Lattice studies of maximally supersymmetric Yang–Mills theories

David Schaich (Bern)

Liverpool Theoretical Physics Seminar, 28 November 2018

& more to come with Simon Catterall, Raghav Jha and Toby Wiseman
Last time...

Lattice field theory is a broadly applicable tool to study strongly coupled quantum field theories.

A high-level summary of lattice field theory

Applications — recent results & future plans

- Composite dark matter
- Dense nuclear matter
- Supersymmetry and holographic duality
- Composite Higgs boson

Outlook
Lattice field theory is a broadly applicable tool to study strongly coupled quantum field theories.

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- **Supersymmetry and holographic duality**
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Outlook
Lattice studies of maximally supersymmetric Yang–Mills theories

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Overview and plan

**WHY:** Lattice supersymmetry

**HOW:** Lattice formulation highlights

**WHAT:**
- Dimensionally reduced (2d) thermodynamics
- Static potential (4d)
- Conformal scaling dimensions

Prospects and future directions
Overview and plan

Central idea
Preserve (some) susy in discrete space-time
→ practical lattice investigations

Goals
1) Reproduce reliable results in perturbative, holographic, etc. regimes

2) Access new domains
Motivations

Lattice field theory promises first-principles predictions for strongly coupled supersymmetric QFTs

BSM  QFT  Holography

(Derek Leinweber)
Supersymmetry is a space-time symmetry

4d Poincaré plus $4\mathcal{N}$ spinor generators $Q^I_\alpha$ and $\bar{Q}^I_{\dot{\alpha}}$, $I = 1, \ldots, \mathcal{N}$

\[ \left\{ Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}} \right\} = 2\delta^{IJ} \sigma^\mu_{\alpha\dot{\alpha}} P_\mu \] broken in discrete space-time

\[ \rightarrow \text{relevant susy-violating operators} \]

Scalar mass, Yukawas, Quartics, Quark mass, Gluino mass
Solution

Preserve susy sub-algebra at non-zero lattice spacing

\[ \Rightarrow \text{correct continuum limit with little or no fine tuning} \]

Equivalent constructions from topological twisting and deconstruction

Review:

arXiv:0903.4881

Need \( 2^d \) supersymmetries in \( d \) dimensions

\[ d = 4 \quad \longrightarrow \quad \text{maximally supersymmetric Yang–Mills} \quad (\mathcal{N} = 4 \text{ SYM}) \]
\( \mathcal{N} = 4 \) SYM in a nutshell

Arguably simplest non-trivial 4d QFT \( \rightarrow \) dualities, amplitudes, \( \ldots \)

SU(\( \mathcal{N} \)) gauge theory with \( \mathcal{N} = 4 \) fermions \( \psi^I \) and 6 scalars \( \phi^{IJ} \), all massless and in adjoint rep.

**Symmetries** relate coeffs of kinetic, Yukawa and \( \phi^4 \) terms

Maximal 16 supersymmetries \( Q^I_\alpha \) and \( \bar{Q}^I_{\dot{\alpha}} \) \( \, \, I = 1, \ldots, 4 \)
transform under global \( SU(4) \sim SO(6) \) \( \text{R symmetry} \)

Conformal \( \rightarrow \beta \) function is zero for all values of \( \lambda = g^2 N \)
Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand $4 \times 4$ matrix of supersymmetries

$$
\begin{pmatrix}
Q_1^\alpha & Q_2^\alpha & Q_3^\alpha & Q_4^\alpha \\
\bar{Q}_1^{\dot{\alpha}} & \bar{Q}_2^{\dot{\alpha}} & \bar{Q}_3^{\dot{\alpha}} & \bar{Q}_4^{\dot{\alpha}}
\end{pmatrix} = Q + Q_\mu \gamma_\mu + Q_{\mu \nu} \gamma_\mu \gamma_\nu + \bar{Q}_\mu \gamma_\mu \gamma_5 + \bar{Q} \gamma_5
$$

\[\rightarrow Q + Q_{a} \gamma_{a} + Q_{ab} \gamma_{a} \gamma_{b}\]

with $a, b = 1, \ldots, 5$

R-symmetry index $\times$ Lorentz index

\[\rightarrow Q\] transform in reps of ‘twisted rotation group’

$$
\text{SO}(4)_{tw} \equiv \text{diag} \left[ \text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R \right] \quad \text{SO}(4)_R \subset \text{SO}(6)_R
$$

