Lattice $\mathcal{N} = 4$ Supersymmetric Yang–Mills

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Quantum Gravity meets Lattice QFT
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& more to come with Simon Catterall, Raghav Jha and Toby Wiseman
Overview and plan

Central idea
Preserve (some) susy in discrete space-time

→ practical lattice investigations

Goals
1) Reproduce reliable results in perturbative, holographic, etc. regimes
2) Access new domains
Overview and plan

Preserve (some) susy in discrete space-time
Reproduce reliable analytic results
Access new domains

Lattice $\mathcal{N} = 4$ SYM formulation highlights

(I) Dimensionally reduced (2d) thermodynamics

(II) 4d static potential Coulomb coefficient

(III) Anomalous dimension of Konishi operator

Open questions and future directions
Motivations

Lattice field theory promises first-principles predictions for strongly coupled supersymmetric QFTs

BSM

QFT

Holography

(Derek Leinweber)
\[ \{ Q^I_{\alpha}, \bar{Q}^J_{\dot{\alpha}} \} = 2 \delta^{IJ} \sigma^{\mu}_{\alpha \dot{\alpha}} P_\mu \quad \text{broken in discrete space-time} \]

\[ \rightarrow \text{relevant susy-violating operators} \]
Solution

Preserve susy sub-algebra at non-zero lattice spacing

⇒ correct continuum limit with little or no fine tuning

Equivalent constructions from topological twisting and deconstruction

Need $2^d$ supersymmetries in $d$ dimensions

$\rightarrow d = 4$ picks out $\mathcal{N} = 4$ SYM
Quick review of $\mathcal{N} = 4$ SYM

Arguably simplest non-trivial 4d QFT

SU($\mathcal{N}$) gauge theory with four fermions $\Psi^I$ and six scalars $\Phi^{IJ}$, all massless and in adjoint rep.

**Symmetries** relate coeffs of kinetic, Yukawa and $\Phi^4$ terms

Maximal 16 supersymmetries $Q^I_{\alpha}$ and $\overline{Q}^I_{\dot{\alpha}}$ ($I = 1, \ldots, 4$)

transform under global $\text{SU}(4) \sim \text{SO}(6)$ $R$ symmetry

Conformal $\rightarrow$ $\beta$ function is zero for any 't Hooft coupling $\lambda = g^2 N$
Topological twisting for $\mathcal{N} = 4$ SYM

Intuitive picture — expand $4 \times 4$ matrix of supersymmetries

\[
\begin{pmatrix}
Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\
\bar{Q}^1_{\dot{\alpha}} & \bar{Q}^2_{\dot{\alpha}} & \bar{Q}^3_{\dot{\alpha}} & \bar{Q}^4_{\dot{\alpha}}
\end{pmatrix} = Q + Q_\mu \gamma_\mu + Q_{\mu\nu} \gamma_\mu \gamma_\nu + \bar{Q}_\mu \gamma_\mu \gamma_5 + \bar{Q} \gamma_5 \\
\rightarrow Q + Q_a \gamma_a + Q_{ab} \gamma_a \gamma_b
\]

with $a, b = 1, \cdots, 5$

R-symmetry index along each row $\times$ Lorentz index along each column

$\rightarrow Q$ transform in reps of ‘twisted rotation group’

$SO(4)_{tw} \equiv \text{diag} \left[ SO(4)_{\text{euc}} \otimes SO(4)_R \right]$

$SO(4)_R \subset SO(6)_R$

Change of variables $\rightarrow Q$ transform with integer spin under $SO(4)_{tw}$
Topological twisting for $\mathcal{N} = 4$ SYM

**Intuitive picture — expand $4 \times 4$ matrix of supersymmetries**

\[
\begin{pmatrix}
Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\
\overline{Q}^1_{\dot{\alpha}} & \overline{Q}^2_{\dot{\alpha}} & \overline{Q}^3_{\dot{\alpha}} & \overline{Q}^4_{\dot{\alpha}}
\end{pmatrix} = Q + Q_\mu \gamma_\mu + Q_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{Q}_\mu \gamma_\mu \gamma_5 + \overline{Q} \gamma_5 \\
\rightarrow Q + Q_a \gamma_a + Q_{ab} \gamma_a \gamma_b
\]

with $a, b = 1, \ldots, 5$

‘Twisted supersymmetries’ $Q$

transform with integer spin under twisted rotation group

\[
SO(4)_{tw} \equiv \text{diag} \left[ \text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R \right] \quad \text{SO}(4)_R \subset \text{SO}(6)_R
\]

