Maximally supersymmetric Yang–Mills on the lattice

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and more to come with Simon Catterall, Raghav Jha and Toby Wiseman
Overview and plan

**Why:** Lattice supersymmetry

**How:** Lattice formulation highlights

**What:** Recent results
   - Dimensionally reduced (2d) thermodynamics
   - Static potential (4d)
   - Conformal scaling dimensions

**Prospects and future directions**
Overview and plan

Central idea
Preserve (some) susy in discrete space-time
→ practical lattice investigations

Goals
1) Reproduce reliable results in perturbative and holographic regimes
2) Access new domains
Motivations

Lattice field theory promises first-principles predictions for strongly coupled supersymmetric QFTs

BSM

QFT

Holography

(Derek Leinweber)
Supersymmetry must be broken on the lattice

Supersymmetry is a space-time symmetry, \((\mathcal{I} = 1, \cdots, \mathcal{N})\)

adding spinor generators \(Q^I_\alpha\) and \(\bar{Q}^I_{\dot{\alpha}}\) to translations, rotations, boosts

\[
\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\delta^{IJ}\sigma^{\mu}_{\alpha\dot{\alpha}} P_\mu
\]

broken in discrete space-time

\[\longrightarrow\] relevant susy-violating operators

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Scalar mass  Yukawas  Quartics  Quark mass  Gluino mass
Supersymmetry need not be *completely* broken on the lattice

Preserve susy sub-algebra at non-zero lattice spacing

\[ \implies \text{correct continuum limit with little or no fine tuning} \]

Equivalent constructions from ‘topological’ twisting and dim’l deconstruction

Need \(2^d\) supersymmetries in \(d\) dimensions

\(d = 4 \implies\) maximally supersymmetric Yang–Mills (\(\mathcal{N} = 4\) SYM)
\[ \mathcal{N} = 4 \text{ SYM in a nutshell} \]

Arguably simplest non-trivial 4d QFT \( \rightarrow \) dualities, amplitudes, \ldots

**SU(4)** gauge theory with \( \mathcal{N} = 4 \) fermions \( \Psi^I \) and 6 scalars \( \Phi^{IJ} \), all massless and in adjoint rep.

**Symmetries** relate coefficients of kinetic, Yukawa and \( \Phi^4 \) terms

Maximal 16 supersymmetries \( Q^I_\alpha \) and \( \overline{Q}^I_\dot{\alpha} \) \( I = 1, \ldots, 4 \)
transform under global \( SU(4) \sim SO(6) \) \( R \) symmetry

**Conformal** \( \rightarrow \) \( \beta \) function is zero for all values of \( \lambda = g^2 N \)
Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand $4 \times 4$ matrix of supersymmetries

\[
\begin{pmatrix}
Q^1_{\alpha} & Q^2_{\alpha} & Q^3_{\alpha} & Q^4_{\alpha} \\
\bar{Q}^1_{\dot{\alpha}} & \bar{Q}^2_{\dot{\alpha}} & \bar{Q}^3_{\dot{\alpha}} & \bar{Q}^4_{\dot{\alpha}}
\end{pmatrix}
= Q + Q_\mu \gamma_\mu + Q_{\mu \nu} \gamma_\mu \gamma_\nu + \bar{Q}_\mu \gamma_\mu \gamma_5 + \bar{Q} \gamma_5
\]

\[
\rightarrow Q + Q_a \gamma_a + Q_{ab} \gamma_a \gamma_b
\]

with $a, b = 1, \cdots, 5$

R-symmetry index $\times$ Lorentz index $\rightarrow$ reps of ‘twisted rotation group’

\[
SO(4)_{tw} \equiv \text{diag} \left[ SO(4)_{\text{euc}} \otimes SO(4)_R \right]
\]

