Lattice $\mathcal{N} = 4$ Supersymmetric Yang–Mills

David Schaich (Bern)

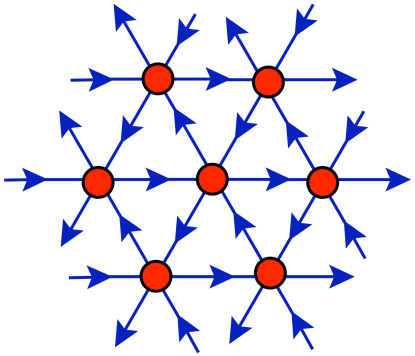
Quantum Gravity meets Lattice QFT

ECT* Trento 5 September 2018

arXiv:1505.03135 arXiv:1611.06561 arXiv:1709.07025

& more to come with Simon Catterall, Raghav Jha and Toby Wiseman

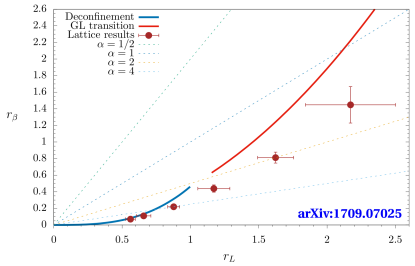
Overview and plan



Central idea

Preserve (some) susy in discrete space-time

$\longrightarrow$ practical lattice investigations

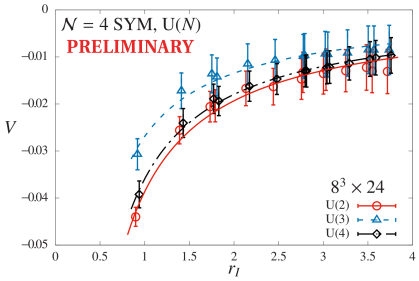


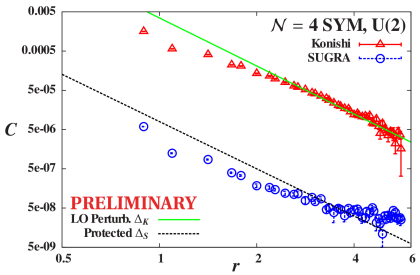
Goals

1) Reproduce reliable results in

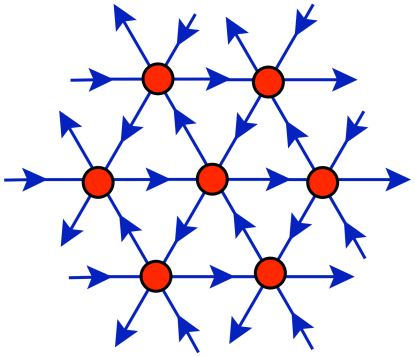
perturbative, holographic, etc. regimes

2) Access new domains





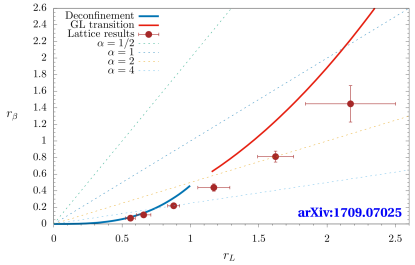
Overview and plan



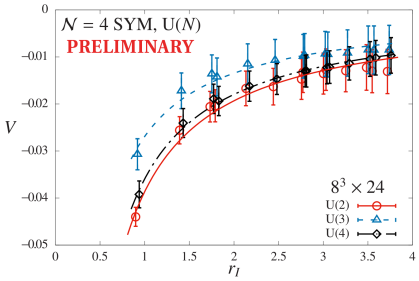
Preserve (some) susy in discrete space-time