Change of variables $\rightarrow Q$ with integer ‘spin’ under $\text{SO}(4)_{tw}$
Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand $4 \times 4$ matrix of supersymmetries

$$
\begin{pmatrix}
Q_{\alpha}^1 & Q_{\alpha}^2 & Q_{\alpha}^3 & Q_{\alpha}^4 \\
\bar{Q}_{\dot{\alpha}}^1 & \bar{Q}_{\dot{\alpha}}^2 & \bar{Q}_{\dot{\alpha}}^3 & \bar{Q}_{\dot{\alpha}}^4
\end{pmatrix} = Q + Q_\mu \gamma_\mu + Q_{\mu\nu} \gamma_\mu \gamma_\nu + \bar{Q}_\mu \gamma_\mu \gamma_5 + \bar{Q} \gamma_5
$$

$$
\rightarrow Q + Q_a \gamma_a + Q_{ab} \gamma_a \gamma_b
$$

with $a, b = 1, \ldots, 5$

Discrete space-time
Can preserve closed sub-algebra

$$\{Q, Q\} = 2Q^2 = 0$$
Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand $4 \times 4$ matrix of supersymmetries

$$
\begin{pmatrix}
Q_1^\alpha & Q_2^\alpha & Q_3^\alpha & Q_4^\alpha \\
\overline{Q}_1^{\dot{\alpha}} & \overline{Q}_2^{\dot{\alpha}} & \overline{Q}_3^{\dot{\alpha}} & \overline{Q}_4^{\dot{\alpha}}
\end{pmatrix}
= Q + Q_\mu \gamma_\mu + Q_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{Q}_\mu \gamma_\mu \gamma_5 + \overline{Q} \gamma_5

\rightarrow Q + Q_a \gamma_a + Q_{ab} \gamma_a \gamma_b

\text{with } a, b = 1, \cdots, 5

Discrete space-time

Can preserve closed subalgebra

$$\{Q, Q\} = 2Q^2 = 0$$
Completing the twist

Fields also transform with integer spin under $SO(4)_{tw}$ — no spinors

$$\psi \text{ and } \bar{\psi} \rightarrow \eta, \psi_a \text{ and } \chi_{ab}$$

$$A_\mu \text{ and } \Phi^I \rightarrow \text{complexified gauge field } A_a \text{ and } \bar{A}_a$$

$$\rightarrow U(N) = SU(N) \otimes U(1) \text{ gauge theory}$$

✓ $Q$ interchanges bosonic $\leftrightarrow$ fermionic d.o.f. with $Q^2 = 0$

$$Q \ A_a = \psi_a \quad Q \ \psi_a = 0$$

$$Q \ \chi_{ab} = -\bar{F}_{ab} \quad Q \ \bar{A}_a = 0$$

$$Q \ \eta = d \quad Q \ d = 0$$

bosonic auxiliary field with e.o.m. $d = \bar{D}_a A_a$
Lattice $\mathcal{N} = 4$ SYM

Lattice theory looks nearly the same despite breaking $Q_a$ and $Q_{ab}$

Covariant derivatives $\rightarrow$ finite difference operators

Complexified gauge fields $A_a \rightarrow$ gauge links $U_a \in \mathfrak{gl}(N, \mathbb{C})$

\[
Q A_a \rightarrow Q U_a = \psi_a \quad \quad Q \psi_a = 0
\]

\[
Q \chi_{ab} = -\bar{F}_{ab} \quad \quad Q \bar{A}_a \rightarrow Q \bar{U}_a = 0
\]

\[
Q \eta = d \quad \quad Q d = 0
\]