$SO(4)_R \subset SO(6)_R$

Change of variables $\rightarrow Q$ transform with integer ‘spin’ under $SO(4)_{tw}$
Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand $4 \times 4$ matrix of supersymmetries

$$
\begin{pmatrix}
Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\
Q^1_{\dot{\alpha}} & Q^2_{\dot{\alpha}} & Q^3_{\dot{\alpha}} & Q^4_{\dot{\alpha}}
\end{pmatrix}
= Q + Q_\mu \gamma_\mu + Q_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{Q}_\mu \gamma_\mu \gamma_5 + \overline{Q} \gamma_5
\rightarrow Q + Q_a \gamma_a + Q_{ab} \gamma_a \gamma_b
$$

with $a, b = 1, \cdots, 5$

Discrete space-time
Can preserve closed sub-algebra

$$\{ Q, Q \} = 2Q^2 = 0$$
Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand $4 \times 4$ matrix of supersymmetries

\[
\begin{pmatrix}
Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\
\bar{Q}^1_{\dot{\alpha}} & \bar{Q}^2_{\dot{\alpha}} & \bar{Q}^3_{\dot{\alpha}} & \bar{Q}^4_{\dot{\alpha}}
\end{pmatrix}
= Q + Q_\mu \gamma_\mu + Q_{\mu \nu} \gamma_\mu \gamma_\nu + \bar{Q}_\mu \gamma_\mu \gamma_5 + \bar{\gamma} \gamma_5
\rightarrow Q + Q_a \gamma_a + Q_{ab} \gamma_a \gamma_b
\text{ with } a, b = 1, \cdots, 5
\]

Discrete space-time
Can preserve closed sub-algebra
\[
\{ Q, Q \} = 2Q^2 = 0
\]
Completing the twist

Fields also transform with integer spin under SO(4)\text{tw} — no spinors

\[
\begin{align*}
\psi \text{ and } \bar{\psi} & \rightarrow \eta, \psi_a \text{ and } \chi_{ab} \\
A_\mu \text{ and } \Phi^I & \rightarrow \text{complexified gauge field } A_a \text{ and } \bar{A}_a \\
& \rightarrow U(N) = SU(N) \otimes U(1) \text{ gauge theory}
\end{align*}
\]

\[Q\text{ interchanges bosonic } \leftrightarrow \text{fermionic d.o.f. with } Q^2 = 0\]

\[
\begin{align*}
Q A_a &= \psi_a \\
Q \chi_{ab} &= -\bar{F}_{ab} \\
Q \eta &= d
\end{align*}
\]

bosonic auxiliary field with e.o.m. \[d = \bar{D}_a A_a\]
Lattice theory looks nearly the same despite breaking $Q_a$ and $Q_{ab}$

Covariant derivatives $\rightarrow$ finite difference operators

Complexified gauge fields $A_a$ $\rightarrow$ gauge links $U_a \in gl(N, \mathbb{C})$

$Q A_a \rightarrow Q U_a = \psi_a$

$Q \chi_{ab} = -\bar{F}_{ab}$

$Q \eta = d$

Geometry: $\eta$ on sites, $\psi_a$ on links, etc.

Supersymmetric lattice action ($QS = 0$) from $Q^2 \cdot = 0$ and Bianchi identity

$$S = \frac{N}{4\lambda_{lat}} \text{Tr} \left[ Q \left( \chi_{ab} F_{ab} + \eta \bar{D}_a U_a - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \bar{D}_c \chi_{de} \right]$$
Five links in four dimensions $\rightarrow$ $A_4^*$ lattice

$A_4^* \sim$ 4d analog of 2d triangular lattice

Basis vectors linearly dependent and non-orthogonal

Large $S_5$ point group symmetry

$S_5$ irreps precisely match onto irreps of twisted SO(4)$_{tw}$

$\psi_a \rightarrow \psi_\mu$, $\bar{\eta}$ is $5 \rightarrow 4 \oplus 1$

$\chi_{ab} \rightarrow \chi_{\mu\nu}$, $\bar{\psi}_\mu$ is $10 \rightarrow 6 \oplus 4$

$S_5 \rightarrow$ SO(4)$_{tw}$ in continuum limit restores $Q_a$ and $Q_{ab}$
Checkpoint

