Lattice field theory for composite dark matter

David Schaich (Liverpool)

Southampton High Energy Theory Seminar, 29 November 2019

PRD 89, 094508  PRL 115, 171803  PRD 92, 075030
and more to come with the Lattice Strong Dynamics Collaboration
Dark matter — we observe it...

![Pie chart showing the composition of the universe according to ESA & Planck]

- Dark Matter: 26.8%
- Ordinary Matter: 4.9%
- Dark Energy: 68.3%
...we don’t yet know what it is
Overview

Composite dark matter is an attractive possibility

Lattice field theory is needed
  to constrain models from experimental results

Dark matter & compositeness

Lattice field theory

Experiments
  Large underground detectors
  High-energy particle colliders
  Gravitational-wave observatories
Gravitational evidence for dark matter

**Rotation** \( \sim 10^3 - 10^6 \) light-years

![Rotation diagram](image1)

**Lensing** \( \sim 10^6 \) light-years

![Lensing image](image2)

**Structure** \( \sim 10^9 \) light-years

![Structure image](image3)

**Cosmic background** \( \sim 10^{10} \) ly

![Cosmic background image](image4)
Non-gravitational dark matter interactions

Three search strategies

**Direct** scattering in underground detectors
Non-gravitational dark matter interactions

Three search strategies

Direct scattering in underground detectors

Collider production at high energies
Non-gravitational dark matter interactions

Three search strategies

**Direct** scattering in underground detectors

**Collider** production at high energies

**Indirect** annihilation into cosmic rays
Non-gravitational dark matter interactions

No clear signals so far
Why we expect non-gravitational interactions

\[ \frac{\Omega_{\text{dark}}}{\Omega_{\text{ordinary}}} \approx 5 \quad \text{... not } 10^5 \text{ or } 10^{-5} \]

Explained by non-gravitational interactions with known particles
Composite dark matter

**Early universe**
Deconfined charged fermions $\rightarrow$ non-gravitational interactions

**Present day**
Confined neutral ‘dark baryons’ $\rightarrow$ no experimental detections
Composite dark matter

Even neutral composites interact, via charged constituents

→ need **lattice calculations** for quantitative predictions
Lattice field theory in a nutshell

Formally \[ \langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \; \mathcal{O}(\Phi) \; e^{-S[\Phi]} \]

Regularize by formulating theory in finite, discrete space-time \( \rightarrow \) the lattice

Spacing between lattice sites ("a") \( \rightarrow \) UV cutoff scale \( \frac{1}{a} \)

Remove cutoff: \( a \rightarrow 0 \) \( (L/a \rightarrow \infty) \)

Hypercubic \( \rightarrow \) automatic symmetries
Numerical lattice field theory calculations

High-performance computing
→ evaluate up to
∼ billion-dimensional integrals

Importance sampling Monte Carlo

Algorithms sample field configurations with probability
\[
\frac{1}{Z} e^{-S[\Phi]}
\]

\[
\langle O \rangle = \frac{1}{Z} \int D\Phi \ O(\Phi) \ e^{-S[\Phi]} \quad \rightarrow \quad \frac{1}{N} \sum_{i=1}^{N} O(\Phi_i) \quad \text{with stat. uncertainty} \quad \propto \frac{1}{\sqrt{N}}
\]
Exploring the range of possible phenomena in strongly coupled field theories
Direct detection of composite dark matter

Charged constituents → **form factors** → experimental signals

**Photon exchange from electromagnetic form factors**

Effective interactions suppressed by powers of dark matter mass

- Magnetic moment $\sim \frac{1}{M_{DM}}$
- Charge radius $\sim \frac{1}{M_{DM}^2}$
- Polarizability $\sim \frac{1}{M_{DM}^3}$
Direct detection of composite dark matter

Charged constituents $\rightarrow$ form factors $\rightarrow$ experimental signals

Higgs exchange from scalar form factor

Can dominate cross section... if $F$ mass comes from Higgs
Direct detection of composite dark matter

Charged constituents $\rightarrow$ **form factors** $\rightarrow$ experimental signals