**Geometry:** $\eta$ on sites, $\psi_a$ on links, etc.

Susy lattice action ($QS = 0$) from $Q^2 \cdot = 0$ and Bianchi identity

\[
S = \frac{N}{4\lambda_{\text{lat}}} \text{Tr} \left[ Q \left( \chi_{ab} \mathcal{F}_{ab} + \eta \mathcal{D}_a U_a - \frac{1}{2} \eta d \right) - \frac{1}{4} \varepsilon_{abcde} \chi_{ab} \mathcal{D}_c \chi_{de} \right]
\]
Five links in four dimensions $\rightarrow A_4^*$ lattice

$A_4^* \sim 4d$ analog of 2d triangular lattice

Basis vectors linearly dependent and non-orthogonal

Large $S_5$ point group symmetry

$S_5$ irreps precisely match onto irreps of twisted SO(4)$_{tw}$

- $\psi_a \rightarrow \psi_\mu, \quad \bar{\eta}$ is $5 \rightarrow 4 \oplus 1$
- $\chi_{ab} \rightarrow \chi_{\mu\nu}, \quad \bar{\psi}_\mu$ is $10 \rightarrow 6 \oplus 4$

$S_5 \rightarrow$ SO(4)$_{tw}$ in continuum limit restores $Q_a$ and $Q_{ab}$
Analytic results for twisted $\mathcal{N} = 4$ SYM on $A_4^*$ lattice

$U(N)$ gauge invariance + $Q$ + $S_5$ lattice symmetries

→ Moduli space preserved to all orders

→ One-loop lattice $\beta$ function vanishes

→ Only one log. tuning to recover continuum $Q_a$ and $Q_{ab}$


Not yet suitable for numerical calculations
Must regulate zero modes and flat directions, especially in $U(1)$ sector
Two deformations stabilize lattice calculations

(i) Add SU($N$) scalar potential \( \propto \mu^2 \sum_a (\text{Tr} \left[ U_a \overline{U}_a \right] - N)^2 \)

**Softly** breaks susy \( \rightarrow \) \( Q \)-violating operators vanish \( \propto \mu^2 \rightarrow 0 \)

Test via Ward identity violations: \( Q \left[ \eta U_a \overline{U}_a \right] \neq 0 \)
Two deformations stabilize lattice calculations

(ii) Constrain U(1) plaquette determinant \( \sim G \sum_{a<b} (\det P_{ab} - 1) \)

Implemented supersymmetrically as Fayet–Iliopoulos \( D \)-term potential

\[ \left\langle \frac{|QO|}{\sqrt{D^2 + F^2}} \right\rangle \]
so that the full improved action becomes

\[
S_{\text{imp}} = S_{\text{exact}}' + S_{\text{closed}} + S_{\text{soft}}' =
\]

\[
\frac{N}{4\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ -\mathcal{F}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}_{+a}^+(n)\psi_b^{+}(n) - \eta(n)\mathcal{D}_{+a}^-(n)\psi_a^{-}(n) \right.
\]

\[
\left. + \frac{1}{2} \left( \mathcal{D}_{a}^-(n)\mathcal{U}_a(n) + G \sum_{a \neq b} (\det \mathcal{P}_{ab}(n) - 1) \mathbb{I}_N \right)^2 \right] - S_{\text{det}}
\]

\[
S_{\text{det}} = \frac{N}{4\lambda_{\text{lat}}} G \sum_n \text{Tr} [\eta(n)] \sum_{a \neq b} (\det \mathcal{P}_{ab}(n)) \text{Tr} [\mathcal{U}_b^{-1}(n)\psi_b(n) + \mathcal{U}_a^{-1}(n + \tilde{\mu}_b)\psi_a(n + \tilde{\mu}_b)]
\]

\[
S_{\text{closed}} = \frac{N}{16\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ \epsilon_{abcde} \chi_{de}^{+}(n + \tilde{\mu}_a + \tilde{\mu}_b + \tilde{\mu}_c)\mathcal{D}_{c}^-(n)\chi_{ab}(n) \right]
\]

\[
S_{\text{soft}}' = \frac{N}{4\lambda_{\text{lat}}} \mu^2 \sum_n \sum_a \left( \frac{1}{N} \text{Tr} [\mathcal{U}_a(n)\mathcal{U}_a(n)] - 1 \right)^2
\]

≥100 inter-node data transfers in the fermion operator — non-trivial.