Analytic results for twisted $\mathcal{N} = 4$ SYM on $A_4^*$ lattice

$U(N)$ gauge invariance + $Q$ + $S_5$ lattice symmetries

$\rightarrow$ Moduli space preserved to all orders

$\rightarrow$ One-loop lattice $\beta$ function vanishes

$\rightarrow$ Only one log. tuning to recover continuum $Q_a$ and $Q_{ab}$


Not yet suitable for numerical calculations

Must regulate zero modes and flat directions, especially in U(1) sector
Two deformations stabilize lattice calculations

(i) Add $\text{SU}(N)$ scalar potential $\propto \mu^2 \sum_a \left( \text{Tr} \left[ U_a \bar{U}_a \right] - N \right)^2$

**Softly** breaks susy $\longrightarrow$ $Q$-violating operators vanish $\propto \mu^2 \rightarrow 0$

Test via Ward identity violations

$Q \left[ \eta U_a \bar{U}_a \right] \neq 0$

\[ \left\langle \frac{|QO|}{\sqrt{D^2 + F^2}} \right\rangle \]
Two deformations stabilize lattice calculations

(ii) Constrain $U(1)$ plaquette determinant $\sim G \sum_{a<b} (\det P_{ab} - 1)$

Implemented supersymmetrically as Fayet–Iliopoulos $D$-term potential

Test via Ward identity violations

$Q \left[ \eta U_a \bar{U}_a \right] \neq 0$

Log–log axes

$\rightarrow \text{violations} \propto (a/L)^2$
so that the full improved action becomes

\[ S_{\text{imp}} = S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \]  \hspace{1cm} (18)\]

\[ S'_{\text{exact}} = \frac{N}{4\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ -F_{ab}(n)F_{ab}(n) - \chi_{ab}(n)D_{[a}^{(+)}\psi_{b]}(n) - \eta(n)D_{a}^{(-)}\psi_{a}(n) \right. \]

\[ + \frac{1}{2} \left( \overline{D}_{a}^{(-)}U_{a}(n) + G \sum_{a \neq b} (\det P_{ab}(n) - 1) \mathbb{I}_{N} \right)^{2} \left. - S_{\text{det}} \right] \]

\[ S_{\text{det}} = \frac{N}{4\lambda_{\text{lat}}} G \sum_n \text{Tr} [\eta(n)] \sum_{a \neq b} [\det P_{ab}(n)] \text{Tr} \left[ U_{b}^{-1}(n)\psi_{b}(n) + U_{a}^{-1}(n + \hat{\mu}_{b})\psi_{a}(n + \hat{\mu}_{b}) \right] \]

\[ S_{\text{closed}} = -\frac{N}{16\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ \epsilon_{abcde} \chi_{de}(n + \hat{\mu}_{a} + \hat{\mu}_{b} + \hat{\mu}_{c})\overline{D}_{c}^{(-)}\chi_{ab}(n) \right], \]

\[ S'_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}} \mu^{2} \sum_n \sum_{a} \left( \frac{1}{N} \text{Tr} \left[ U_{a}(n)\overline{U}_{a}(n) \right] - 1 \right)^{2} \]

\[ \gtrsim 100 \text{ inter-node data transfers in the fermion operator} \quad \text{— non-trivial} \ldots \]

Public parallel code to reduce barriers to entry: [github.com/daschaich/susy](https://github.com/daschaich/susy)

(i) Thermodynamics on \((r_L \times r_\beta)\) 2-torus

Dimensionally reduce to (deconfined) 2d \(\mathcal{N} = (8, 8)\) SYM with four scalar \(Q\)

Low temperatures \(t = 1/r_\beta \leftrightarrow\) black holes in dual supergravity

For decreasing \(r_L\) \textbf{at large} \(N\)

homogeneous black string (D1)

\(\longrightarrow\) localized black hole (D0)

“spatial deconfinement”

signalled by Wilson line \(P_L\)
Spatial deconfinement transition signals

Peaks in Wilson line susceptibility match change in its magnitude $|\text{PL}|$, grow with size of SU($N$) gauge group, comparing $N = 6, 9, 12$