Can preserve closed subalgebra $\{ Q, Q \} = 2Q^2 = 0$ on the lattice
Susy subalgebra from twisted $\mathcal{N} = 4$ SYM

Fields also transform with integer spin under $\text{SO}(4)_{tw}$ — no spinors

\[ Q_\alpha \text{ and } \bar{Q}_{\dot{\alpha}} \rightarrow Q, Q_a \text{ and } Q_{ab} \]
\[ \Psi \text{ and } \bar{\Psi} \rightarrow \eta, \psi_a \text{ and } \chi_{ab} \]
\[ A_\mu \text{ and } \Phi^I \rightarrow \text{complexified gauge field } A_a \text{ and } \bar{A}_a \]
\[ \rightarrow U(N) = \text{SU}(N) \otimes \text{U}(1) \text{ gauge theory} \]

Twisted-scalar supersymmetry $Q$

correctly interchanges bosonic $\leftrightarrow$ fermionic d.o.f. with $Q^2 = 0$

\[ Q A_a = \psi_a \]
\[ Q \chi_{ab} = -\bar{F}_{ab} \]
\[ Q \eta = d \]
\[ Q \psi_a = 0 \]
\[ Q \chi_{ab} = 0 \]
\[ Q \bar{A}_a = 0 \]
\[ Q d = 0 \]

\[ \text{bosonic auxiliary field with e.o.m. } d = \bar{D}_a A_a \]
Lattice $\mathcal{N} = 4$ SYM

Lattice theory looks nearly the same despite breaking $Q_a$ and $Q_{ab}$

Covariant derivatives $\rightarrow$ finite difference operators

Complexified gauge fields $A_a \rightarrow$ gauge links $U_a \in \mathfrak{gl}(N, \mathbb{C})$

\[ Q A_a \rightarrow Q U_a = \psi_a \]
\[ Q \chi_{ab} = -\bar{F}_{ab} \]
\[ Q \eta = d \]

\[ Q \psi_a = 0 \]
\[ Q \bar{A}_a \rightarrow Q \bar{U}_a = 0 \]
\[ Q d = 0 \]

Geometry: $\eta$ on sites, $\psi_a$ on links, etc.

Susy lattice action ($QS = 0$) from $Q^2 \cdot = 0$ and Bianchi identity

\[ S = \frac{N}{4\lambda_{\text{lat}}} \text{Tr} \left[ Q \left( \chi_{ab} \mathcal{F}_{ab} + \eta \overline{D}_a U_a - \frac{1}{2} \eta d \right) - \frac{1}{4} \varepsilon_{abcde} \chi_{ab} \overline{D}_c \chi_{de} \right] \]
Five links in four dimensions $\rightarrow A_4^*$ lattice

$A_4^* \sim$ 4d analog of 2d triangular lattice

Basis vectors linearly dependent and non-orthogonal

Large $S_5$ point group symmetry

$S_5$ irreps precisely match onto irreps of twisted SO(4)$_{tw}$

\[
5 = 4 \oplus 1 : \quad \psi_a \rightarrow \psi_\mu, \quad \bar{\eta} \\
10 = 6 \oplus 4 : \quad \chi_{ab} \rightarrow \chi_{\mu \nu}, \quad \bar{\psi}_\mu
\]

$S_5 \rightarrow SO(4)_{tw}$ in continuum limit restores $Q_a$ and $Q_{ab}$
Analytic results for twisted $\mathcal{N} = 4$ SYM on $A_4^*$ lattice

U($N$) gauge invariance $+$ $Q$ $+$ $S_5$ lattice symmetries

$\rightarrow$ Moduli space preserved to all orders

$\rightarrow$ One-loop lattice $\beta$ function vanishes

$\rightarrow$ Only one log. tuning to recover continuum $Q_a$ and $Q_{ab}$


Not quite suitable for numerical calculations
Must regulate zero modes and flat directions, especially in U(1) sector
Two deformations in lattice action