Public parallel code to reduce barriers to entry

\[\text{github.com/daschaich/susy}\]

Evolved from MILC QCD code, user guide in \[\text{arXiv:1410.6971}\]
Dimensionally reduce to 2d $\mathcal{N} = (8,8)$ SYM with four scalar $Q$

Low temperatures $t = 1/r_\beta \longleftrightarrow$ black holes in dual supergravity

For decreasing $r_L$ at large $N$

homogeneous black string (D1) $\longrightarrow$ localized black hole (D0)

“spatial deconfinement” signalled by Wilson line $P_L$
2d $\mathcal{N} = (8, 8)$ SYM lattice phase diagram results

Good agreement with high-temp. bosonic QM
Consistent with holography at low temperatures

Example spatial deconfinement transition in Wilson line

Fix aspect ratio $\alpha = r_L/r_\beta = 4$, scan in $r_\beta = r_L/\alpha$
Dual black hole thermodynamics

Holography: bosonic action $\leftrightarrow$ dual black hole internal energy

$\propto t^3$ for large-$r_L$ D1 phase

$\propto t^{3.2}$ for small-$r_L$ D0 phase

Lattice results consistent with holography for sufficiently low $t \lesssim 0.4$

Need larger $N > 16$ to avoid instabilities at lower temperatures
(ii) 4d $\mathcal{N} = 4$ SYM static potential $V(r)$

Static probes $\rightarrow r \times T$ Wilson loops $W(r, T) \propto e^{-V(r) T}$

Coulomb gauge trick reduces $A_4^*$ lattice complications
Static potential is Coulombic at all $\lambda$

Fits to confining $V(r) = A - C/r + \sigma r \quad \rightarrow \quad \text{vanishing string tension } \sigma$

$\implies$ Fit to just $V(r) = A - C/r$ to extract Coulomb coefficient $C(\lambda)$

Discretization artifacts reduced by tree-level improved analysis
Coupling dependence of Coulomb coefficient

Continuum perturbation theory $\rightarrow C(\lambda) = \lambda/(4\pi) + O(\lambda^2)$

Holography $\rightarrow C(\lambda) \propto \sqrt{\lambda}$ for $N \rightarrow \infty$ and $\lambda \rightarrow \infty$ with $\lambda \ll N$

Consistent with leading-order perturbation theory for $\lambda_{\text{lat}} \leq 2$
(iii) Konishi operator scaling dimension

\[ \mathcal{O}_K(x) = \sum_I \text{Tr} \left[ \Phi^I(x) \Phi^I(x) \right] \] is simplest conformal primary operator

Scaling dimension \( \Delta_K(\lambda) = 2 + \gamma_K(\lambda) \) investigated through perturbation theory (\& S duality), holography, conformal bootstrap

\[ C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K} \]

‘SUGRA’ is 20’ op., \( \Delta_S = 2 \)

Will compare:
- Direct power-law decay
- Finite-size scaling
- Monte Carlo RG
(iii) Konishi operator scaling dimension

Lattice scalars $\varphi(n)$ from polar decomposition of complexified links

$$\mathcal{U}_a(n) \rightarrow e^{\varphi_a(n)} \mathcal{U}_a(n)$$

$$\mathcal{O}_{\text{lat}}^K(n) = \sum_a \text{Tr}[\varphi_a(n)\varphi_a(n)] - \text{vev}$$

$$\mathcal{O}_{\text{lat}}^S(n) \sim \text{Tr}[\varphi_a(n)\varphi_b(n)]$$

$$C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

'SUGRA' is 20' op., $\Delta_S = 2$

Will compare:
- Direct power-law decay
- Finite-size scaling
- Monte Carlo RG
Scaling dimensions from MCRG stability matrix

Lattice system: \( H = \sum_i c_i O_i \) (infinite sum)