Agreement for $16 \times 4$ vs. $24 \times 6$ lattices (aspect ratio $\alpha = r_L/r_\beta = 4$)

\[ \text{arXiv:1709.07025} \]
Lattice $2d \mathcal{N} = (8, 8)$ SYM phase diagram

Large $\alpha = r_L/r_\beta \gtrsim 4 \rightarrow$ good agreement with high-temperature bosonic QM

Small $\alpha \lesssim 2 \rightarrow$ harder to control uncertainties with $6 \leq N \leq 16$

Overall consistent with holography

Comparing multiple lattice sizes

Controlled extrapolations are work in progress
Dual black hole thermodynamics

Dual black hole energy from 2d $\mathcal{N} = (8,8)$ SYM

$\propto t^3$ for large-$r_L$ D1 phase

$\propto t^{3.2}$ for small-$r_L$ D0 phase

Lattice results consistent with holography for sufficiently low $t \lesssim 0.4$

\begin{align*}
\alpha = 2 & \quad \text{SU(12), 16nt8} \quad \text{SU(16), 16nt8} \quad \text{SU(12), 24nt12} \\
\alpha = 1/2 & \quad \text{SU(9), 6nt12} \quad \text{SU(12), 6nt12} \quad \text{SU(12), 8nt16} \quad \text{SU(16), 8nt16}
\end{align*}

\text{D1 gravity prediction} \quad \text{D0 gravity prediction}
(ii) $4d$ $\mathcal{N} = 4$ SYM static potential $V(r)$

Static probes $\rightarrow r \times T$ Wilson loops $W(r, T) \propto e^{-V(r)T}$

Coulomb gauge trick reduces $A_4^*$ lattice complications
Static potential is Coulombic at all $\lambda$

Fits to confining $V(r) = A - C/r + \sigma r \rightarrow$ vanishing string tension $\sigma$

$\Rightarrow$ Fit to just $V(r) = A - C/r$

to extract Coulomb coefficient $C(\lambda)$

Discretization artifacts reduced by tree-level improved analysis
Coupling dependence of Coulomb coefficient

Continuum perturbation theory $\rightarrow C(\lambda) = \lambda/(4\pi) + O(\lambda^2)$

Holography $\rightarrow C(\lambda) \propto \sqrt{\lambda}$ for $N \to \infty$ and $\lambda \to \infty$ with $\lambda \ll N$

For $\lambda_{\text{lat}} \leq 2$, consistent with leading-order perturbation theory
(iii) Konishi operator scaling dimension

\[ \mathcal{O}_K(x) = \sum_i \text{Tr} [\Phi^I(x)\Phi^I(x)] \] is simplest conformal primary operator

Scaling dimension \( \Delta_K(\lambda) = 2 + \gamma_K(\lambda) \) investigated through perturbation theory (& S duality), holography, conformal bootstrap

\[ C_K(r) \equiv \mathcal{O}_K(x + r)\mathcal{O}_K(x) \propto r^{-2\Delta_K} \]

‘SUGRA’ is 20’ op., \( \Delta_S = 2 \)

Will compare:
- Direct power-law decay
- Finite-size scaling
- Monte Carlo RG
(iii) Konishi operator scaling dimension

Lattice scalars $\varphi(n)$ from polar decomposition $U_a(n) \rightarrow e^{\varphi_a(n)} U_a(n)$

$$O_{\text{lat}}^K(n) = \sum_a \text{Tr}[\varphi_a(n)\varphi_a(n)] - \text{vev}$$

$$O_{\text{lat}}^S(n) \sim \text{Tr}[\varphi_a(n)\varphi_b(n)]$$

$$C_K(r) \equiv O_K(x + r)O_K(x) \propto r^{-2\Delta_K}$$

‘SUGRA’ is 20’ op., $\Delta_S = 2$

Will compare:
- Direct power-law decay
- Finite-size scaling
- Monte Carlo RG
Scaling dimensions from MCRG stability matrix

Lattice system: \( H = \sum_i c_i \mathcal{O}_i \) (infinite sum)

Couplings flow under RG blocking \( \longrightarrow \) \( H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)} \)

Fixed point \( \longrightarrow \) \( H^* = R_b H^* \) with couplings \( c_i^* \)