**SU($N$) scalar potential** \( \propto \mu^2 \sum_a (\text{Tr} [U_a \bar{U}_a] - N)^2 \)

**Softly breaks susy** \( \rightarrow Q\)-violating operators vanish \( \propto \mu^2 \rightarrow 0 \)

**U(1) plaquette determinant** \( \sim G \sum_{a<b} (\det P_{ab} - 1) \)

Implemented supersymmetrically as Fayet–Iliopoulos $D$-term potential

**Test via Ward identity violations:** \( Q \left[ \eta U_a \bar{U}_a \right] \neq 0 \)
so that the full improved action becomes

$$S_{\text{imp}} = S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}}$$

$$S'_{\text{exact}} = \frac{N}{4\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ -\mathcal{F}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}_{[a}^{(+)}\psi_{b]}(n) - \eta(n)\overline{\mathcal{D}}_{a}^{-}\psi_a(n) \\
+ \frac{1}{2} \left( \overline{\mathcal{D}}_{a}^{-}\mathcal{U}_a(n) + G \sum_{a \neq b} (\text{det} \mathcal{P}_{ab}(n) - 1) \mathbb{I}_N \right)^2 \right] - S_{\text{det}}$$

$$S_{\text{det}} = \frac{N}{4\lambda_{\text{lat}}} G \sum_n \text{Tr} \left[ \eta(n) \sum_{a \neq b} [\text{det} \mathcal{P}_{ab}(n)] \text{Tr} \left[ \mathcal{U}_b^{-1}(n)\psi_b(n) + \mathcal{U}_a^{-1}(n + \tilde{\mu}_b)\psi_a(n + \tilde{\mu}_b) \right] \right]$$

$$S_{\text{closed}} = -\frac{N}{16\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ \epsilon_{abcd} \chi_{de}(n + \tilde{\mu}_a + \tilde{\mu}_b + \tilde{\mu}_c)\overline{\mathcal{D}}_{c}^{(-)}\chi_{ab}(n) \right],$$

$$S'_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}} \mu^2 \sum_n \sum_a \left( \frac{1}{N} \text{Tr} \left[ \mathcal{U}_a(n)\overline{\mathcal{U}}_a(n) \right] - 1 \right)^2$$

$\sim 100$ inter-node data transfers in the fermion operator — non-trivial…

Reduce barriers to entry  $\longrightarrow$ public parallel code at

github.com/daschaich/susy

Evolved from MILC lattice QCD code, presented in arXiv:1410.6971
Dimensionally reduce to 2d $\mathcal{N} = (8, 8)$ SYM with four scalar $Q$,
study low temperatures $t = 1/r_\beta \leftrightarrow$ black holes in dual supergravity

For decreasing $r_L$ at large $N$
homogeneous black string (D1)
$\longrightarrow$ localized black hole (D0)

“spatial deconfinement” signalled by Wilson line $P_L$
$\mathcal{N} = (8, 8)$ SYM lattice phase diagram results

- Good agreement with high-temp. bosonic QM
- Consistent with holography at low temperatures

Example spatial deconfinement transition in Wilson line

- Fixing aspect ratio $\alpha = r_L/r_\beta = 4$, scanning in $r_\beta = r_L/\alpha$
Dual black hole thermodynamics

Holography: bosonic action $\leftrightarrow$ dual black hole internal energy

$\propto t^3$ for large-$r_L$ D1 phase

$\propto t^{3.2}$ for small-$r_L$ D0 phase

Lattice results consistent with holography for sufficiently low $t \lesssim 0.4$

Need larger $N > 16$ to avoid instabilities at lower temperatures
(II) Static potential $V(r)$

Static probes $\rightarrow r \times T$ Wilson loops $W(r, T) \propto e^{-V(r) T}$

Coulomb gauge trick reduces $A_4^*$ lattice complications
Static potential is Coulombic at all \( \lambda \)

Fits to confining \( V(r) = A - C/r + \sigma r \longrightarrow \) vanishing string tension \( \sigma \)

\[ \Rightarrow \text{Fit to just } V(r) = A - C/r \text{ to extract Coulomb coefficient } C(\lambda) \]

Discretization artifacts reduced by tree-level improved analysis
Coupling dependence of Coulomb coefficient

Continuum perturbation theory $\rightarrow C(\lambda) = \frac{\lambda}{4\pi} + O(\lambda^2)$

Holography $\rightarrow C(\lambda) \propto \sqrt{\lambda}$ for $N \to \infty$ and $\lambda \to \infty$ with $\lambda \ll N$