Couplings flow under RG blocking \( \rightarrow H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} O_i^{(n)} \)

Fixed point \( \rightarrow H^* = R_b H^* \) with couplings \( c_i^* \)

Linear expansion around fixed point \( \rightarrow \) stability matrix \( T_{ik}^* \)

\[
c_i^{(n)} - c_i^* = \sum_k \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \bigg|_{H^*} \left( c_k^{(n-1)} - c_k^* \right) \equiv \sum_k T_{ik}^* \left( c_k^{(n-1)} - c_k^* \right)
\]

Correlators of \( O_i, O_k \) \( \rightarrow \) elements of stability matrix [Swendsen, 1979]

Eigenvalues of \( T_{ik}^* \) \( \rightarrow \) scaling dimensions of corresponding operators
Preliminary $\Delta_K$ results from Monte Carlo RG

Analyzing both $\mathcal{O}_{K}^{\text{lat}}$ and $\mathcal{O}_{S}^{\text{lat}}$

Imposing protected $\Delta_S = 2$

$\rightarrow \Delta_K(\lambda)$ looks perturbative

Systematic uncertainties from different amounts of smearing

Complication from twisting $\text{SO}(4)_R \subset \text{SO}(6)_R$

$\mathcal{O}_{K}^{\text{lat}}$ mixes with $\text{SO}(4)_R$-singlet part of $\text{SO}(6)_R$-nonsinglet $\mathcal{O}_S$

Working on variational analyses to disentangle operators
Future: Pushing $\mathcal{N} = 4$ SYM to stronger coupling

✓ Reproduce reliable 4d results in perturbative regime

$\rightarrow$ Check holographic predictions and access new domains

Sign problem seems to become obstruction

$$\langle O \rangle = \frac{1}{Z} \int [dU][d\overline{U}] \ O \ e^{-S_B[U,\overline{U}]} \ \text{pf} \ D[U,\overline{U}]$$

Complex pfaffian $\text{pf} \ D = |\text{pf} \ D| e^{i\alpha}$ complicates importance sampling

We phase quench, $\text{pf} \ D \rightarrow |\text{pf} \ D|$, need to reweight $\langle O \rangle = \frac{\langle O e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}}$
\( \mathcal{N} = 4 \) SYM sign problem

Fix \( \lambda_{\text{lat}} = g_{\text{lat}}^2 N = 0.5 \)

Pfaffian nearly real positive for all accessible volumes

\[
\text{Fix } 4^4 \text{ volume}
\]

Fluctuations increase with coupling

Signal-to-noise becomes obstruction for \( \lambda_{\text{lat}} \gtrsim 4 \)
Future: Lattice superQCD (in 2d & 3d)

Preserve twisted supersymmetry sub-algebra

Proposed by Matsuura [0805.4491] and Sugino [0807.2683],

first numerical study by Catterall & Veernala [1505.00467]

2-slice lattice SYM

with $U(N) \times U(F)$ gauge group

Adj. fields on each slice

Bi-fundamental in between

Decouple $U(F)$ slice

$\rightarrow U(N)$ SQCD in $d - 1$ dims.

with $F$ fund. hypermultiplets

$U(N_c)$ SYM Adjoint Model

$[U_\mu, \bar{U}_\mu, (\eta, \psi_\mu, \chi_{\mu\nu})]$
Dynamical susy breaking in 2d lattice superQCD

Auxiliary field e.o.m. \( \rightarrow \) Fayet–Iliopoulos \( D \)-term potential

\[
d = \overline{D}_a U_a + \sum_{i=1}^{F} \phi_i \overline{\phi}_i - r I_N \quad \rightarrow \quad S_D \propto \text{Tr} \left[ \left( \sum_i \phi_i \overline{\phi}_i - r I_N \right)^2 \right]
\]

Zero out \( N \) diagonal elements via \( F \) scalar vevs

or else susy breaking, \( \langle Q \eta \rangle = \langle d \rangle \neq 0 \leftarrow \langle 0 \mid H \mid 0 \rangle > 0 \)
Recap: An exciting time for lattice supersymmetry

✓ Preserve (some) susy in discrete space-time

→ practical lattice $\mathcal{N} = 4$ SYM, public code available

Reproduce reliable analytic results

✓ 2d $\mathcal{N} = (8, 8)$ SYM thermodynamics consistent with holography

✓ Perturbative static potential Coulomb coefficient $C(\lambda)$
  and Konishi operator conformal scaling dimension $\Delta_K(\lambda)$

Access new domains

→ Tackling a sign problem at stronger couplings

→ Lower-dimensional superQCD and more...
Thank you!