Simple first case: Dark matter like a “more-neutral neutron”
SU(3) with weak singlets $\rightarrow$ no Higgs-exchange interaction

Investigate leading photon-exchange contributions

- Magnetic moment $\sim \frac{1}{M_{DM}}$
- Charge radius $\sim \frac{1}{M_{DM}^2}$
Magnetic moment and charge radius

\[
\langle DM(p') \left| \Gamma_\mu(q^2) \right| DM(p) \rangle \sim F_1(q^2) \gamma_\mu + F_2(q^2) \frac{i\sigma_{\mu\nu}q^\nu}{2M_{DM}}, \quad q = p' - p
\]

Electric charge: \( F_1(0) = 0 \)

Magnetic moment: \( F_2(0) \)

Charge radius: \(-6 \left. \frac{dF_1(q^2)}{dq^2} \right|_{q^2=0} + \frac{3F_2(0)}{2M_{DM}^2}\)
Resulting direct detection constraints

Lattice calculations of magnetic moment and charge radius → event rate vs. dark matter mass

XENON100 → $M_B \gtrsim 10$ TeV

XENON1T → $M_B \gtrsim 30$ TeV [1805.12562]

Little effect from varying model params
Magnetic moment dominates event rate

Charge radius contributions (dashed) are suppressed \( \sim 1/M_{DM}^2 \)

Symmetries can forbid both magnetic moment and charge radius

\[ N_f = 2 \quad N_f = 6 \quad \text{XENON100 [1207.5988], 95% CL exclusion} \]

\[ M_X = M_B \text{ [TeV]} \]

More freedom for resulting model
Smarter second case: Stealth Dark Matter

SU(4) composite dark matter with four $F$

- Scalar particle $\rightarrow$ no magnetic moment $\checkmark$
- $+/-$ charge symmetry $\rightarrow$ no charge radius $\checkmark$

(Tiny) Coupling to Higgs needed for nucleosynthesis

Polarizability $\sim 1/M_{DM}^3$ dominates direct detection

$\rightarrow$ Unavoidable lower bound on broad class of composite dark matter models
‘Stealth’ composites constructed from conspicuous constituents

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<thead>
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<th>Direct detection cross section ((pb))</th>
<th>Radar cross section ((m^2))</th>
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‘Stealth’ composites constructed from conspicuous constituents

Direct detection cross section (pb)

- Neutrino
  \[ \sigma \sim 10^{-2} \]

- SUSY neutralino
  \[ 10^{-6} \lesssim \sigma \lesssim 10^{-5} \]

- Stealth Dark Matter
  \[ \sigma \sim \left( \frac{200 \text{ GeV}}{M_{DM}} \right)^6 \times 10^{-9} \]

Radar cross section (m^2)

- Neutrino
  \[ 747 \]
  \[ \sigma \sim 10^2 \]

- Falcon
  \[ \sigma \sim 10^{-2} \]

- Stealth F-22
  \[ \sigma < 10^{-3} \]
Polarizability of Stealth Dark Matter

Unavoidable lower bound on broad class of composite dark matter models

Nuclear physics very complicated with large uncertainties

Polarizability is dependence of lattice $M_{DM}$ on external field $\mathcal{E}$
Lower bound on direct detection

Results specific
to Xenon detectors

Uncertainty dominated
by Xenon nuclear physics

Shaded region is complementary constraint from particle colliders
The dark matter is the only stable composite particle, **not** the lightest.

Main constraints from much lighter **charged** “Π”

→ standard ‘missing energy’ searches not efficient
“Particularly tricky” at the LHC: Current bounds only \( M_\Pi \gtrsim 130 \) GeV similar to \( M_\Pi \gtrsim 100 \) GeV from LEP searches for SUSY tau-partner

Lattice calculation of \( M_{DM}/M_\Pi \rightarrow M_{DM} \gtrsim 300 \) GeV

More form factors to compute: \( F_1(4M_\Pi^2) \) for \( \Pi \) and decay constant \( F_\Pi \)
Gravitational waves

Gravitational-wave observatories opening new window on cosmology

First-order confinement transition $\rightarrow$ stochastic background of grav. waves