Consistent with leading-order perturbation theory for $\lambda_{\text{lat}} \leq 2$
(III) Konishi operator scaling dimension

\[ O_K(x) = \sum_I \text{Tr} [\Phi^I(x)\Phi^I(x)] \] is simplest conformal primary operator

Scaling dimension \( \Delta_K(\lambda) = 2 + \gamma_K(\lambda) \) investigated through perturbation theory (& S duality), holography, conformal bootstrap

\[ C_K(r) \equiv O_K(x+r)O_K(x) \propto r^{-2\Delta_K} \]

‘SUGRA’ is 20’ op., \( \Delta_S = 2 \)

Will compare:
- Direct power-law decay
- Finite-size scaling
- Monte Carlo RG
(III) Konishi operator scaling dimension

Lattice scalars \( \varphi(n) \) from polar decomposition of complexified links

\[
U_a(n) \rightarrow e^{\varphi_a(n)} U_a(n)
\]

\[
\mathcal{O}_{\text{lat}}^K(n) = \sum_a \text{Tr} [\varphi_a(n) \varphi_a(n)] - \text{vev}
\]

\[
\mathcal{O}_{\text{lat}}^S(n) \sim \text{Tr} [\varphi_a(n) \varphi_b(n)]
\]

\[
C_K(r) \equiv \mathcal{O}_K(x+r) \mathcal{O}_K(x) \propto r^{-2\Delta_K}
\]

‘SUGRA’ is 20’ op., \( \Delta_S = 2 \)

Will compare:
- Direct power-law decay
- Finite-size scaling
- Monte Carlo RG
Scaling dimensions from MCRG stability matrix

**Lattice system:** \( H = \sum_i c_i O_i \) (infinite sum)

Couplings flow under RG blocking \( \rightarrow H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} O_i^{(n)} \)

Fixed point \( \rightarrow H^* = R_b H^* \) with couplings \( c_i^* \)

Linear expansion around fixed point \( \rightarrow \) stability matrix \( T_{ik}^* \)

\[
c_i^{(n)} - c_i^* = \sum_k \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \bigg|_{H^*} \left( c_k^{(n-1)} - c_k^* \right) \equiv \sum_k T_{ik}^* \left( c_k^{(n-1)} - c_k^* \right)
\]

Correlators of \( O_i, O_k \) \( \rightarrow \) elements of stability matrix [Swendsen, 1979]

Eigenvalues of \( T_{ik}^* \) \( \rightarrow \) scaling dimensions of corresponding operators
Preliminary $\Delta_K$ results from Monte Carlo RG

Analyzing both $\mathcal{O}^{\text{lat}}_K$ and $\mathcal{O}^{\text{lat}}_S$

Imposing protected $\Delta_S = 2$

$\longrightarrow \Delta_K(\lambda)$ looks perturbative

Systematic uncertainties from different amounts of smearing

Complication: Twisting involves only $\text{SO}(4)_R \subset \text{SO}(6)_R$

$\longrightarrow$ Lattice Konishi op. mixes with $\text{SO}(4)_R$-singlet part of $\text{SO}(6)_R$-nonsinglet SUGRA op.

Working on variational analyses to disentangle operators
Future: Pushing $\mathcal{N} = 4$ SYM to stronger coupling

- Reproduce reliable (4d) results in perturbative regime
  - Check holographic predictions and access new domains

**Sign problem** seems to become obstruction

![Graph showing Re($e^{i\alpha}$) vs $\lambda_{\text{lat}}$]
Quick review of sign problem

\[ \langle O \rangle = \frac{1}{\mathcal{Z}} \int [dU][d\overline{U}] \ O \ e^{-S_B[U,\overline{U}]} \ \text{pf} \mathcal{D}[U,\overline{U}] \]

Complex pfaffian \( \text{pf} \mathcal{D} = |\text{pf} \mathcal{D}| e^{i\alpha} \) complicates importance sampling

We phase quench, \( \text{pf} \mathcal{D} \rightarrow |\text{pf} \mathcal{D}| \), need to reweight \( \langle O \rangle = \frac{\langle O e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}} \)

![Graph showing Re(e^{i\alpha}) vs. \lambda_{lat} for \( \mathcal{N} = 4 \) SYM, U(2)]
The Pfaffian is nearly real positive for all accessible volumes (at fixed $\lambda_{\text{lat}} = 0.5$).