Linear expansion around fixed point \( \longrightarrow \) stability matrix \( T_{ik}^* \)

\[
    c_i^{(n)} - c_i^* = \sum_k \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \bigg|_{H^*} \left( c_k^{(n-1)} - c_k^* \right) \equiv \sum_k T_{ik}^* \left( c_k^{(n-1)} - c_k^* \right)
\]

Correlators of \( \mathcal{O}_i, \mathcal{O}_k \) \( \longrightarrow \) elements of stability matrix [Swendsen, 1979]

Eigenvalues of \( T_{ik}^* \) \( \longrightarrow \) scaling dimensions of corresponding operators
Preliminary $\Delta_K$ results from Monte Carlo RG

Analyzing both $\mathcal{O}_{K}^{\text{lat}}$ and $\mathcal{O}_{S}^{\text{lat}}$

Imposing protected $\Delta_S = 2$

$\rightarrow \Delta_K(\lambda)$ looks perturbative

Systematic uncertainties from different amounts of smearing

Complication from twisting $\text{SO}(4)_R \subset \text{SO}(6)_R$

$\mathcal{O}_{K}^{\text{lat}}$ mixes with $\text{SO}(4)_R$-singlet part of $\text{SO}(6)_R$-nonsinglet $\mathcal{O}_S$

$\rightarrow$ disentangle via variational analyses
Future: Pushing $\mathcal{N} = 4$ SYM to stronger coupling

✓ Reproduce reliable 4d results in perturbative regime

→ Check holographic predictions and access new domains

Sign problem seems to become obstruction

\[
\langle \mathcal{O} \rangle = \frac{1}{Z} \int [dU][d\bar{U}] \mathcal{O} \ e^{-S_{B[U,\bar{U}]} \ pf D[U,\bar{U}]}
\]

Complex pfaffian \( pf D = |pf D| e^{i\alpha} \) complicates importance sampling

We phase quench, \( pf D \longrightarrow |pf D| \), need to reweight \( \langle \mathcal{O} \rangle = \frac{\langle \mathcal{O} e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}} \)
$N = 4$ SYM sign problem

**Fix $\lambda_{\text{lat}} = g_{\text{lat}}^2 N = 0.5$**

Pfaffian nearly real positive
for all accessible volumes

**Fix $4^4$ volume**

Fluctuations increase with coupling

Signal-to-noise becomes obstruction for $\lambda_{\text{lat}} \gtrsim 4$
Preserve twisted supersymmetry sub-algebra in 2d or 3d

2-slice lattice SYM
with $U(N) \times U(F)$ gauge group
Adj. fields on each slice
Bi-fundamental in between

Decouple $U(F)$ slice
$\rightarrow U(N)$ SQCD in $d - 1$ dims.
with $F$ fund. hypermultiplets

$U(N_c)$ SYM Adjoint Model

$[U_\mu, \bar{U}_\mu, (\eta, \psi_\mu, \chi_{\mu\nu})]$

Frozen (Non-dynamical)

$U(N_F)$ SYM Adjoint Model

$arXiv:1505.00467$
Dynamical susy breaking in 2d lattice superQCD

**U(N) superQCD with $F$ fundamental hypermultiplets**

Spontaneous susy breaking requires $N > F$

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**Graphs:**
- **Left graph:**
  - Title: $N > F$: spontaneous susy breaking, light goldstino
  - $\ln |C(t)|$ vs. time $t$ for different lattices:
    - $16x6$: $N_c < N_f$
    - $16x6$: $N_c > N_f$
    - $16x8$: $N_c < N_f$
    - $16x8$: $N_c > N_f$
    - $16x12$: $N_c < N_f$
    - $16x12$: $N_c > N_f$
    - $16x14$: $N_c < N_f$
    - $16x14$: $N_c > N_f$
  - **Note:** arXiv:1505.00467

- **Right graph:**
  - Title: Extracted Mass
  - $Y$-intercept = 0.040006
  - $\frac{1}{L}$ vs. Light Goldstino Mass
  - **Note:** arXiv:1505.00467
Recap: An exciting time for lattice supersymmetry

