Lattice studies of maximally supersymmetric Yang–Mills theories

David Schaich (Liverpool)

Perimeter Quantum Fields & Strings Seminar, 10 January 2020

Overview and plan

Why: Lattice supersymmetry

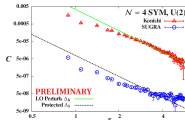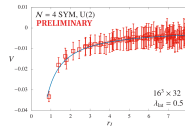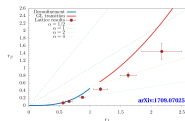**How:** Lattice formulation highlights

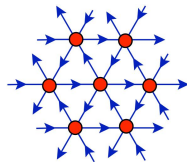**What:** Recent results

Dimensionally reduced (2d) thermodynamics

Static potential (4d)

Conformal scaling dimensions

Prospects and future directions

Overview and plan

Central idea

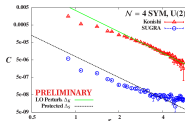Preserve (some) susy in discrete space-time

$\longrightarrow$ practical lattice investigations

Goals

Reproduce reliable results

in perturbative and holographic regimes

Access new domains
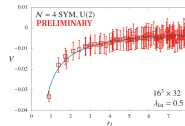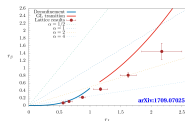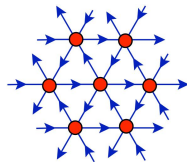
Lattice field theory in a nutshell

Formally $\quad \langle \mathcal{O} \rangle \; = \; \dfrac{1}{\mathcal{Z}} \displaystyle\int \mathcal{D}\Phi \;\; \mathcal{O}(\Phi) \; e^{-S[\Phi]}$

Regularize by formulating theory in finite, discrete, euclidean space-time
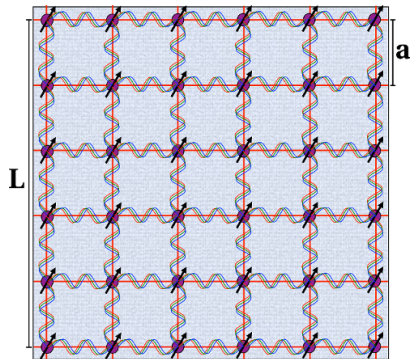
P. Vranas LLNL

Spacing between lattice sites ("*a*")
$\qquad\qquad\qquad \longrightarrow$ UV cutoff scale $1/a$

Remove cutoff: $\quad a \to 0 \quad (L/a \to \infty)$

Discrete $\longrightarrow$ continuous symmetries $\checkmark$

Motivations

Lattice field theory promises first-principles predictions
for strongly coupled supersymmetric QFTs

BSM   **QFT**   **Holography**

(Derek Leinweber)
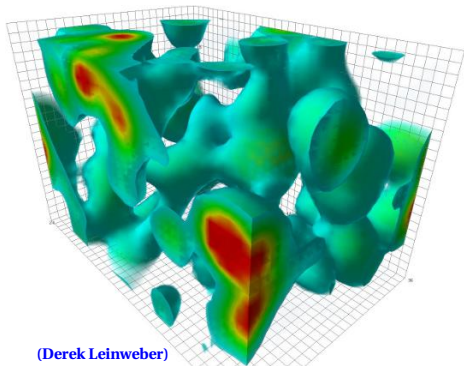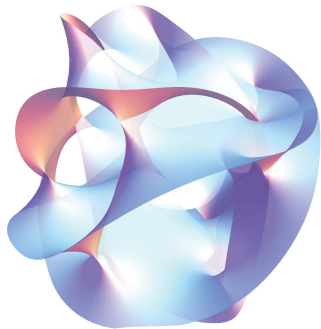
Supersymmetry must be broken on the lattice

Supersymmetry is a space-time symmetry, $(I = 1, \cdots, \mathcal{N})$
  adding spinor generators Q_α^I and $\overline{Q}_{\dot\alpha}^I$ to translations, rotations, boosts

$\left\{ Q_\alpha^I, \overline{Q}_{\dot\alpha}^J \right\} = 2\delta^{IJ}\sigma_{\alpha\dot\alpha}^\mu P_\mu$ broken in discrete space-time

$\longrightarrow$ relevant susy-violating operators

Scalar mass   **Yukawas**   **Quartics**   **Quark mass**   **Gluino mass**

Supersymmetry need not be *completely* broken on the lattice

Preserve susy sub-algebra at non-zero lattice spacing
$\implies$ correct continuum limit with little or no fine tuning

Equivalent constructions from 'topological' twisting and dim'l deconstruction

Review:
    arXiv:0903.4881

Need 2^d supersymmetries in d dimensions

$d = 4 \longrightarrow \mathcal{N} = 4$ super-Yang–Mills (SYM)

Twisting $\mathcal{N} = 4$ SYM
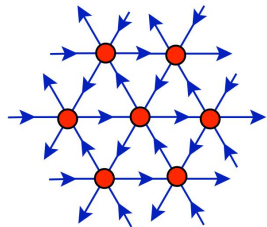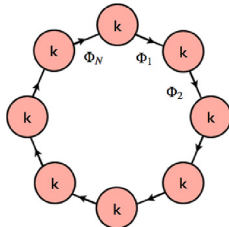
Intuitive picture — expand 4×4 matrix of supersymmetries

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}} \gamma_5$$
$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$
$$\text{with } a, b = 1, \cdots, 5$$

R-symmetry index $\times$ Lorentz index $\implies$ reps of 'twisted rotation group'

$$\mathsf{SO}(4)_{tw} \equiv \mathsf{diag}\left[\mathsf{SO}(4)_{\text{euc}} \otimes \mathsf{SO}(4)_R\right] \qquad \qquad \mathsf{SO}(4)_R \subset \mathsf{SO}(6)_R$$

Change of variables $\longrightarrow$ $\mathcal{Q}$ transform with integer 'spin' under $\mathsf{SO}(4)_{tw}$

Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand 4×4 matrix of supersymmetries

$$\begin{pmatrix} Q_\alpha^1 & Q_\alpha^2 & Q_\alpha^3 & Q_\alpha^4 \\ \overline{Q}_{\dot{\alpha}}^1 & \overline{Q}_{\dot{\alpha}}^2 & \overline{Q}_{\dot{\alpha}}^3 & \overline{Q}_{\dot{\alpha}}^4 \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}} \gamma_5$$
$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$
$$\text{with } a, b = 1, \cdots, 5$$

Discrete space-time

Can preserve closed sub-algebra

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$$

Twisting $\mathcal{N} = 4$ SYM

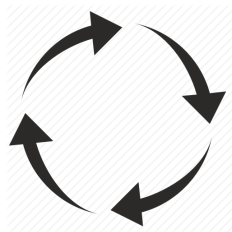Intuitive picture — expand 4×4 matrix of supersymmetries

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$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b$$
$$\text{with } a, b = 1, \cdots, 5$$

Discrete space-time

Can preserve closed sub-algebra

$$\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$$

Completing the twist

> Fields also transform with integer spin under $SO(4)_{tw}$ — no spinors
>
> $$\Psi \text{ and } \overline{\Psi} \longrightarrow \eta, \ \psi_a \text{ and } \chi_{ab}$$
>
> $$A_\mu \text{ and } \Phi^I \longrightarrow \text{complexified gauge field } \mathcal{A}_a \text{ and } \overline{\mathcal{A}}_a$$
> $$\longrightarrow U(N) = SU(N) \otimes U(1) \text{ gauge theory}$$