The expectation value $\langle e^{i\alpha} \rangle_{pq}$ is extremely sensitive to boundary conditions.

But other $\langle O \rangle_{pq}$ are not!
Future: Lattice superQCD (in 2d & 3d)

Preserve twisted supersymmetry sub-algebra on the lattice

Proposed by Matsuura [0805.4491] and Sugino [0807.2683],
first numerical study by Catterall & Veernala [1505.00467]

2-slice lattice SYM
with $U(N) \times U(F)$ gauge group
Adj. fields on each slice
Bi-fundamental in between

Decouple $U(F)$ slice
$\longrightarrow$ $U(N)$ SQCD in $d - 1$ dims.
with $F$ fund. hypermultiplets
Dynamical susy breaking in 2d lattice superQCD

Auxiliary field e.o.m. $\rightarrow$ Fayet–Iliopoulos $D$-term potential

$$d = \overline{D}_a U_a + \sum_{i=1}^{F} \phi_i \bar{\phi}_i + r \Pi_N$$

$$\rightarrow$$

$$S_D \propto \sum_{i=1}^{F} \left( \text{Tr} \left[ \phi_i \bar{\phi}_i + r \Pi_N \right] \right)^2$$

Zero out $N$ diagonal elements via $F$ scalar vevs

or else susy breaking, $\langle Q \eta \rangle = \langle d \rangle \neq 0 \leftrightarrow \langle 0 | H | 0 \rangle > 0$

arXiv:1505.00467
Recap: An exciting time for lattice supersymmetry

✓ Preserve (some) susy in discrete space-time
  → practical lattice $\mathcal{N} = 4$ SYM, public code available

Reproduce reliable analytic results

✓ 2d $\mathcal{N} = (8, 8)$ SYM thermodynamics consistent with holography

✓ Perturbative static potential Coulomb coefficient $C(\lambda)$
  and Konishi operator conformal scaling dimension $\Delta_K(\lambda)$

Access new domains

→ Understanding the sign problem at stronger couplings

→ Lower-dimensional superQCD and more...
Thank you!

Collaborators
Simon Catterall, Raghav Jha, Toby Wiseman
also Georg Bergner, Poul Damgaard, Joel Giedt, Anosh Joseph

Funding and computing resources
Backup: Breakdown of Leibniz rule on the lattice

\[ \{ Q_\alpha, \overline{Q}_{\dot{\alpha}} \} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = 2i\sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \text{ is problematic} \]

\[ \implies \text{try finite difference } \partial \phi(x) \rightarrow \Delta \phi(x) = \frac{1}{a} [\phi(x + a) - \phi(x)] \]

Crucial difference between \( \partial \) and \( \Delta \)

\[ \Delta [\phi \eta] = a^{-1} [\phi(x + a)\eta(x + a) - \phi(x)\eta(x)] \]

\[ = [\Delta \phi] \eta + \phi \Delta \eta + a [\Delta \phi] \Delta \eta \]

Only recover Leibniz rule \( \partial [\phi \eta] = [\partial \phi] \eta + \phi \partial \eta \) when \( a \rightarrow 0 \)

\[ \implies \text{“discrete supersymmetry” breaks down on the lattice} \]
Backup: Complexified gauge field from twisting

Why combine $A_\mu$ and $\Phi^I \rightarrow$ complexified gauge field $A_a$ and $\overline{A}_a$?

This is source of $U(N) = SU(N) \otimes U(1)$ that complicates lattice action

Schematically, under $SO(d)_{tw} = \text{diag}[SO(d)_{\text{euc}} \otimes SO(d)_R]$

$A_\mu \sim \text{vector} \otimes \text{scalar} \rightarrow \text{vector}$

$\Phi^I \sim \text{scalar} \otimes \text{vector} \rightarrow \text{vector}$

Easiest to see by dimensionally reducing from 5d

$A_a = A_a + i\Phi_a \rightarrow (A_\mu, \phi) + i(\Phi_\mu, \overline{\phi})$
Backup: $A_4^*$ lattice from five dimensions

Again dimensionally reduce, treating all five gauge links symmetrically.

Start with hypercubic lattice in 5d momentum space.