Collaborators
Simon Catterall, Raghav Jha, Toby Wiseman
also Georg Bergner, Poul Damgaard, Joel Giedt, Anosh Joseph

Funding and computing resources
Lattice field theory in a nutshell

Formally

\[ \langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \, \mathcal{O}(\Phi) \, e^{-S[\Phi]} \]

... but infinite-dimensional integrals in general intractable

Formulate theory in finite, discrete space-time \( \rightarrow \) the lattice

Spacing between lattice sites ("\( a \)"") \( \rightarrow \) UV cutoff scale \( 1/a \)

Remove cutoff: \( a \rightarrow 0 \) \( (L/a \rightarrow \infty) \)

Hypercubic \( \rightarrow \) automatic symmetries
Numerical lattice field theory calculations

High-performance computing
→ evaluate up to
∼ billion-dimensional integrals

Importance sampling Monte Carlo

Algorithms sample field configurations with probability

\[ \langle O \rangle = \frac{1}{Z} \int D\Phi \ O(\Phi) \ e^{-S[\Phi]} \]

\[ \rightarrow \frac{1}{N} \sum_{i=1}^{N} O(\Phi_i) \text{ with statistical uncertainty } \propto \frac{1}{\sqrt{N}} \]
Backup: Breakdown of Leibniz rule on the lattice

\[ \{ Q_\alpha, \overline{Q}_{\dot{\alpha}} \} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = 2i\sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \] is problematic

\[ \implies \text{try finite difference} \quad \partial \phi(x) \rightarrow \Delta \phi(x) = \frac{1}{a} [\phi(x + a) - \phi(x)] \]

Crucial difference between \( \partial \) and \( \Delta \)

\[ \Delta [\phi \eta] = a^{-1} [\phi(x + a)\eta(x + a) - \phi(x)\eta(x)] \]

\[ = [\Delta \phi] \eta + \phi \Delta \eta + a [\Delta \phi] \Delta \eta \]

Only recover Leibniz rule \( \partial [\phi \eta] = [\partial \phi] \eta + \phi \partial \eta \) when \( a \rightarrow 0 \)

\[ \implies \text{“discrete supersymmetry” breaks down on the lattice} \]
Backup: Complexified gauge field from twisting

Combining $A_\mu$ and $\Phi^I \longrightarrow A_a$ and $\overline{A}_a$

$\longrightarrow U(N) = SU(N) \otimes U(1)$ gauge theory

Complicates lattice action but needed so that $Q A_a = \psi_a$

Further motivation: Under $SO(d)_{tw} = \text{diag}[SO(d)_{euc} \otimes SO(d)_R]$

$A_\mu \sim \text{vector} \otimes \text{scalar} = \text{vector}$

$\Phi^I \sim \text{scalar} \otimes \text{vector} = \text{vector}$

Easiest to see in 5d (then dimensionally reduce)

$A_a = A_a + i \Phi_a \longrightarrow (A_\mu, \phi) + i(\Phi_\mu, \overline{\phi})$
Backup: $A^*_4$ lattice from five dimensions

Again dimensionally reduce, treating all five gauge links symmetrically

Start with hypercubic lattice in 5d momentum space

Symmetric constraint $\sum_a \partial_a = 0$

projects to 4d momentum space

Result is $A_4$ lattice

$\rightarrow$ dual $A^*_4$ lattice in real space
Backup: Restoration of $Q_a$ and $Q_{ab}$ supersymmetries

"$Q + \text{discrete } R_a \subset \text{SO}(4)_{tw} = Q_a \text{ and } Q_{ab}$"