$\Rightarrow$ Lattice studies of stealth dark matter phase transition
Phase diagram expectations

Pure-gauge transition is first order
Becomes stronger as $N$ increases

First-order transition persists for sufficiently heavy fermions

Preliminary: Seem to need $M_P/M_V \gtrsim 0.9$

Form factor calculations considered
$0.55 \leq M_P/M_V \leq 0.77$
From first-order transition to gravitational wave signal

First-order transition $\rightarrow$ gravitational wave background will be produced

Four key parameters

Transition temperature $T_* \lesssim T_c$

Vacuum energy fraction from **latent heat**

Bubble nucleation rate (transition duration)

Bubble wall speed

BSM transitions $\rightarrow$ low frequencies requiring space-based observatories
Next step: Latent heat $\Delta \epsilon$

First-order transition $\rightarrow$ gravitational wave background will be produced

Vacuum energy fraction

$$\alpha \approx \frac{30}{4N(N^2 - 1)} \frac{\Delta \epsilon}{\pi^2 T^4}$$

Latent heat $\Delta \epsilon$

is change in energy density

at transition

\[
\Delta \epsilon \approx \frac{30}{4N(N^2 - 1)} \frac{\Delta \epsilon}{\pi^2 T^4}
\]
Recapitulation and outlook

Composite dark matter is an attractive possibility

Lattice field theory is needed
to constrain models from experimental results

Minimize EM form factors for direct detection

\[\rightarrow\text{Stealth Dark Matter}\]

Collider constraints on dark sector

Future searches for gravitational waves

And more: relic abundance; indirect detection; \ldots
Thank you!

Lattice Strong Dynamics Collaboration
Especially Graham Kribs, Ethan Neil, Enrico Rinaldi

Funding and computing resources
Backup: Thermal freeze-out for relic density

Requires non-gravitational interactions with known particles

\[ \text{DM} \leftrightarrow \text{SM \ for \ } T \gtrsim M_{DM} \]

\[ \text{DM} \rightarrow \text{SM \ for \ } T \lesssim M_{DM} \quad \Rightarrow \quad \text{rapid depletion of \ } \Omega_{DM} \]

Hubble expansion \quad \Rightarrow \quad \text{dilution} \rightarrow \text{freeze-out}

2 \rightarrow 2 \text{ scattering relates coupling and mass, } \quad 200\alpha \sim \frac{M_{DM}}{100 \gev}

Strong \ \alpha \sim 16 \quad \rightarrow \quad \text{‘natural’ mass scale } M_{DM} \sim 300 \text{ TeV}

Smaller \ M_{DM} \gtrsim 1 \text{ TeV possible from } 2 \rightarrow n \text{ scattering or asymmetry}
Idea: Dark matter relic density related to baryon asymmetry

\[ \Omega_D \approx 5\Omega_B \]
\[ \Rightarrow M_D n_D \approx 5M_B n_B \]

\[ n_D \sim n_B \quad \Rightarrow \quad M_D \sim 5M_B \approx 5 \text{ GeV} \]

High-dim. interactions relate baryon# and DM# violation

\[ M_D \gg M_B \quad \Rightarrow \quad n_B \gg n_D \sim \exp\left[-\frac{M_D}{T_s}\right] \quad T_s \sim 200 \text{ GeV} \]

EW sphaleron processes above \( T_s \) distribute asymmetries

Both require non-gravitational interactions with known particles
Backup: Hybrid Monte Carlo (HMC) algorithm

Goal: Sample field configurations $\Phi$ with probability $\frac{1}{Z} e^{-S[\Phi]}$

HMC is Markov process based on Metropolis–Rosenbluth–Teller

Fermions $\rightarrow$ extensive action computation

$\Rightarrow$ Global updates via fictitious molecular dynamics

1. Introduce fictitious random momenta and “MD time” $\tau$

2. Inexact MD evolution along trajectory in $\tau \rightarrow$ new configuration

3. Accept/reject test on MD discretization error

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Backup: More details about form factors

