/N_t$

Modular transformation into fundamental domain

$\rightarrow$ some skewed tori actually rectangular

Also need to stabilize compactified links to ensure broken center symmetries

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David Schaich (Liverpool)
Backup: Stabilizing compactified links

Add potential $\propto \text{Tr} \left[ (\varphi - I_N)^\dagger (\varphi - I_N) \right]$ to break center symmetry in reduced dir(s) (~Kaluza–Klein rather than Eguchi–Kawai reduction)
Backup: High-temperature \((t \gtrsim 1)\) 3d maximal SYM

Wilson line eigenvalue phases localized rather than uniform (left)

Thermodynamics consistent with weak-coupling expectation \(\propto t^3\) (right)
Backup: Static potential is Coulombic at all $\lambda$

String tension $\sigma$ from fits to confining form $V(r) = A - C/r + \sigma r$

Slightly negative values flatten $V(r_i)$ for $r_i \lesssim L/2$

$\sigma \rightarrow 0$ as accessible range of $r_i$ increases on larger volumes
Discretization artifacts visible at short distances where Coulomb term in \( V(r) = A - C/r \) is most significant.
Backup: Tree-level improvement

Classic trick to reduce discretization artifacts in static potential

Associate $V(r_\nu)$ data with ‘$r_i$’ from Fourier transform of gluon propagator

Recall $\frac{1}{4\pi^2 r^2} = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{e^{i r_{\nu} k_{\nu}}}{k^2}$ where $\frac{1}{k^2} = G(k_{\nu})$ in continuum

$A^*_4$ lattice $\rightarrow \frac{1}{r_i^2} = 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \frac{\cos(\nu \hat{k}_{\nu})}{4 \sum_{\mu=1}^{4} \sin^2 \left(\hat{k} \cdot \hat{e}_\mu / 2\right)}$

Tree-level lattice propagator from arXiv:1102.1725

$\hat{e}_\mu$ are $A^*_4$ lattice basis vectors;

momenta $\hat{k} = \frac{2\pi}{L} \sum_{\mu=1}^{4} n_\mu \hat{g}_\mu$ depend on dual basis vectors
Backup: Tree-level-improved static potential

\[ \frac{1}{r_i^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4\hat{k}}{(2\pi)^4} \frac{\cos \left(i r_{\nu} \hat{k}_{\nu}\right)}{4 \sum_{\mu=1}^{4} \sin^2 \left(\hat{k} \cdot \hat{e}_\mu / 2\right)} \]

\[ \rightarrow \text{significantly reduced discretization artifacts} \]
Lattice system: \[ H = \sum_i c_i O_i \] (infinite sum)

Couplings flow under RG blocking \[ \rightarrow H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} O_i^{(n)} \]

Conformal fixed point \[ \rightarrow H^* = R_b H^* \text{ with couplings } c_i^* \]

Linear expansion around fixed point \[ \rightarrow \text{stability matrix } T_{ik}^* \]

\[ c_i^{(n)} - c_i^* = \sum_k \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \bigg|_{H^*} \left( c_k^{(n-1)} - c_k^* \right) \equiv \sum_k T_{ik}^* \left( c_k^{(n-1)} - c_k^* \right) \]

Correlators of \[ O_i, O_k \rightarrow \text{elements of stability matrix} \] [Swendsen, 1979]

Eigenvalues of \[ T_{ik}^* \rightarrow \text{scaling dimensions of corresponding operators} \]
Backup: Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve $Q$ and $S_5$ symmetries $\leftrightarrow$ geometric structure

Simple transformation constructed in arXiv:1408.7067

$$U'_a(n') = \xi U_a(n) U_a(n + \hat{\mu}_a)$$  \hspace{1cm} \eta'(n') = \eta(n)$$

$$\psi'_a(n') = \xi [\psi_a(n) U_a(n + \hat{\mu}_a) + U_a(n) \psi_a(n + \hat{\mu}_a)]$$  \hspace{1cm} \text{etc.}$$

Doubles lattice spacing $a \longrightarrow a' = 2a$, with tunable rescaling factor $\xi$

Scalar fields from polar decomposition $U(n) = e^{\varphi(n)} U(n)$

$\Longrightarrow$ shift $\varphi \longrightarrow \varphi + \log \xi$ to keep blocked $U$ unitary

$Q$-preserving RG transformation needed

to show only one log. tuning to recover continuum $Q_a$ and $Q_{ab}$
Smear to enlarge (MCRG or variational) operator basis

APE-like smearing: \[ (1 - \alpha) \rightarrow \sum \cap \alpha \delta \]

staples built from unitary parts of links but no final unitarization

Average plaquette stable upon smearing (right), minimum plaquette steadily increases (left)
Backup: Dynamical susy breaking in 2d lattice superQCD

U(\(N\)) superQCD with \(F\) fundamental hypermultiplets

Observe spontaneous susy breaking only for \(N > F\), as expected

Catterall–Veernala, arXiv:1505.00467
Spontaneous susy breaking means $\langle 0 | H | 0 \rangle > 0$ or equivalently $\langle QO \rangle \neq 0$

Twisted superQCD auxiliary field e.o.m. $\leftrightarrow$ Fayet–Iliopoulos $D$-term potential

$$d = \overline{D}a U_a + \sum_{i=1}^{F} \phi_i \overline{\phi}_i - r I_N \quad \leftrightarrow \quad \text{Tr} \left[ \left( \sum_i \phi_i \overline{\phi}_i - r I_N \right)^2 \right] \in H$$

Have $F \times N$ scalar vevs to zero out $N \times N$ matrix

$\rightarrow N > F$ suggests susy breaking, $\langle 0 | H | 0 \rangle > 0 \leftrightarrow \langle Q\eta \rangle = \langle d \rangle \neq 0$