Test $R_a$ on Wilson loops $\tilde{\mathcal{W}}_{ab} \equiv R_a \mathcal{W}_{ab}$,

Tune coeff. $c_2$ of $d^2$ term to ensure restoration in continuum.
Backup: Problem with SU($N$) flat directions

$\mu^2/\lambda_{\text{lat}}$ too small $\rightarrow U_a$ can move far from continuum form $I_N + A_a$

Example: $\mu = 0.2$ and $\lambda_{\text{lat}} = 2.5$ on $8^3 \times 24$ volume

Left: Bosonic action stable $\sim 18\%$ off its supersymmetric value

Right: (Complexified) Polyakov loop wanders off to $\sim 10^9$
Backup: Problem with U(1) flat directions

Monopole condensation $\rightarrow$ confined lattice phase
not present in continuum $\mathcal{N} = 4$ SYM

Around the same $2\lambda_{\text{lat}} \approx 2\ldots$

**Left:** Polyakov loop falls towards zero

**Center:** Plaquette determinant falls towards zero

**Right:** Density of U(1) monopole world lines becomes non-zero
Backup: Regulating SU(\(N\)) flat directions

Add soft \(Q\)-breaking scalar potential to lattice action

\[
S = \frac{N}{4\lambda_{\text{lat}}} \left[ Q \left( \chi_{ab} F_{ab} + \eta \overline{D}_a U_a - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcd} \chi_{ab} \overline{D}_c \chi_{de} + \mu^2 V \right]
\]

\[
V = \sum_a \left( \frac{1}{N} \text{Tr} \left[ U_a \overline{U}_a \right] - 1 \right)^2 \text{ lifts SU}(N) \text{ flat directions},
\]

\[
\text{ensures } U_a = \mathbb{I}_N + A_a \text{ in continuum limit}
\]

Correct continuum limit requires \(\mu^2 \to 0\)

to restore \(Q\) and recover moduli space

Typically scale \(\mu \propto 1/L\) in \(L \to \infty\) continuum extrapolation
Backup: Poorly regulating U(1) flat directions

Until 2015 we added another soft $Q$-breaking term

$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}} \mu^2 \sum_a \left( \frac{1}{N} \text{Tr} \left[ U_a \overline{U}_a \right] - 1 \right)^2 + \kappa \sum_{a<b} |\text{det} P_{ab} - 1|^2$$

More sensitivity to $\kappa$ than to $\mu^2$

Showing $Q$ Ward identity from bosonic action

$$\langle s_B \rangle = \frac{9N^2}{2}$$
Backup: Better regulating U(1) flat directions

\[
S = \frac{N}{4 \lambda_{\text{lat}}} \left[ Q \left( \chi_{ab} F_{ab} + \nabla^a \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \nabla_c \chi_{de} + \mu^2 V \right]
\]

\[
\eta \left\{ \mathcal{D}_a U_a + G \sum_{a < b} [\det P_{ab} - 1] \mathbb{I}_N \right\}
\]

\(Q\) Ward identity violations scale \(\propto 1/N^2\) (left) and \(\propto (a/L)^2\) (right)

\(~\) effective ‘\(O(a)\) improvement’ since \(Q\) forbids all dim-5 operators
Backup: Supersymmetric moduli space modification

Method to impose $Q$-invariant constraints applicable to generic site operator $\mathcal{O}(n)$ [arXiv:1505.03135]

Modify auxiliary field equations of motion $\mathcal{M}$ moduli space

$$d(n) = \overline{D}_a^{(-)} U_a(n) \quad \rightarrow \quad d(n) = \overline{D}_a^{(-)} U_a(n) + G\mathcal{O}(n)_{\Pi_N}$$

However, both U(1) and SU($N$) $\in \mathcal{O}(n)$ over-constrains system

![Graphs showing numerical data for $\mathcal{N} = 4$ SYM, U(2) with various symbols and lines indicating unimproved, over-constrained, and improved results for $\langle s_B \rangle - 18$ vs $\lambda_{\text{lat}}$ and $1/a/L$ vs $\lambda_{\text{lat}}$ with $\lambda_{\text{lat}} = 1$.]}
**Backup: $\mathcal{N} = (8, 8)$ SYM Wilson line eigenvalues**