### Photon exchange via electromagnetic form factors

Interactions suppressed by powers of confinement scale $\Lambda \sim M_{DM}$

**Dimension 5:** Magnetic moment  
$$ \langle X \sigma_{\mu\nu} X \rangle F^{\mu\nu} / \Lambda $$

**Dimension 6:** Charge radius  
$$ \langle XX \rangle \nu_\mu \partial_\nu F^{\mu\nu} / \Lambda^2 $$

**Dimension 7:** Polarizability  
$$ \langle XX \rangle \nu_\mu \nu_\nu F^{\mu\alpha} F^{\nu\alpha} / \Lambda^3 $$

### Higgs exchange via scalar form factors

Higgs couples through $\sigma$ terms  
$$ \langle B \left| m_\psi \overline{\psi} \psi \right| B \rangle $$

Produces rapid charged ‘$\Pi$’ decay needed for Big Bang nucleosynthesis
Backup: More details about SU(3) composite dark matter model

Same SU(3) gauge group as QCD

Re-analyze existing data sets:
- $32^3 \times 64$ lattices, domain wall fermions

Scan relatively heavy fermion masses $m_F \rightarrow 0.55 \lesssim M\Pi / M_V \lesssim 0.75$

Compare $N_F = 2$ or 6 degenerate flavors with same $M_{B_0} \equiv \lim_{m_F \rightarrow 0} M_B$

Unlike QCD, fermions are all SU(2)$_L$ singlets $\rightarrow Q = Y$

Half have $Q_p = 2/3$, half $Q_M = -1/3$

Dark matter candidate is singlet “dark baryon” $B = \text{PMM}$
Backup: Form factor calculations on the lattice

\[ R(\tau, T, p, p') \sim \langle DM(p') | \Gamma_\mu(q^2) | DM(p) \rangle + \mathcal{O}(e^{-\Delta \tau}, e^{-\Delta T}, e^{-\Delta(T-\tau)}) \]
Backup: Electromagnetic form factor results

**Magnetic moment** \( \kappa \)

- \( N_f = 2 \)
- \( N_f = 6 \)

**Charge radius** \( \langle r^2 \rangle \)

- \( N_f = 2 \)
- \( N_f = 6 \)

Little dependence on \( N_F \) or on \( m_F \sim M_B/M_{B_0} \)

\( \kappa \) comparable to neutron’s \( \kappa_N = -1.91 \)

\( \langle r^2 \rangle \) smaller than neutron’s \( \langle r^2 \rangle_N \approx -38 \) (related to larger \( M_\Pi/M_V \))

Insert into standard event rate formulas...
Rate = \frac{M_{\text{detector}}}{M_T} \frac{\rho_{DM}}{M_{DM}} \int_{E_{\text{min}}}^{E_{\text{max}}} dE_R \, \text{Acc}(E_R) \left\langle v_{DM} \frac{d\sigma}{dE_R} \right\rangle_f

\frac{d\sigma}{dE_R} = \frac{|\mathcal{M}_{SI}|^2 + |\mathcal{M}_{SD}|^2}{16\pi (M_{DM} + M_T)^2 E_R^{\text{max}}}

E_R^{\text{max}} = \frac{2M_{DM}^2 M_T \nu_{col}^2}{(M_{DM} + M_T)^2}

From magnetic moment \( \kappa \) and charge radius \( \langle r^2 \rangle \)

\frac{|\mathcal{M}_{SI}|^2}{e^4 [ZF_c(Q)]^2} = \left( \frac{M_T}{M_{DM}} \right)^2 \left[ \frac{4}{9} M_{DM}^4 \langle r^2 \rangle^2 + \frac{\kappa^2 (M_T + M_{DM})^2 (E_R^{\text{max}} - E_R)}{M_T E_R} \right]

|\mathcal{M}_{SD}|^2 = e^4 \frac{2}{3} \left( \frac{J + 1}{J} \right) \left[ \left( A_{\mu_T}^{\mu_n} \right) F_s(Q) \right]^2 \kappa^2
Backup: Event rate formulas and lattice input