**Symmetric** constraint $\sum_a \partial_a = 0$ projects to 4d momentum space.

Result is $A_4$ lattice $\rightarrow$ dual $A_4^*$ lattice in real space.
Backup: Restoration of $Q_a$ and $Q_{ab}$ supersymmetries

\[ Q + \text{discrete } R_a \subset \text{SO}(4)_{tw} = Q_a \text{ and } Q_{ab} \]

Test $R_a$ on Wilson loops $\tilde{\mathcal{W}}_{ab} \equiv R_a \mathcal{W}_{ab}$,

tune coeff. $c_2$ of $d^2$ term to ensure restoration in continuum

Results from arXiv:1411.0166 to be revisited with improved action
Backup: Problem with SU($N$) flat directions

$\mu^2/\lambda_{\text{lat}}$ too small $\rightarrow U_a$ can move far from continuum form $\mathbb{1}_N + A_a$

Example: $\mu = 0.2$ and $\lambda_{\text{lat}} = 2.5$ on $8^3 \times 24$ volume

Left: Bosonic action stable $\sim 18\%$ off its supersymmetric value

Right: (Complexified) Polyakov loop wanders off to $\sim 10^9$
Backup: Problem with U(1) flat directions

Monopole condensation $\rightarrow$ confined lattice phase
not present in continuum $\mathcal{N} = 4$ SYM

Around the same $2\lambda_{\text{lat}} \approx 2\ldots$

**Left:** Polyakov loop falls towards zero

**Center:** Plaquette determinant falls towards zero

**Right:** Density of U(1) monopole world lines becomes non-zero
Backup: Regulating SU(N) flat directions

Add soft $Q$-breaking scalar potential to lattice action

$$S = \frac{N}{4\lambda_{\text{lat}}} \left[ Q \left( \chi_{ab} F_{ab} + \eta \overline{D}_a U_a - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \overline{D}_c \chi_{de} + \mu^2 V \right]$$

$$V = \sum_a \left( \frac{1}{N} \text{Tr} [U_a \overline{U}_a] - 1 \right)^2$$ lifts SU(N) flat directions,

ensures $U_a = \mathbb{I}_N + A_a$ in continuum limit

Correct continuum limit requires $\mu^2 \to 0$

to restore $Q$ and recover physical flat directions

Typically scale $\mu \propto 1/L$ in $L \to \infty$ continuum extrapolation
Backup: Poorly regulating $U(1)$ flat directions

Until 2015 we added another soft $Q$-breaking term

$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}} \mu^2 \sum_a \left( \frac{1}{N} \text{Tr} [U_a \overline{U}_a] - 1 \right)^2 + \kappa \sum_{a<b} |\text{det} P_{ab} - 1|^2$$

More sensitivity to $\kappa$ than to $\mu^2$

Showing $Q$ Ward identity from bosonic action

$$\langle s_B \rangle = \frac{9N^2}{2}$$

$N = 4$ SYM, $U(2)$

$4^4$
Backup: Better regulating U(1) flat directions

\[ S = \frac{N}{4\lambda_{\text{lat}}} \left[ \mathcal{Q} \left( \chi_{ab} \mathcal{F}_{ab} + \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \mathcal{D}_c \chi_{de} + \mu^2 V \right] \]

\[ \eta \left\{ \mathcal{D}_a \mathcal{U}_a + G \sum_{a < b} [\det \mathcal{P}_{ab} - 1] \mathbb{I}_N \right\} \]

\( \mathcal{Q} \) Ward identity violations scale \( \propto 1/N^2 \) (left) and \( \propto (a/L)^2 \) (right)

\( \sim \) effective ‘\( O(a) \) improvement’ since \( \mathcal{Q} \) forbids all dim-5 operators
Backup: Supersymmetric moduli space modification

Method to impose $Q$-invariant constraints applicable to generic site operator $\mathcal{O}(n)$ \[\text{[arXiv:1505.03135]}\]

Modify auxiliary field equations of motion $\rightarrow$ moduli space

\[
d(n) = \overline{D}_a^{(-)} U_a(n) \quad \rightarrow \quad d(n) = \overline{D}_a^{(-)} U_a(n) + G\mathcal{O}(n)^I_N
\]

However, both U(1) and SU($N$) $\in \mathcal{O}(n)$ over-constrains system
Backup: $\mathcal{N} = (8, 8)$ SYM Wilson line eigenvalues