$\checkmark$ $\mathcal{Q}$ interchanges bosonic $\longleftrightarrow$ fermionic d.o.f. with $\mathcal{Q}^2 = 0$

$$\mathcal{Q} \, \mathcal{A}_a = \psi_a \qquad\qquad \mathcal{Q} \, \psi_a = 0$$
$$\mathcal{Q} \, \chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q} \, \overline{\mathcal{A}}_a = 0$$
$$\mathcal{Q} \, \eta = d \qquad\qquad \mathcal{Q} \, d = 0$$

$\nwarrow$ bosonic auxiliary field with e.o.m. $d = \overline{\mathcal{D}}_a \mathcal{A}_a$

Lattice $\mathcal{N} = 4$ SYM

Lattice theory looks nearly the same despite breaking $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Covariant derivatives $\longrightarrow$ finite difference operators

Complexified gauge fields $\mathcal{A}_a \longrightarrow$ gauge links $\mathcal{U}_a \in \mathfrak{gl}(N, \mathbb{C})$

$$\mathcal{Q}\,\mathcal{A}_a \longrightarrow \mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad\qquad\qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{A}}_a \longrightarrow \mathcal{Q}\,\overline{\mathcal{U}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad\qquad\qquad\qquad \mathcal{Q}\,d = 0$$

Geometry: η on sites, ψ_a on links, etc.

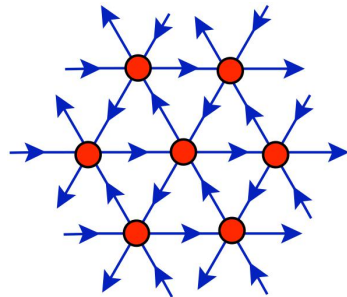Supersymmetric lattice action $(\mathcal{Q}S = 0)$ from $\mathcal{Q}^2 \cdot = 0$ and Bianchi identity

$$S = \frac{N}{4\lambda_{\text{lat}}}\text{Tr}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de}\right]$$

Five links in four dimensions $\longrightarrow A_4^*$ lattice

$A_4^* \sim$ 4d analog of 2d triangular lattice

Basis vectors linearly dependent and non-orthogonal

Large S_5 point group symmetry

S_5 irreps precisely match onto irreps of twisted SO(4)$_{tw}$

$$\psi_a \longrightarrow \psi_\mu, \ \overline{\eta} \qquad \text{is} \qquad \mathbf{5} \longrightarrow \mathbf{4} \oplus \mathbf{1}$$

$$\chi_{ab} \longrightarrow \chi_{\mu\nu}, \ \overline{\psi}_\mu \qquad \text{is} \qquad \mathbf{10} \longrightarrow \mathbf{6} \oplus \mathbf{4}$$

$S_5 \longrightarrow$ SO(4)$_{tw}$ in continuum limit restores $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Checkpoint

Analytic results for twisted $\mathcal{N} = 4$ SYM on A_4^* lattice

U(N) gauge invariance + $\mathcal{Q}$ + S_5 lattice symmetries

$\longrightarrow$ Moduli space preserved to all orders

$\longrightarrow$ One-loop lattice β function vanishes

$\longrightarrow$ Only one log. tuning to recover continuum $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

[arXiv:1102.1725, arXiv:1306.3891, arXiv:1408.7067]

Not yet suitable for numerical calculations

Must regulate zero modes and flat directions, especially in U(1) sector

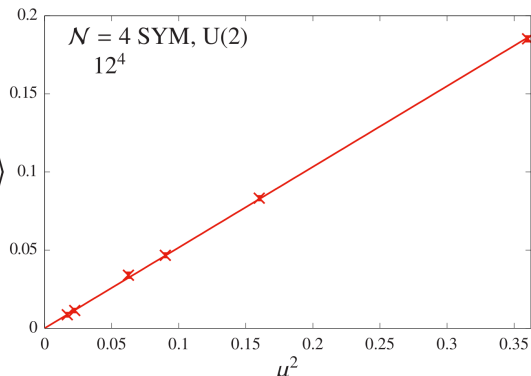## Two deformations stabilize lattice calculations

(i) Add SU(N) scalar potential $\propto \mu^2 \sum_a \left(\text{Tr} \left[\mathcal{U}_a \overline{\mathcal{U}}_a \right] - N \right)^2$

Softly breaks susy $\longrightarrow$ $\mathcal{Q}$-violating operators vanish $\propto \mu^2 \to 0$

Test via Ward identity violations
$$\mathcal{Q} \left[\eta \mathcal{U}_a \overline{\mathcal{U}}_a \right] \neq 0$$

$\mathcal{N} = 4$ SYM, U(2)
12^4

$\left\langle \frac{|QO|}{\sqrt{D^2+F^2}} \right\rangle$ vs μ^2

Two deformations stabilize lattice calculations

(ii) Constrain U(1) plaquette determinant $\sim G \sum_{a<b} (\det \mathcal{P}_{ab} - 1)$

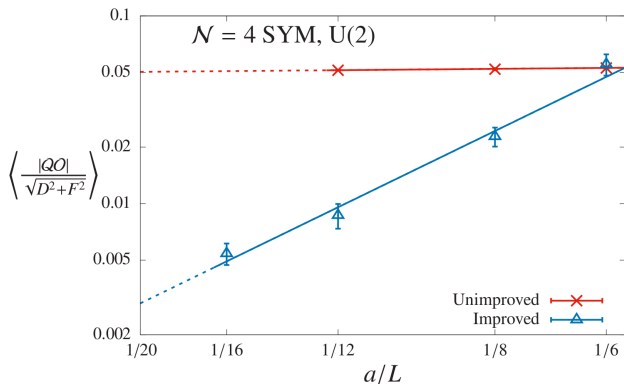Implemented supersymmetrically as Fayet–Iliopoulos D-term potential

Test via Ward identity violations
$$\mathcal{Q}\left[\eta \mathcal{U}_a \overline{\mathcal{U}}_a\right] \neq 0$$

Log–log axes
$\longrightarrow$ violations $\propto (a/L)^2$

Advertisement: Public code for lattice $\mathcal{N} = 4$ SYM

so that the full improved action becomes

$$S_{\text{imp}} = S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \tag{18}$$

$$S'_{\text{exact}} = \frac{N}{4\lambda_{\text{lat}}} \sum_n \text{Tr}\Bigg[-\overline{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi_{b]}(n) - \eta(n)\overline{\mathcal{D}}^{(-)}_a\psi_a(n)$$

$$+ \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_a\mathcal{U}_a(n) + G\sum_{a\neq b}(\det\mathcal{P}_{ab}(n) - 1)\,\mathbb{I}_N\right)^2\Bigg] - S_{\text{det}}$$

$$S_{\text{det}} = \frac{N}{4\lambda_{\text{lat}}}G\sum_n \text{Tr}\,[\eta(n)]\sum_{a\neq b}[\det\mathcal{P}_{ab}(n)]\,\text{Tr}\,\left[\mathcal{U}^{-1}_b(n)\psi_b(n) + \mathcal{U}^{-1}_a(n+\widehat{\mu}_b)\psi_a(n+\widehat{\mu}_b)\right]$$

$$S_{\text{closed}} = -\frac{N}{16\lambda_{\text{lat}}}\sum_n \text{Tr}\,\left[\epsilon_{abcde}\,\chi_{de}(n+\widehat{\mu}_a+\widehat{\mu}_b+\widehat{\mu}_c)\overline{\mathcal{D}}^{(-)}_c\chi_{ab}(n)\right],$$

$$S'_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}}\mu^2\sum_n\sum_a\left(\frac{1}{N}\text{Tr}\,\left[\mathcal{U}_a(n)\overline{\mathcal{U}}_a(n)\right] - 1\right)^2$$

$\gtrsim$100 inter-node data transfers in the fermion operator — non-trivial. . .