✓ Preserve (some) susy in discrete space-time
  \[ \rightarrow \text{practical lattice } \mathcal{N} = 4 \text{ SYM, public code available} \]

Reproduce reliable analytic results

✓ 2d \( \mathcal{N} = (8,8) \) SYM thermodynamics consistent with holography
✓ Perturbative static potential Coulomb coefficient \( C(\lambda) \)
   and Konishi operator conformal scaling dimension \( \Delta_K(\lambda) \)

Access new domains \[ \rightarrow \text{sign problem, lower-dim’l superQCD and more…} \]
Thank you!

Collaborators
Simon Catterall, Raghav Jha, Toby Wiseman
also Georg Bergner, Poul Damgaard, Joel Giedt, Anosh Joseph

Funding and computing resources
Backup: Lattice field theory in a nutshell

Formally \( \langle O \rangle = \frac{1}{Z} \int D\Phi \ O(\Phi) \ e^{-S[\Phi]} \)

Regularize by formulating theory in finite, discrete space-time \( \rightarrow \) the lattice

Spacing between lattice sites ("a") \( \rightarrow \) UV cutoff scale \( 1/a \)

Remove cutoff: \( a \rightarrow 0 \ (L/a \rightarrow \infty) \)

Hypercubic \( \rightarrow \) automatic symmetries
Importance sampling Monte Carlo

Algorithms sample field configurations with probability $\frac{1}{\mathcal{Z}} e^{-S[\Phi]}$

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \ O(\Phi) \ e^{-S[\Phi]} \quad \rightarrow \quad \frac{1}{N} \sum_{i=1}^{N} O(\Phi_i) \text{ with stat. uncertainty } \propto \frac{1}{\sqrt{N}}$$
Backup: Breakdown of Leibniz rule on the lattice

\[ \left\{ Q_\alpha, \overline{Q}_{\dot{\alpha}} \right\} = 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu = 2i\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \] is problematic

\[ \implies \text{try finite difference} \quad \partial \phi(x) \longrightarrow \Delta \phi(x) = \frac{1}{a} [\phi(x + a) - \phi(x)] \]

**Crucial difference between \( \partial \) and \( \Delta \)**

\[ \Delta [\phi \eta] = a^{-1} [\phi(x + a)\eta(x + a) - \phi(x)\eta(x)] \]

\[ = [\Delta \phi] \eta + \phi \Delta \eta + a [\Delta \phi] \Delta \eta \]

Full supersymmetry requires Leibniz rule \( \partial [\phi \eta] = [\partial \phi] \eta + \phi \partial \eta \)

only recoverd in \( a \to 0 \) continuum limit for any local finite difference
Backup: Complexified gauge field from twisting

Combining $A_\mu$ and $\Phi^I \longrightarrow A_a$ and $\bar{A}_a$

produces $U(N) = SU(N) \otimes U(1)$ gauge theory

Complicates lattice action but needed so that $Q A_a = \psi_a$

Further motivation: Under $SO(d)_{tw} = \text{diag}[SO(d)_{\text{euc}} \otimes SO(d)_R]$

$A_\mu \sim \text{vector} \otimes \text{scalar} = \text{vector}$

$\Phi^I \sim \text{scalar} \otimes \text{vector} = \text{vector}$

Easiest to see in 5d (then dimensionally reduce)

$A_a = A_a + i\Phi_a \longrightarrow (A_\mu, \phi) + i(\Phi_\mu, \bar{\phi})$
Backup: $A_4^*$ lattice from five dimensions

Again dimensionally reduce, treating all five gauge links symmetrically

Start with hypercubic lattice in 5d momentum space

**Symmetric** constraint $\sum_a \partial_a = 0$

projects to 4d momentum space

Result is $A_4$ lattice

$\rightarrow$ dual $A_4^*$ lattice in real space
Backup: Restoration of $Q_a$ and $Q_{ab}$ supersymmetries

“$Q + \text{discrete } R_a \subset SO(4)_{tw} = Q_a \text{ and } Q_{ab}$”