Rate = \frac{M_{\text{detector}}}{M_T} \frac{\rho_{DM}}{M_{DM}} \int_{E_{\text{min}}}^{E_{\text{max}}} dE_R \text{Acc}(E_R) \left\langle v_{DM} \frac{d\sigma}{dE_R} \right\rangle_f

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E_R^{\text{max}} = \frac{2M_{DM}^2 M_T v_{col}^2}{(M_{DM} + M_T)^2}

From polarizability \: C_F

\sigma_{SI} = \frac{Z^4}{A^2} \frac{144\pi \alpha_{em}^4 \tilde{M}_{n,DM}^2}{M_{DM}^6 R^2} C_F^2 \propto \frac{Z^4}{A^2} \text{ per nucleon}
Backup: More details about SU(4) Stealth Dark Matter

Quenched SU(4) lattice ensembles

Lattice volumes up to $64^3 \times 128$, several lattice spacings to check systematic effects

**Flavor combinations**

- $S=0$
- $S=1$
- $S=2$

Dark matter candidate is spin-zero baryon $\rightarrow$ no magnetic moment

Need at least two flavors to anti-symmetrize $\rightarrow$ no charge radius
Backup: Even more details about SU(4) Stealth Dark Matter

Mass terms \( m_V (F_1 F_2 + F_3 F_4) + y (F_1 \cdot HF_4 + F_2 \cdot H^\dagger F_3) + \text{h.c.} \)

Vector-like masses evade Higgs-exchange direct detection bounds

Higgs couplings \( \rightarrow \) charged meson decay before Big Bang nucleosynthesis

Both required \( \rightarrow \) four flavors
Backup: Stealth Dark Matter mass scales

Lattice studies focus on $m_\psi \simeq \Lambda_{DM}$ where effective theories least reliable

$m_\psi \simeq \Lambda_{DM}$ could arise dynamically

Collider constraints on $M_{DM}$ become stronger as $m_\psi$ decreases
Backup: Effective Higgs interaction

\[ M_H = 125 \text{ GeV} \quad \rightarrow \quad \text{Higgs exchange can dominate direct detection} \]

\[
\sigma_H^{(SI)} \propto \left| \frac{\tilde{M}_{DM,N}}{M_H^2} \right|^2 y_\psi \left\langle DM \left| \bar{\psi} \psi \right| DM \right\rangle y_q \left\langle N \left| \bar{q} q \right| N \right\rangle
\]

Quark \( y_q = \frac{m_q}{v} \)

Dark \( y_\psi = \alpha \frac{m_\psi}{v} \) suppressed by \( \alpha \equiv \left. \frac{v}{m_\psi} \frac{\partial m_\psi(h)}{\partial h} \right|_{h=v} = \frac{yv}{yv + m_V} \)

Determine using Feynman–Hellmann theorem \( \left\langle DM \left| \bar{\psi} \psi \right| DM \right\rangle = \frac{\partial M_{DM}}{\partial m_\psi} \)
Backup: Feynman–Hellmann theorem

\[ m_\psi \bar{\psi} \psi \] is the only term in the Hamiltonian that depends on \( m_\psi \)

\[
\implies \left\langle B \left| \frac{\partial \hat{H}}{\partial m_\psi} \right| B \right\rangle = \left\langle B \left| \bar{\psi} \psi \right| B \right\rangle
\]

Since \( \hat{H} |B\rangle = M_B |B\rangle \) and \( \langle B| \hat{H} = \langle B| M_B \) we have

\[
\frac{\partial}{\partial m_\psi} M_B = \frac{\partial}{\partial m_\psi} \left\langle B \left| \hat{H} \right| B \right\rangle = \left\langle B \left| \frac{\partial \hat{H}}{\partial m_\psi} \right| B \right\rangle + \left\langle B \left| \hat{H} \left| \frac{\partial B}{\partial m_\psi} \right\rangle + \left\langle B \left| \frac{\partial \hat{H}}{\partial m_\psi} \right| B \right\rangle \right.
\]

\[
= M_B \langle \frac{\partial B}{\partial m_\psi} \rangle + M_B \langle B \frac{\partial B}{\partial m_\psi} \rangle + \langle B \bar{\psi} \psi | B \rangle
\]

\[
= M_B \frac{\partial}{\partial m_\psi} \langle B B \rangle + \langle B \bar{\psi} \psi | B \rangle = \langle B \bar{\psi} \psi | B \rangle \quad \square
\]
Backup: Lattice results for Higgs exchange constrain $\alpha$

$$\sigma_H^{(SI)} \propto |y_\psi \langle DM | \bar{\psi} \psi | DM \rangle|^2$$

Matrix element $\propto \frac{\partial M_{DM}}{\partial m_\psi}$
(Feynman–Hellmann)