Public parallel code to reduce barriers to entry: github.com/daschaich/susy

Evolved from MILC QCD code, user guide in arXiv:1410.6971

Dimensionally reduce to (deconfined) 2d $\mathcal{N} = (8,8)$ SYM with four scalar $\mathcal{Q}$

Low temperatures $t = 1/r_\beta \longleftrightarrow$ black holes in dual supergravity
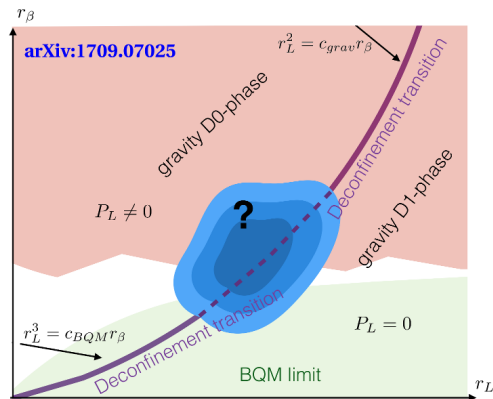
For decreasing r_L at large N

homogeneous black string (D1)
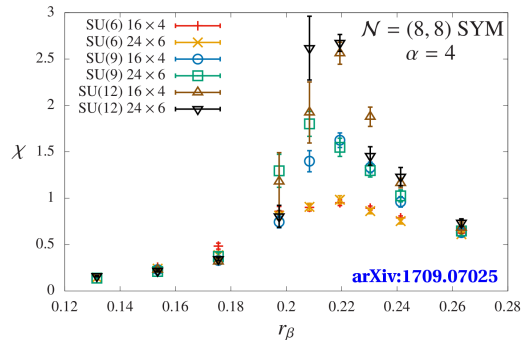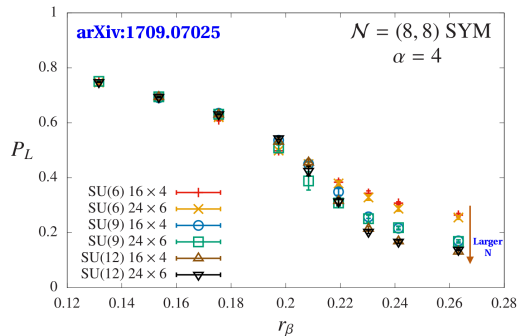$\qquad \longrightarrow$ localized black hole (D0)

$\updownarrow$

"spatial deconfinement"
$\qquad$ signalled by Wilson line P_L
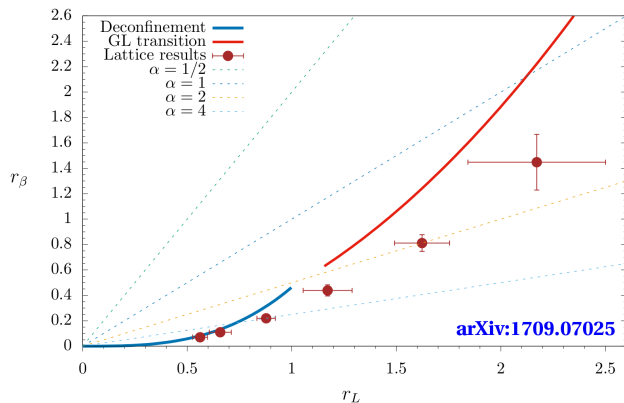
Spatial deconfinement transition signals

Peaks in Wilson line susceptibility match change in its magnitude $|P_L|$,

grow with size of SU(N) gauge group, comparing $N = 6, 9, 12$

Agreement for 16×4 vs. 24×6 lattices (aspect ratio $\alpha = r_L/r_\beta = 4$)

Lattice 2d $\mathcal{N} = (8, 8)$ SYM phase diagram

Large $\alpha = r_L / r_\beta \gtrsim 4 \longrightarrow$ good agreement with high-temperature bosonic QM

Small $\alpha \lesssim 2 \longrightarrow$ harder to control uncertainties with $6 \leq N \leq 16$

Overall consistent with holography

Comparing multiple lattice sizes

Controlled extrapolations
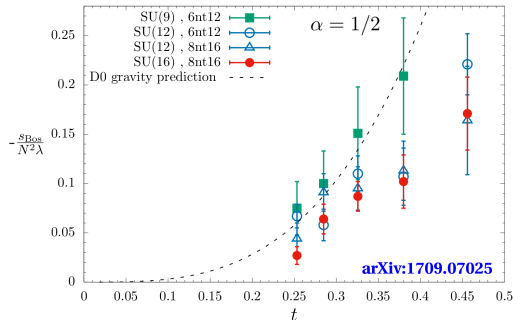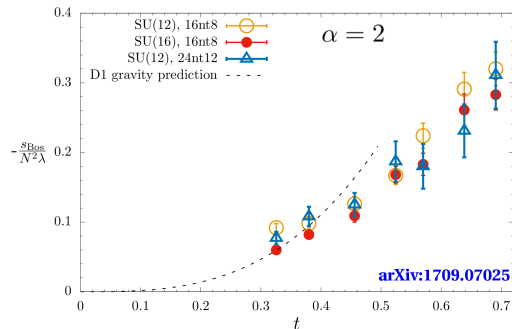   are work in progress

arXiv:1709.07025

Dual black hole thermodynamics

Dual black hole energy from 2d $\mathcal{N} = (8,8)$ SYM

$\propto t^3$ for large-r_L D1 phase $\qquad\qquad$ $\propto t^{3.2}$ for small-r_L D0 phase
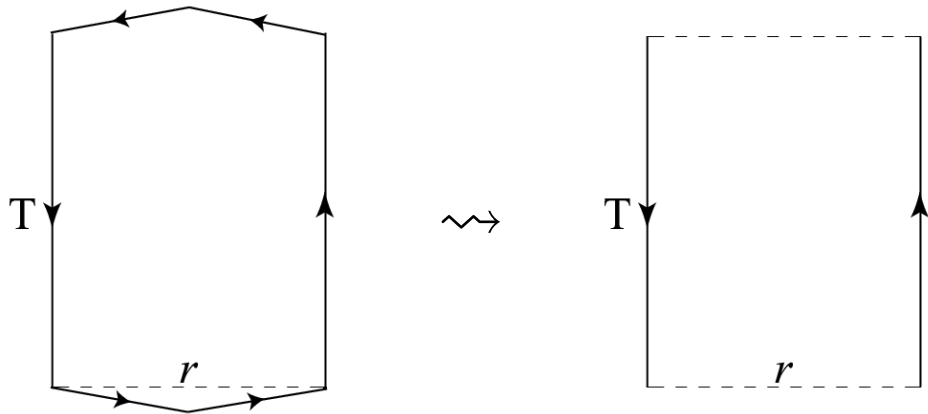
Lattice results consistent with holography for sufficiently low $t \lesssim 0.4$

(ii) 4d $\mathcal{N} = 4$ SYM static potential $V(r)$

Static probes $\longrightarrow$ $r \times T$ Wilson loops $\qquad W(r, T) \propto e^{-V(r)\, T}$