Test $R_a$ on Wilson loops $\tilde{\mathcal{W}}_{ab} \equiv R_a \mathcal{W}_{ab}$

Tune coeff. $c_2$ of $d^2$ term to ensure restoration in continuum
Backup: Problem with SU($N$) flat directions

$\mu^2/\lambda_{\text{lat}}$ too small $\rightarrow \mathcal{U}_a$ can move far from continuum form $\mathbb{I}_N + \mathcal{A}_a$

Example: $\mu = 0.2$ and $\lambda_{\text{lat}} = 2.5$ on $8^3 \times 24$ volume

**Left:** Bosonic action stable $\sim 18\%$ off its supersymmetric value

**Right:** (Complexified) Polyakov loop wanders off to $\sim 10^9$
Backup: Problem with U(1) flat directions

Monopole condensation $\rightarrow$ confined lattice phase not present in continuum

Around the same $2\lambda_{\text{lat}} \approx 2.\ldots$

Left: Polyakov loop falls towards zero

Center: Plaquette determinant falls towards zero

Right: Density of U(1) monopole world lines becomes non-zero
Backup: Regulating SU($N$) flat directions

Add soft $Q$-breaking scalar potential to lattice action

$$S = \frac{N}{4 \lambda_{\text{lat}}} \left[ Q \left( \chi_{ab} F_{ab} + \eta \overline{D}_a U_a - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \overline{D}_c \chi_{de} + \mu^2 V \right]$$

$$V = \sum_a \left( \frac{1}{N} \text{Tr} [U_a \overline{U}_a] - 1 \right)^2$$ lifts SU($N$) flat directions,

ensures $U_a = I_N + A_a$ in continuum limit

Correct continuum limit requires $\mu^2 \rightarrow 0$ to restore $Q$ and recover moduli space

Typically scale $\mu \propto 1/L$ in $L \rightarrow \infty$ continuum extrapolation
Backup: Poorly regulating U(1) flat directions

In earlier work we added another soft $Q$-breaking term

$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}} \mu^2 \sum_a \left( \frac{1}{N} \text{Tr} \left[ U_a \overline{U}_a \right] - 1 \right)^2 + \kappa \sum_{a < b} |\det P_{ab} - 1|^2$$

More sensitivity to $\kappa$ than to $\mu^2$

Showing $Q$ Ward identity from bosonic action

$$\langle s_B \rangle = \frac{9N^2}{2}$$
Backup: Better regulating $\text{U}(1)$ flat directions

$$S = \frac{N}{4\lambda_{\text{lat}}} \left[ Q \left( \chi_{ab} F_{ab} + \eta \left\{ \overline{D}_a U_a + G \sum_{a<b} [\det P_{ab} - 1] \mathbb{I}_N \right\} - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \overline{D}_c \chi_{de} + \mu^2 V \right]$$

$Q$ Ward identity violations scale $\propto 1/N^2$ (left) and $\propto (a/L)^2$ (right)

$\sim$ effective ‘$O(a)$ improvement’ since $Q$ forbids all dim-5 operators
Modify auxiliary field equations of motion \[ \rightarrow \text{ moduli space} \]

\[
d(n) = \overline{D}_{i}^{(-)} U_{i}(n) \quad \rightarrow \quad d(n) = \overline{D}_{i}^{(-)} U_{i}(n) + G \mathcal{O}(n) \mathbb{I}_{N}
\]

However, both U(1) and SU(N) \( \in \mathcal{O}(n) \) over-constrains system.
Backup: Dimensional reduction to 2d $\mathcal{N} = (8,8)$ SYM

Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_x = N_y = 1$

$A_4^* \rightarrow A_2^*$ (triangular) lattice

Torus **skewed** depending on $\alpha = N_t / L$

Modular transformation into fundamental domain  
$\rightarrow$ some skewed tori actually rectangular

Also need to stabilize compactified links  
to ensure broken center symmetries

*David Schaich (Liverpool)*

Lattice MSYM

*Southampton, 27 November 2019*
Backup: 2d $\mathcal{N} = (8, 8)$ SYM Wilson line eigenvalues