Stealth Dark Matter:
$$0.15 \lesssim \frac{m_\psi}{M_{DM}} \frac{\partial M_{DM}}{\partial m_\psi} \lesssim 0.34$$

Larger than QCD
$$0.04 \lesssim \frac{m_q}{M_N} \frac{\partial M_N}{\partial m_q} \lesssim 0.08$$
Backup: Bounds on effective Higgs coupling

Higgs-exchange cross section $\rightarrow$ maximum $\alpha$ allowed by LUX [1310.8214]

Maximum $\alpha$ depends on $M_\Pi/M_V$ and dark matter mass

Smaller $M_\Pi/M_V \leftrightarrow m_F \rightarrow$ stronger constraints from colliders

Effective Higgs interaction tightly constrained

$\alpha \lesssim 0.3$ for $M_\Pi/M_V \gtrsim 0.55 \rightarrow$ fermion masses must be mainly vector-like
Backup: More about Stealth Dark Matter at the LHC

LHC can search for $\Pi^+\Pi^- \rightarrow t\bar{b} + \bar{t}b$ in addition to $\tau^+\tau^- + E_T$

Should eventually surpass $M_\Pi \gtrsim 100$ GeV from LEP
Backup: Indirect detection

Lattice results for composite spectrum

Predict $\gamma$-rays from splitting between baryons with spin $S = 0, 1$ and $2$

Much more challenging future work

$\text{DM–\overline{DM}}$ annihilation into (many) lighter $\Pi$ that then decay
Backup: Volume and discretization effects

Baryon masses vs. $L$ at fixed lattice spacing (set by $\beta \approx 8/g_0^2$) and fermion mass
Backup: Volume and discretization effects

Edinburgh-style plot of $\frac{M_{S0}}{M_V}$ vs. $\frac{M_\Pi}{M_V}$ and line of constant physics (LCP)
Backup: Volume and discretization effects

Lattice spacing and discretization effects for $\frac{M_{S2,S1}}{M_{S0}}$ on line of constant physics

![Graphs showing lattice spacing and discretization effects for different values of L.](image)
Backup: Large-$N$ predictions for SU(4) baryons

Tune $(\beta, m_F)$ to match SU(3) $M_\Pi$ and $M_V$ (dashed)

Rotor spectrum for spin-$J$ baryons: $M(N, J) = NM_0 + C + B\frac{J(J+1)}{N} + \mathcal{O}\left(\frac{1}{N^2}\right)$

Fit $M_0$, $C$ and $B$ with nucleon, $\Delta$ and spin-0 baryon masses

$\rightarrow$ predictions for $S = 1, 2$ baryons (diamonds)
Backup: Pure gauge checks — Bulk and thermal transitions

Try to avoid bulk transition for small $N_T$ → use $\beta_A = -\beta_F/4$

Still need $N_T > 4$ for clear separation between bulk & thermal transitions
Try to avoid bulk transition for small $N_T \rightarrow$ use $\beta_A = -\beta_F/4$

Still need $N_T > 4$ for clear separation between bulk & thermal transitions
Backup: Pure gauge checks — Bulk and thermal transitions

Try to avoid bulk transition for small $N_T \rightarrow$ use $\beta_A = -\beta_F/4$

Still need $N_T > 4$ for clear separation between bulk & thermal transitions
Backup: Pure gauge checks — Order of thermal transition

- Two peaks in Polyakov loop magnitude histogram $\rightarrow$ first-order transition $\checkmark$

- Hysteresis not clearly visible even in pure-gauge case