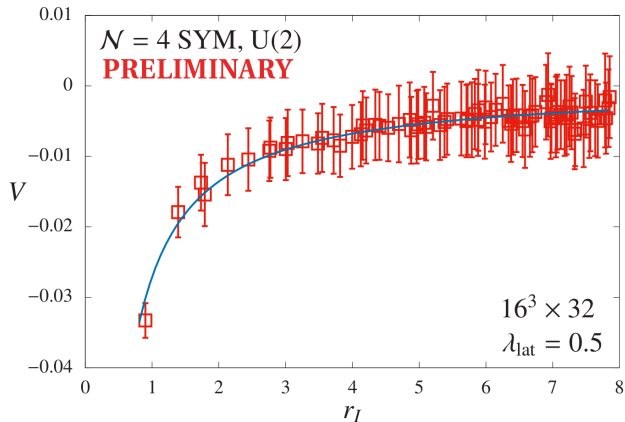Coulomb gauge trick reduces A_4^* lattice complications

Static potential is Coulombic at all λ

Fits to confining $V(r) = A - C/r + \sigma r \longrightarrow$ vanishing string tension σ

$\Longrightarrow$ Fit to just $V(r) = A - C/r$
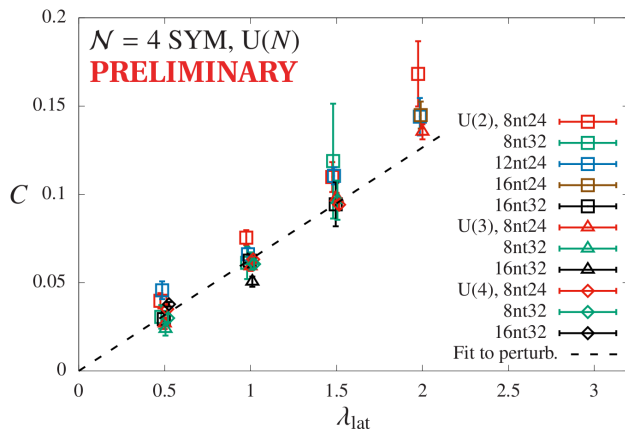to extract Coulomb coefficient $C(\lambda)$

Discretization artifacts reduced
by tree-level improved analysis

$\mathcal{N} = 4$ SYM, U(2)
PRELIMINARY

$16^3 \times 32$
$\lambda_{\text{lat}} = 0.5$

Coupling dependence of Coulomb coefficient

Continuum perturbation theory $\longrightarrow$ $C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

Holography $\longrightarrow$ $C(\lambda) \propto \sqrt{\lambda}$ for $N \to \infty$ and $\lambda \to \infty$ with $\lambda \ll N$

For $\lambda_{\text{lat}} \leq 2$, consistent with leading-order perturbation theory

(iii) Konishi operator scaling dimension

$\mathcal{O}_K(x) = \sum_I \text{Tr}\left[\Phi^I(x)\Phi^I(x)\right]$ is simplest conformal primary operator

Scaling dimension $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$ investigated through
perturbation theory (& S duality), holography, conformal bootstrap

$C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$

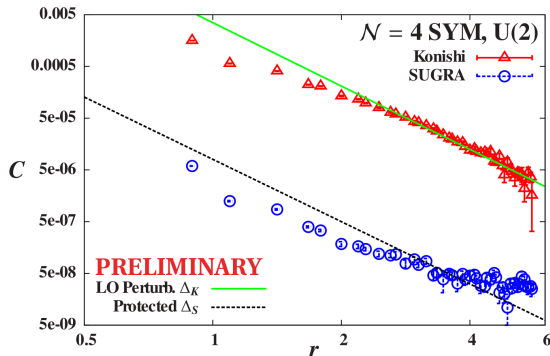'SUGRA' is 20′ op., $\Delta_S = 2$

Will compare:
    Direct power-law decay
    Finite-size scaling
    Monte Carlo RG

(iii) Konishi operator scaling dimension

Lattice scalars $\varphi(n)$ from polar decomposition $\mathcal{U}_a(n) = e^{\varphi_a(n)} U_a(n)$

$$\mathcal{O}_K^{\text{lat}}(n) = \sum_a \text{Tr}\left[\varphi_a(n)\varphi_a(n)\right] - \text{vev} \qquad \mathcal{O}_S^{\text{lat}}(n) \sim \text{Tr}\left[\varphi_a(n)\varphi_b(n)\right]$$

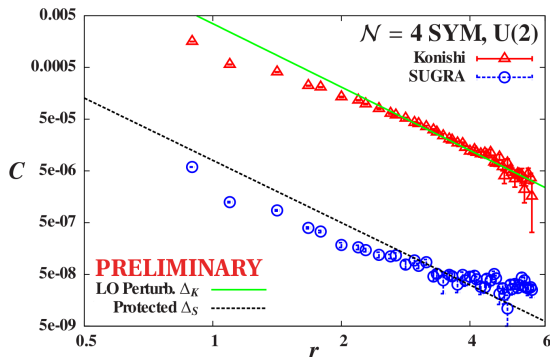$C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$

'SUGRA' is 20' op., $\Delta_S = 2$

Will compare:
- Direct power-law decay
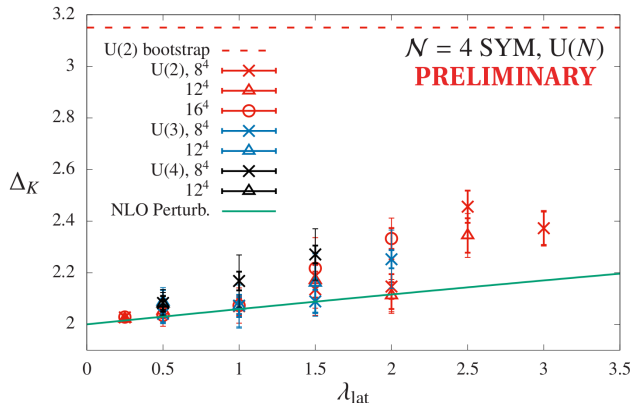- Finite-size scaling
- Monte Carlo RG

Preliminary Δ_K results from Monte Carlo RG

Analyzing both $\mathcal{O}_K^{\text{lat}}$ and $\mathcal{O}_S^{\text{lat}}$

Imposing protected $\Delta_S = 2$
$\longrightarrow \Delta_K(\lambda)$ looks perturbative

Systematic uncertainties from
different amounts of smearing

Complication from twisting $SO(4)_R \subset SO(6)_R$

$\mathcal{O}_K^{\text{lat}}$ mixes with $SO(4)_R$-singlet part of $SO(6)_R$-nonsinglet $\mathcal{O}_S$
$\longrightarrow$ disentangle via variational analyses

Future: Pushing $\mathcal{N} = 4$ SYM to stronger coupling

✓ Reproduce reliable 4d results in perturbative regime

$\longrightarrow$ Check holographic predictions and access new domains

Sign problem seems to become obstruction

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [d\mathcal{U}][d\overline{\mathcal{U}}] \ \mathcal{O} \ e^{-S_B[\mathcal{U}, \overline{\mathcal{U}}]} \ \text{pf} \, \mathcal{D}[\mathcal{U}, \overline{\mathcal{U}}]$$
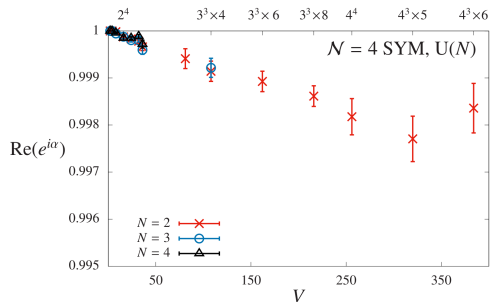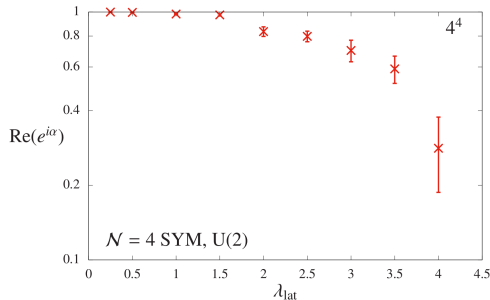
Complex pfaffian $\text{pf} \, \mathcal{D} = |\text{pf} \, \mathcal{D}| e^{i\alpha}$ complicates importance sampling