Check ‘spatial deconfinement’ through Wilson line eigenvalue phases

Left: $\alpha = 2$ distributions more extended as $N$ increases $\rightarrow$ D1 black string

Right: $\alpha = 1/2$ distributions more compact as $N$ increases $\rightarrow$ D0 black hole
Backup: Static potential is Coulombic at all $\lambda$

String tension $\sigma$ from fits to confining form $V(r) = A - C/r + \sigma r$

Slightly negative values flatten $V(r_I)$ for $r_I \lesssim L/2$

$\sigma \to 0$ as accessible range of $r_I$ increases on larger volumes
Discretization artifacts visible at short distances where Coulomb term in \( V(r) = A - \frac{C}{r} \) is most significant

Danger of distorting Coulomb coefficient \( C \)
Backup: Tree-level improvement

Classical trick to reduce discretization artifacts in static potential

Associate $V(r)$ data with $r$ from Fourier transform of gluon propagator

Recall

$$\frac{1}{4\pi^2 r^2} = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} e^{ir \cdot k} \frac{k^2}{k^2} \quad \text{where} \quad \frac{1}{k^2} = G(k) \quad \text{in continuum}$$

$$A_4^* \quad \text{lattice} \quad \rightarrow \quad \frac{1}{r_i^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \frac{\cos (i r_i \cdot \hat{k})}{4 \sum_{\mu=1}^{4} \sin^2 \left( \frac{\hat{k} \cdot \hat{e}_\mu}{2} \right)}$$

Tree-level lattice propagator from arXiv:1102.1725

$\hat{e}_\mu$ are $A_4^*$ lattice basis vectors;

momenta $\hat{k} = \frac{2\pi}{L} \sum_{\mu=1}^{4} n_\mu \hat{g}_\mu$ depend on dual basis vectors
Backup: Tree-level-improved static potential

Significantly reduced discretization artifacts
Backup: Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve $Q$ and $S_5$ symmetries $\leftrightarrow$ geometric structure

Simple transformation constructed in arXiv:1408.7067

\[
\begin{align*}
U'_a(n') &= \xi U_a(n)U_a(n + \hat{\mu}_a) \\
\psi'_a(n') &= \xi [\psi_a(n)U_a(n + \hat{\mu}_a) + U_a(n)\psi_a(n + \hat{\mu}_a)]
\end{align*}
\]

Doubles lattice spacing $a \rightarrow a' = 2a$, with tunable rescaling factor $\xi$

Scalar fields from polar decomposition $U(n) = e^{\varphi(n)}U(n)$

$\Rightarrow$ shift $\varphi \rightarrow \varphi + \log \xi$ to keep blocked $U$ unitary

$Q$-preserving RG transformation needed

to show only one log. tuning to recover continuum $Q_a$ and $Q_{ab}$
Backup: Smearing for Konishi analyses

Smear to enlarge (MCRG or variational) operator basis

APE-like smearing: $$\rightarrow (1 - \alpha) \rightarrow + \frac{\alpha}{8} \sum \nabla,$$

staples built from unitary parts of links but no final unitarization

Average plaquette stable upon smearing (right),
minimum plaquette steadily increases (left)
Spontaneous susy breaking means $\langle 0 \mid H \mid 0 \rangle > 0$ or equivalently $\langle Q O \rangle \neq 0$

Twisted superQCD auxiliary field e.o.m. $\leftrightarrow$ Fayet–Iliopoulos $D$-term potential

$$d = \overline{D}_a U_a + \sum_{i=1}^F \phi_i \overline{\phi}_i - r I \mathbb{1}_N \leftrightarrow \text{Tr} \left[ \left( \sum_i \phi_i \overline{\phi}_i - r I \mathbb{1}_N \right)^2 \right] \in H$$

Have $F$ scalar vevs to zero out $N$ diagonal elements

$\rightarrow N > F$ suggests susy breaking, $\langle 0 \mid H \mid 0 \rangle > 0$ $\leftrightarrow$ $\langle Q \eta \rangle = \langle d \rangle \neq 0$