Check ‘spatial deconfinement’ through Wilson line eigenvalue phases

**Left:** $\alpha = 2$ distributions more extended as $N$ increases

$\rightarrow$ dual gravity describes homogeneous black string (D1 phase)

**Right:** $\alpha = 1/2$ distributions more compact as $N$ increases

$\rightarrow$ dual gravity describes localized black hole (D0 phase)
Backup: Static potential is Coulombic at all $\lambda$

String tension $\sigma$ from fits to confining form $V(r) = A - C/r + \sigma r$

Slightly negative values flatten $V(r_I)$ for $r_I \lesssim L/2$

$\sigma \to 0$ as accessible range of $r_I$ increases on larger volumes
Discretization artifacts visible at short distances
where Coulomb term in \( V(r) = A - C/r \) is most significant

**Right:** Fluctuations around Coulomb fit highlight artifacts

Danger of distorting Coulomb coefficient \( C \)
Backup: Tree-level improvement

Classic trick to reduce discretization artifacts in static potential
(Lang & Rebbi ’82; Sommer ’93; Necco ’03)

Associate $V(r)$ data with $r$ from Fourier transform of gluon propagator

Recall
\[
\frac{1}{4\pi^2 r^2} = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{e^{i r \cdot k}}{k^2} \text{ where } \frac{1}{k^2} = G(k) \text{ in continuum}
\]

\[
A^*_4 \text{ lattice} \rightarrow \frac{1}{r^2} \equiv \frac{1}{r_I^2} = 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4\hat{k}}{(2\pi)^4} \frac{\cos \left( i r_I \cdot \hat{k} \right)}{4 \sum_{\mu=1}^{4} \sin^2 \left( \hat{k} \cdot \hat{e}_{\mu} / 2 \right)}
\]

Tree-level lattice propagator from arXiv:1102.1725

$\hat{e}_{\mu}$ are $A^*_4$ lattice basis vectors;

momenta $\hat{k} = \frac{2\pi}{L} \sum_{\mu=1}^{4} n_{\mu} \hat{g}_{\mu}$ depend on dual basis vectors
Backup: Tree-level-improved static potential

Tree-level improvement significantly reduces discretization artifacts

\[
\mathcal{N} = 4 \text{ SYM, } U(N)
\]

\[
8^3 \times 24
\]

\[
V
\]

\[
V
\]

\[
\frac{rV-A}{C}
\]

\[
\frac{rV-A}{C}
\]

\[
\text{PRELIMINARY}
\]

\[
\text{PRELIMINARY}
\]

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Lattice $\mathcal{N} = 4$ SYM

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Backup: Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve $Q$ and $S_5$ symmetries $\leftrightarrow$ geometric structure

Simple transformation constructed in arXiv:1408.7067

$$U'_a(n') = \xi U_a(n)U_a(n + \hat{\mu}_a)$$  $$\eta'(n') = \eta(n)$$

$$\psi'_a(n') = \xi [\psi_a(n)U_a(n + \hat{\mu}_a) + U_a(n)\psi_a(n + \hat{\mu}_a)]$$  etc.

Doubles lattice spacing $a \rightarrow a' = 2a$, with tunable rescaling factor $\xi$

Scalar fields from polar decomposition $U(n) = e^{\varphi(n)}U(n)$

$\rightarrow$ shift $\varphi \rightarrow \varphi + \log \xi$ to keep blocked $U$ unitary

This $Q$-preserving RG transformation needed

to show only one log. tuning to recover continuum $Q_a$ and $Q_{ab}$
Backup: Smearing for Konishi analyses

Smear to enlarge (MCRG or variational) operator basis

APE-like smearing: \[ (1 - \alpha) \rightarrow \sum_\Box + \frac{\alpha}{8} \sum \cap, \]

staples built from unitary parts of links but no final unitarization

(unitarized smearing — e.g. stout — doesn’t affect Konishi)

Average plaquette stable upon smearing (right),
minimum plaquette steadily increases (left)
Backup: Dimensional reduction to $\mathcal{N} = (8, 8)$ SYM

Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_x = N_y = 1$

$A_4^* \rightarrow A_2^* \text{ (triangular) lattice}$

Torus **skewed** depending on $\alpha = N_t/L$

Modular trans. into fund. domain
$\rightarrow$ some skewed tori actually rectangular

Also need to stabilize compactified links
to ensure broken center symmetries

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