We phase quench, $\text{pf} \, \mathcal{D} \longrightarrow |\text{pf} \, \mathcal{D}|$, need to reweight $\langle \mathcal{O} \rangle = \dfrac{\langle \mathcal{O} e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}}$

$\mathcal{N} = 4$ SYM sign problem

Fix $\lambda_{lat} = g_{lat}^2 N = 0.5$

Pfaffian nearly real positive

for all accessible volumes

Fix 4^4 volume

Fluctuations increase with coupling

Signal-to-noise

becomes obstruction for $\lambda_{lat} \gtrsim 4$

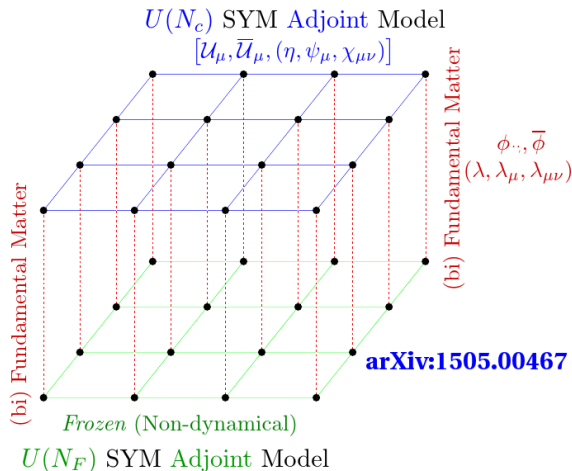Preserve twisted supersymmetry sub-algebra in 2d or 3d

2-slice lattice SYM
with U(N) × U(F) gauge group

Adj. fields on each slice
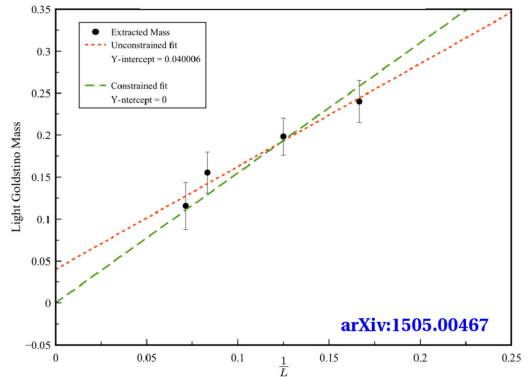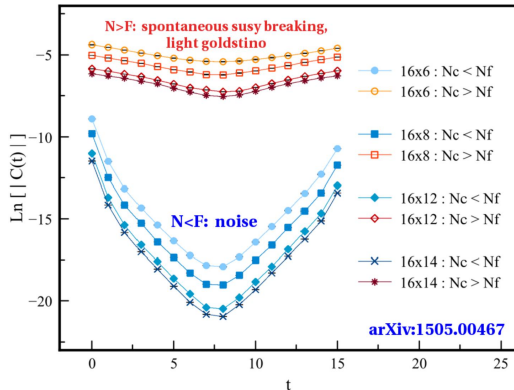
Bi-fundamental in between

Decouple U(F) slice

$\longrightarrow$ U(N) SQCD in $d-1$ dims.
with F fund. hypermultiplets

Dynamical susy breaking in 2d lattice superQCD

U(N) superQCD with F fundamental hypermultiplets

Spontaneous susy breaking requires $N > F$

Recap: An exciting time for lattice supersymmetry
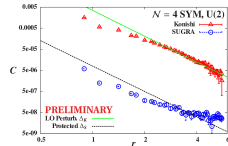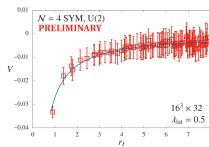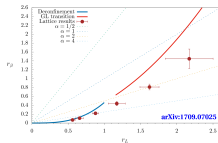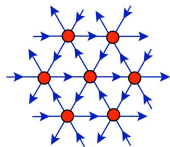
✓ Preserve (some) susy in discrete space-time

$\longrightarrow$ practical lattice $\mathcal{N} = 4$ SYM, public code available

Reproduce reliable analytic results

  ✓ 2d $\mathcal{N} = (8, 8)$ SYM thermodynamics consistent with holography

  ✓ Perturbative $\mathcal{N} = 4$ SYM static potential Coulomb coefficient $C(\lambda)$

and Konishi operator scaling dimension $\Delta_K(\lambda)$

Access new domains $\longrightarrow$ sign problem, lower-dim'l superQCD and more...

Thank you!

UK Research and Innovation

Backup: Numerical lattice field theory calculations

High-performance computing
$\longrightarrow$ evaluate up to
$\sim$billion-dimensional integrals

Importance sampling Monte Carlo

Algorithms sample field configurations with probability $\dfrac{1}{\mathcal{Z}} e^{-S[\Phi]}$

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \ \mathcal{O}(\Phi) \ e^{-S[\Phi]} \longrightarrow \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(\Phi_i) \ \text{with stat. uncertainty} \ \propto \frac{1}{\sqrt{N}}$$

Backup: Breakdown of Leibniz rule on the lattice

$\left\{ Q_\alpha, \overline{Q}_{\dot{\alpha}} \right\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = 2i\sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu$ is problematic

$\implies$ try finite difference $\partial\phi(x) \longrightarrow \Delta\phi(x) = \frac{1}{a}\left[\phi(x+a) - \phi(x)\right]$

Crucial difference between ∂ and Δ

$$\Delta\left[\phi\eta\right] = a^{-1}\left[\phi(x+a)\eta(x+a) - \phi(x)\eta(x)\right]$$
$$= \left[\Delta\phi\right]\eta + \phi\Delta\eta + a\left[\Delta\phi\right]\Delta\eta$$

Full supersymmetry requires Leibniz rule $\partial\left[\phi\eta\right] = \left[\partial\phi\right]\eta + \phi\partial\eta$

only recoverd in $a \to 0$ continuum limit for any local finite difference

Backup: $\mathcal{N} = 4$ SYM in a nutshell

Arguably simplest non-trivial 4d QFT $\longrightarrow$ dualities, amplitudes, ...

SU(N) gauge theory with $\mathcal{N} = 4$ fermions Ψ^{I} and 6 scalars Φ^{IJ},

all massless and in adjoint rep.