Check ‘spatial deconfinement’ through Wilson line eigenvalue phases

**Left:** $\alpha = 2$ distributions more extended as $N$ increases

$\rightarrow$ dual gravity describes homogeneous black string (D1 phase)

**Right:** $\alpha = 1/2$ distributions more compact as $N$ increases

$\rightarrow$ dual gravity describes localized black hole (D0 phase)
Backup: Dimensional reduction to 2d $\mathcal{N} = (8, 8)$ SYM

Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_x = N_y = 1$

$A_4^* \longrightarrow A_2^*$ (triangular) lattice

Torus \textbf{skewed} depending on $\alpha = N_t/L$

Modular trans. into fund. domain

$\longrightarrow$ some skewed tori actually rectangular

Also need to stabilize compactified links to ensure broken center symmetries

\textbf{arXiv:1709.07025}
Backup: Static potential is Coulombic at all $\lambda$

String tension $\sigma$ from fits to confining form $V(r) = A - C/r + \sigma r$

Slightly negative values flatten $V(r_i)$ for $r_i \lesssim L/2$

$\sigma \to 0$ as accessible range of $r_i$ increases on larger volumes
Discretization artifacts visible at short distances

where Coulomb term in $V(r) = A - C/r$ is most significant

Right: Fluctuations around Coulomb fit highlight artifacts

Danger of distorting Coulomb coefficient $C$
Backup: Tree-level improvement

Classic trick to reduce discretization artifacts in static potential
(Lang & Rebbi ’82; Sommer ’93; Necco ’03)

Associate \( V(r) \) data with \( r \) from Fourier transform of gluon propagator

Recall

\[
\frac{1}{4\pi^2 r^2} = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{e^{i r \cdot k}}{k^2} \quad \text{where} \quad \frac{1}{k^2} = G(k) \quad \text{in continuum}
\]

\[
A_4^* \text{ lattice} \quad \rightarrow \quad \frac{1}{r_i^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \frac{\cos \left( i r_i \cdot \hat{k} \right)}{4 \sum_{\mu=1}^{4} \sin^2 \left( \hat{k} \cdot \hat{e}_\mu / 2 \right)}
\]

Tree-level lattice propagator from arXiv:1102.1725

\( \hat{e}_\mu \) are \( A_4^* \) lattice basis vectors;

momenta \( \hat{k} = \frac{2\pi}{L} \sum_{\mu=1}^{4} n_\mu \hat{g}_\mu \) depend on dual basis vectors
Backup: Tree-level-improved static potential

Significantly reduced discretization artifacts
Backup: Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve $Q$ and $S_5$ symmetries $\leftrightarrow$ geometric structure

Simple transformation constructed in arXiv:1408.7067

$$U'_a(n') = \xi U_a(n) U_a(n + \hat{\mu}_a)$$
$$\psi'_a(n') = \xi [\psi_a(n) U_a(n + \hat{\mu}_a) + U_a(n) \psi_a(n + \hat{\mu}_a)]$$ etc.

Doubles lattice spacing $a \rightarrow a' = 2a$, with tunable rescaling factor $\xi$

Scalar fields from polar decomposition $U(n) = e^{\phi(n)} U(n)$

$\rightarrow$ shift $\phi \rightarrow \phi + \log \xi$ to keep blocked $U$ unitary

$Q$-preserving RG transformation needed

to show only one log. tuning to recover continuum $Q_a$ and $Q_{ab}$
Backup: Smearing for Konishi analyses

Smear to enlarge (MCRG or variational) operator basis

APE-like smearing: \( \to (1 - \alpha) \to + \frac{\alpha}{8} \sum \Box \), staples built from unitary parts of links but no final unitarization (unitarized smearing — e.g. stout — doesn’t affect Konishi)

Average plaquette stable upon smearing (right), minimum plaquette steadily increases (left)