Symmetries relate coefficients of kinetic, Yukawa and Φ^4 terms

Maximal 16 supersymmetries Q_α^{I} and $\overline{Q}_{\dot{\alpha}}^{\mathrm{I}}$ $\mathrm{I} = 1, \cdots, 4$

transform under global SU(4) $\sim$ SO(6) R symmetry

Conformal $\longrightarrow$ β function is zero for all values of $\lambda = g^2 N$

Backup: Complexified gauge field from twisting

Combining A_μ and $\Phi^I \longrightarrow \mathcal{A}_a$ and $\overline{\mathcal{A}}_a$

produces $U(N) = SU(N) \otimes U(1)$ gauge theory

Complicates lattice action but needed so that $\mathcal{Q}\, \mathcal{A}_a = \psi_a$

Further motivation: Under $SO(d)_{tw} = \text{diag}\big[SO(d)_{\text{euc}} \otimes SO(d)_R\big]$

$$A_\mu \sim \text{vector} \otimes \text{scalar} = \text{vector}$$
$$\Phi^I \sim \text{scalar} \otimes \text{vector} = \text{vector}$$

Easiest to see in 5d (then dimensionally reduce)

$$\mathcal{A}_a = A_a + i\Phi_a \longrightarrow (A_\mu, \phi) + i(\Phi_\mu, \overline{\phi})$$
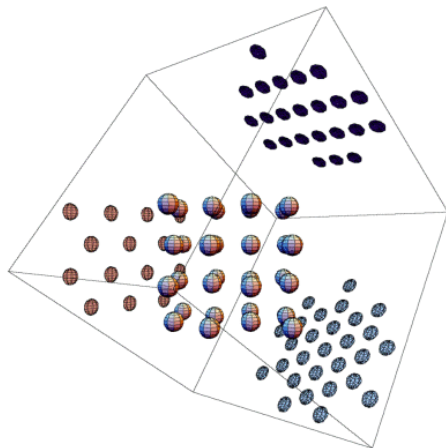
Backup: A_4^* lattice from five dimensions

Again dimensionally reduce, treating all five gauge links symmetrically

Start with hypercubic lattice
in 5d momentum space

Symmetric constraint $\sum_a \partial_a = 0$
projects to 4d momentum space

Result is A_4 lattice
$\longrightarrow$ dual A_4^* lattice in position space

"$\mathcal{Q} + $ discrete $R_a \subset $ SO(4)$_{tw} = \mathcal{Q}_a$ and $\mathcal{Q}_{ab}$"

[arXiv:1306.3891]

Test R_a on Wilson loops $\widetilde{\mathcal{W}}_{ab} \equiv R_a \mathcal{W}_{ab}$

Tune coeff. c_2 of d^2 term in action
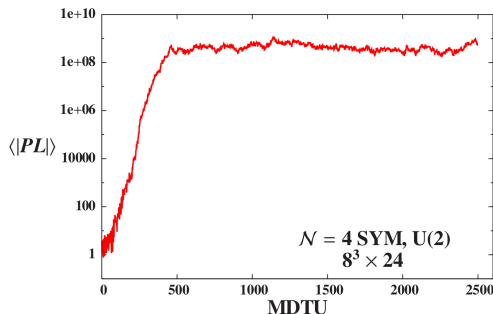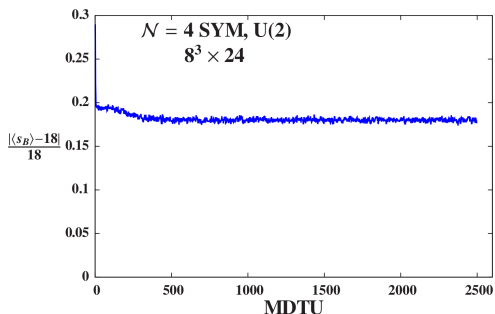   to ensure restoration in continuum
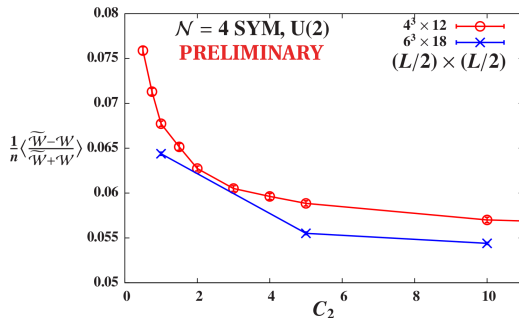
Backup: Problem with SU(N) flat directions

$\mu^2/\lambda_{\text{lat}}$ too small $\longrightarrow$ $\mathcal{U}_a$ can move far from continuum form $\mathbb{I}_N + \mathcal{A}_a$

Example: $\mu = 0.2$ and $\lambda_{\text{lat}} = 2.5$ on $8^3 \times 24$ volume
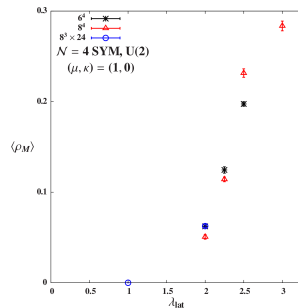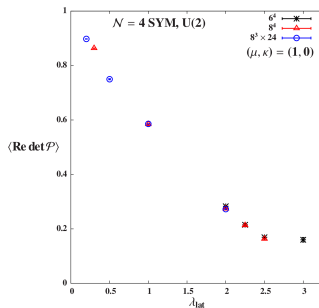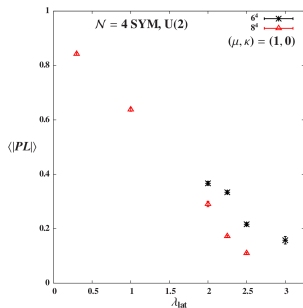
Left: Bosonic action stable $\sim$18% off its supersymmetric value
Right: (Complexified) Polyakov loop wanders off to $\sim 10^9$

Backup: Problem with U(1) flat directions

Monopole condensation $\longrightarrow$ confined lattice phase not present in continuum

Around the same $2\lambda_{\text{lat}} \approx 2$...

  Left: Polyakov loop falls towards zero
  Center: Plaquette determinant falls towards zero
  Right: Density of U(1) monopole world lines becomes non-zero

Backup: Regulating SU(*N*) flat directions

Add soft $\mathcal{Q}$-breaking scalar potential to lattice action

$$S = \frac{N}{4\lambda_{\text{lat}}} \left[\mathcal{Q} \left(\chi_{ab} \mathcal{F}_{ab} + \eta \overline{\mathcal{D}}_a \mathcal{U}_a - \frac{1}{2}\eta d \right) - \frac{1}{4}\epsilon_{abcde} \, \chi_{ab} \overline{\mathcal{D}}_c \, \chi_{de} + \mu^2 V \right]$$

$V = \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a \overline{\mathcal{U}}_a\right] - 1 \right)^2$ lifts SU(*N*) flat directions,

ensures $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$ in continuum limit

Correct continuum limit requires $\mu^2 \to 0$ to restore $\mathcal{Q}$ and recover moduli space

Typically scale $\mu \propto 1/L$ in $L \to \infty$ continuum extrapolation

Backup: Poorly regulating U(1) flat directions

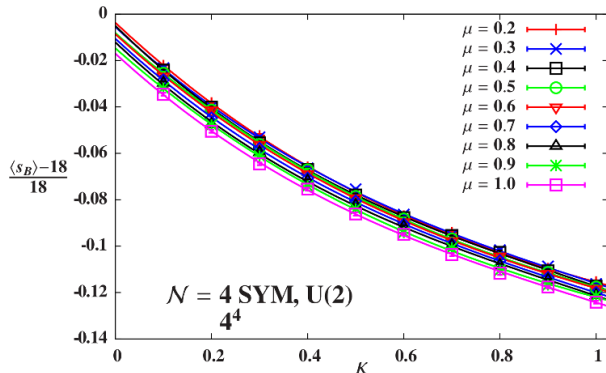In earlier work we added another soft $\mathcal{Q}$-breaking term

$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}}\mu^2 \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2 + \kappa \sum_{a<b} |\det \mathcal{P}_{ab} - 1|^2$$

More sensitivity to κ than to μ^2

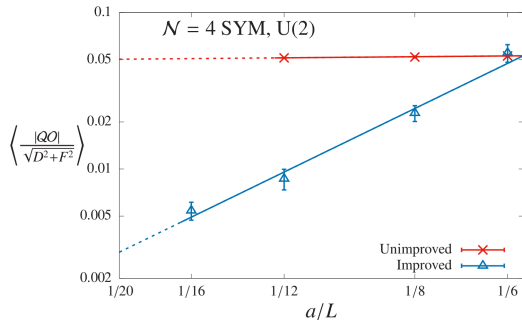Showing $\mathcal{Q}$ Ward identity
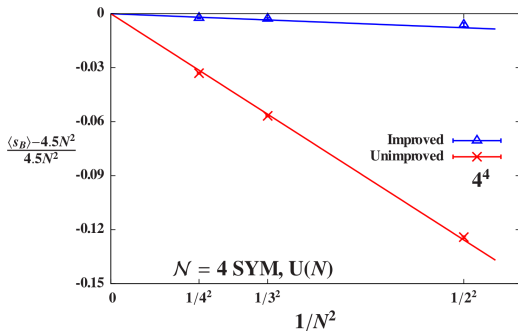 from bosonic action

$$\langle s_B \rangle = 9N^2/2$$

Backup: Better regulating U(1) flat directions

$$S = \frac{N}{4\lambda_{\text{lat}}} \left[\mathcal{Q} \left(\chi_{ab}\mathcal{F}_{ab} + \eta\left\{ \overline{\mathcal{D}}_a \mathcal{U}_a + G\sum_{a<b}[\det \mathcal{P}_{ab} - 1]\,\mathbb{I}_N \right\} - \frac{1}{2}\eta d \right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de} + \mu^2 V \right]$$

$\mathcal{Q}$ Ward identity violations scale $\propto 1/N^2$ (**left**) and $\propto (a/L)^2$ (**right**)

$\sim$ effective '$O(a)$ improvement' since $\mathcal{Q}$ forbids all dim-5 operators

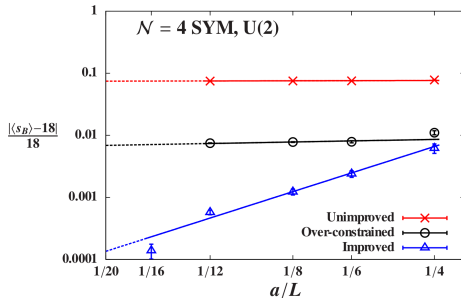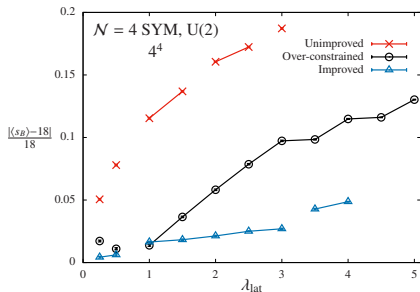# Backup: Supersymmetric moduli space modification [arXiv:1505.03135]

Method to impose $\mathcal{Q}$-invariant constraints on generic site operator $\mathcal{O}(n)$

Modify auxiliary field equations of motion $\longrightarrow$ moduli space
$$d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) \qquad \longrightarrow \qquad d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) + G\mathcal{O}(n)\mathbb{I}_N$$

However, both U(1) and SU(N) $\in \mathcal{O}(n)$ over-constrains system

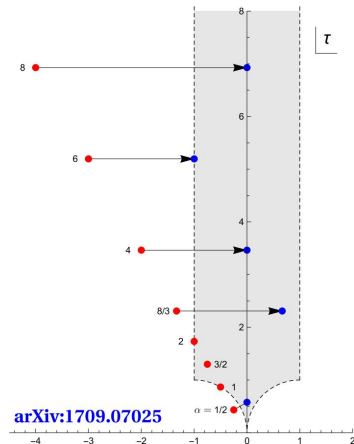# Backup: Dimensional reduction to 2d $\mathcal{N} = (8, 8)$ SYM

Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_x = N_y = 1$

$A_4^* \longrightarrow A_2^*$ (triangular) lattice
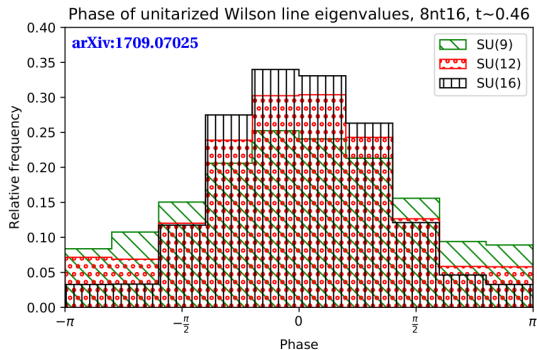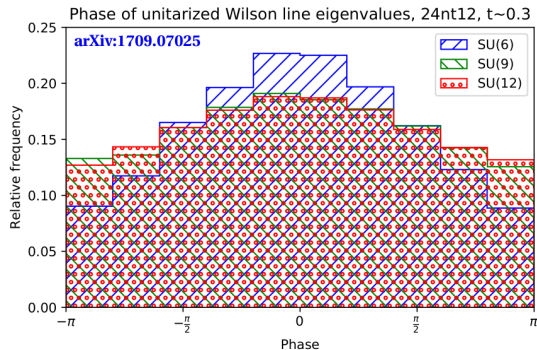
Torus **skewed** depending on $\alpha = L/N_t$

Modular transformation into fundamental domain
$\longrightarrow$ some skewed tori actually rectangular

Also need to stabilize compactified links
to ensure broken center symmetries

arXiv:1709.07025

Backup: 2d $\mathcal{N} = (8,8)$ SYM Wilson line eigenvalues

Check 'spatial deconfinement' through Wilson line eigenvalue phases

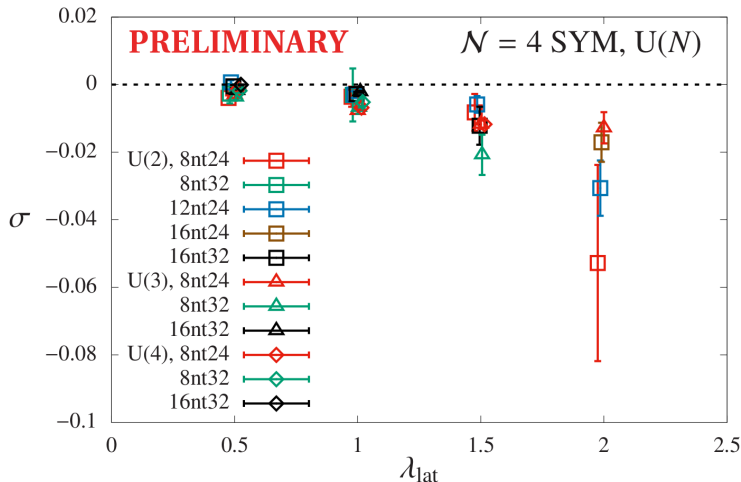**Left:** $\alpha = 2$ distributions more extended as *N* increases $\longrightarrow$ D1 black string

Right: $\alpha = 1/2$ distributions more compact as *N* increases $\longrightarrow$ D0 black hole

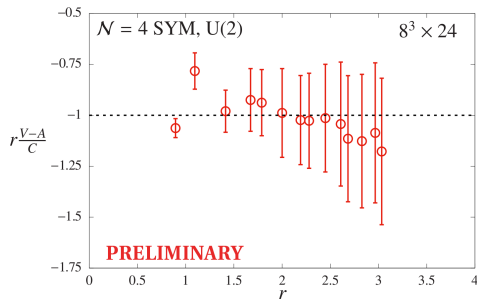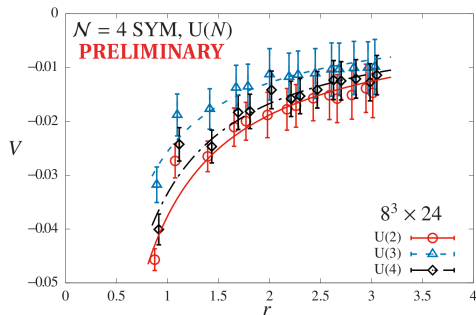String tension σ from fits to confining form $V(r) = A - C/r + \sigma r$

Slightly negative values flatten $V(r_I)$ for $r_I \lesssim L/2$

$\sigma \to 0$ as accessible range of r_I increases on larger volumes

Backup: Discretization artifacts in static potential

Discretization artifacts visible at short distances
where Coulomb term in $V(r) = A - C/r$ is most significant

Danger of distorting Coulomb coefficient C

Backup: Tree-level improvement

Classic trick to reduce discretization artifacts in static potential

Associate $V(r_\nu)$ data with 'r_I' from Fourier transform of gluon propagator

Recall $\dfrac{1}{4\pi^2 r^2} = \displaystyle\int_{-\pi}^{\pi} \dfrac{d^4k}{(2\pi)^4}\, \dfrac{e^{ir_\nu k_\nu}}{k^2}$ where $\dfrac{1}{k^2} = G(k_\nu)$ in continuum

$$A_4^* \text{ lattice } \longrightarrow \frac{1}{r_I^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4\widehat{k}}{(2\pi)^4} \frac{\cos\left(ir_\nu \widehat{k}_\nu\right)}{4\sum_{\mu=1}^{4}\sin^2\left(\widehat{k}\cdot\widehat{e}_\mu\,/\,2\right)}$$

Tree-level lattice propagator from arXiv:1102.1725

$\widehat{e}_\mu$ are A_4^* lattice basis vectors;

momenta $\widehat{k} = \frac{2\pi}{L}\sum_{\mu=1}^{4} n_\mu \widehat{g}_\mu$ depend on dual basis vectors

Backup: Tree-level-improved static potential

$$\frac{1}{r_I^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4\widehat{k}}{(2\pi)^4} \frac{\cos\left(ir_\nu\widehat{k}_\nu\right)}{4\sum_{\mu=1}^{4}\sin^2\left(\widehat{k}\cdot\widehat{e}_\mu / 2\right)}$$

$\longrightarrow$ significantly reduced discretization artifacts
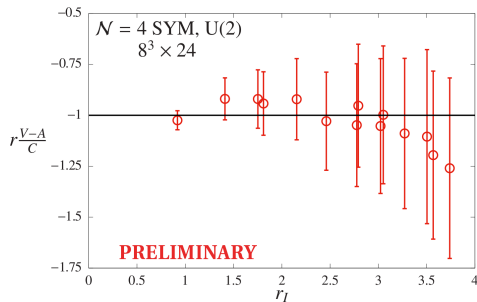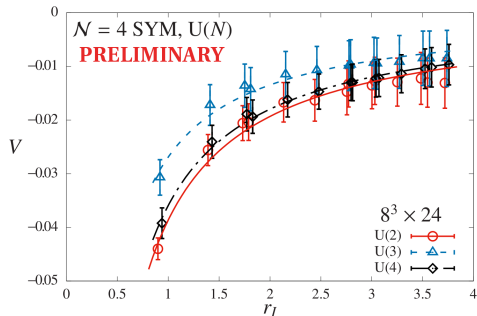
Backup: Scaling dimensions from MCRG stability matrix

Lattice system: $H = \sum_i c_i \, \mathcal{O}_i$ (infinite sum)

Couplings flow under RG blocking $\longrightarrow H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Conformal fixed point $\longrightarrow H^\star = R_b H^\star$ with couplings $c_i^\star$

Linear expansion around fixed point $\longrightarrow$ stability matrix $T_{ik}^\star$

$$c_i^{(n)} - c_i^\star = \sum_k \left. \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \right|_{H^\star} \left(c_k^{(n-1)} - c_k^\star \right) \equiv \sum_k T_{ik}^\star \left(c_k^{(n-1)} - c_k^\star \right)$$

Correlators of $\mathcal{O}_i$, $\mathcal{O}_k$ $\longrightarrow$ elements of stability matrix [Swendsen, 1979]

Eigenvalues of $T_{ik}^\star$ $\longrightarrow$ scaling dimensions of corresponding operators

Backup: Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve $\mathcal{Q}$ and S_5 symmetries $\longleftrightarrow$ geometric structure

Simple transformation constructed in arXiv:1408.7067
$$\mathcal{U}'_a(n') = \xi \mathcal{U}_a(n)\mathcal{U}_a(n + \widehat{\mu}_a) \qquad\qquad \eta'(n') = \eta(n)$$
$$\psi'_a(n') = \xi\left[\psi_a(n)\mathcal{U}_a(n + \widehat{\mu}_a) + \mathcal{U}_a(n)\psi_a(n + \widehat{\mu}_a)\right] \qquad \text{etc.}$$

Doubles lattice spacing $a \longrightarrow a' = 2a$, with tunable rescaling factor ξ

Scalar fields from polar decomposition $\mathcal{U}(n) = e^{\varphi(n)}U(n)$
$$\implies \text{shift} \quad \varphi \longrightarrow \varphi + \log\xi \quad \text{to keep blocked } U \text{ unitary}$$

$\mathcal{Q}$-preserving RG transformation needed
to show only one log. tuning to recover continuum $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$
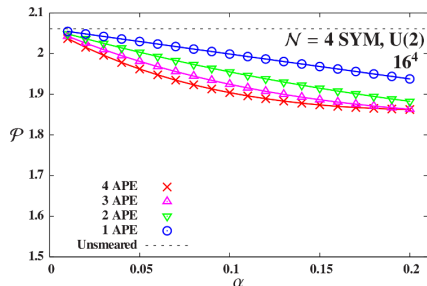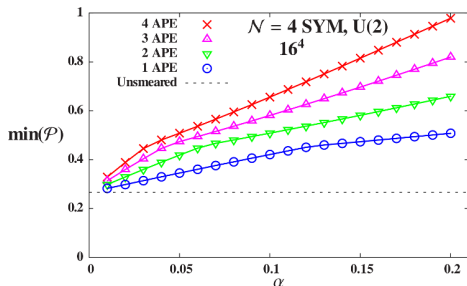
Backup: Smearing for Konishi analyses

Smear to enlarge (MCRG or variational) operator basis

APE-like smearing: $\quad$ — $\quad \longrightarrow \quad (1 - \alpha)$— $\quad + \quad \frac{\alpha}{8} \sum \sqcap$,

$\qquad\qquad$ staples built from unitary parts of links but no final unitarization

Average plaquette stable upon smearing (**right**),

$\qquad\qquad\qquad\qquad\qquad$ minimum plaquette steadily increases (**left**)

Backup: More on dynamical susy breaking

Spontaneous susy breaking means $\langle 0|H|0\rangle > 0$ or equivalently $\langle \mathcal{QO}\rangle \neq 0$

Twisted superQCD auxiliary field e.o.m. $\longleftrightarrow$ Fayet–Iliopoulos D-term potential

$$d = \overline{\mathcal{D}}_a \mathcal{U}_a + \sum_{i=1}^{F} \phi_i \overline{\phi}_i - r\mathbb{I}_N \qquad \longleftrightarrow \qquad \text{Tr}\left[\left(\sum_i \phi_i \overline{\phi}_i - r\mathbb{I}_N\right)^2\right] \in H$$

Have $F \times N$ scalar vevs to zero out $N \times N$ matrix
$\longrightarrow N > F$ suggests susy breaking, $\langle 0|H|0\rangle > 0 \longleftrightarrow \langle \mathcal{Q}\eta\rangle = \langle d\rangle \neq 0$