Reproduce reliable analytic results

Access new domains

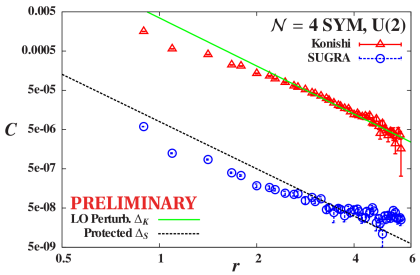


Lattice $\mathcal{N} = 4$ SYM formulation highlights



(I) Dimensionally reduced (2d) thermodynamics

(II) 4d static potential Coulomb coefficient



(III) Anomalous dimension of Konishi operator

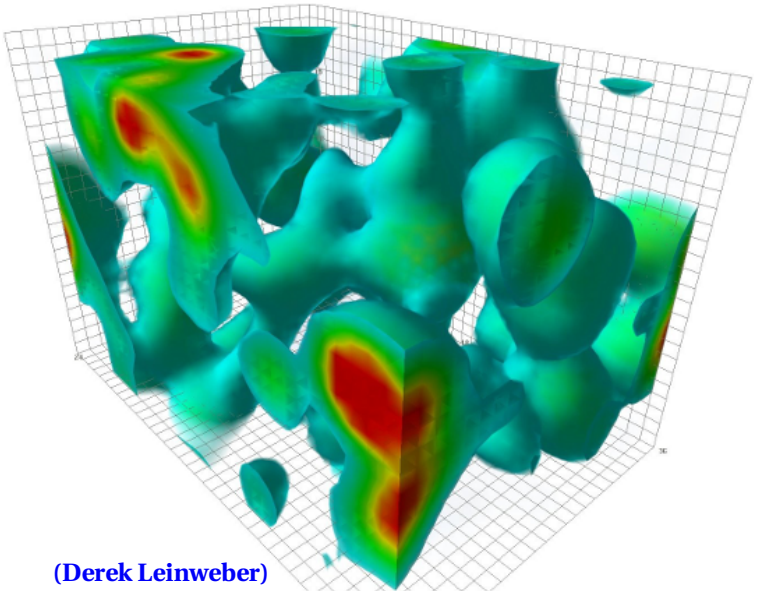
Open questions and future directions

Motivations

Lattice field theory promises first-principles predictions
for strongly coupled supersymmetric QFTs

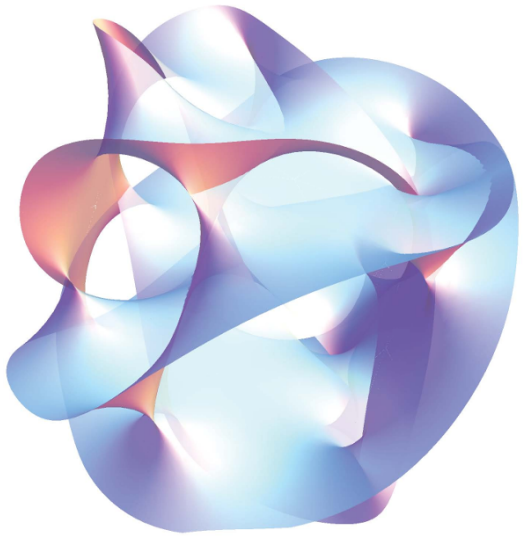
BSM

QFT



(Derek Leinweber)

Holography



Obstruction

$\left\{ Q^{\mathrm{I}}_{\alpha}, \overline{Q}^{\mathrm{J}}_{\dot{\alpha}} \right\} = 2\delta^{\mathrm{IJ}} \sigma^{\mu}_{\alpha\dot{\alpha}} P_{\mu}$ broken in discrete space-time

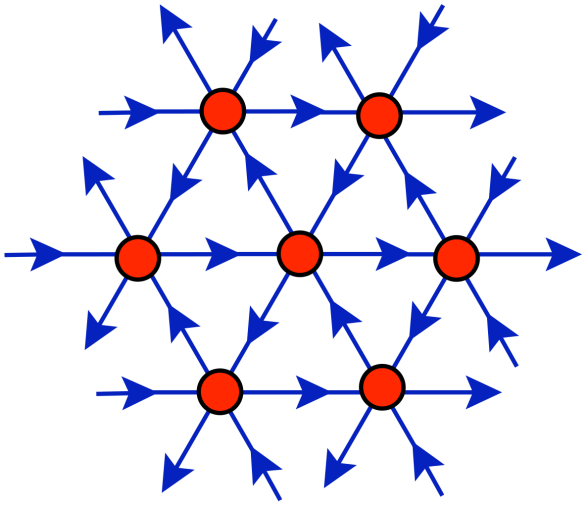
$\longrightarrow$ relevant susy-violating operators

Solution

Preserve susy sub-algebra at non-zero lattice spacing

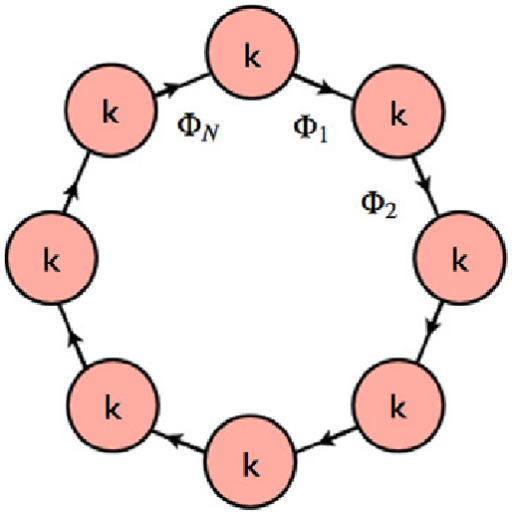
$\Longrightarrow$ correct continuum limit with little or no fine tuning

Equivalent constructions from topological twisting and deconstruction



Review:

arXiv:0903.4881



Need 2^d supersymmetries in d dimensions

$\longrightarrow d = 4$ picks out $\mathcal{N} = 4$ SYM

Quick review of $\mathcal{N} = 4$ SYM

Arguably simplest non-trivial 4d QFT

SU(N) gauge theory with four fermions Ψ^{I} and six scalars Φ^{IJ},
all massless and in adjoint rep.

Symmetries relate coeffs of kinetic, Yukawa and Φ^4 terms

Maximal 16 supersymmetries Q^{I}_{α} and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ $\quad (\mathrm{I} = 1, \cdots, 4)$
transform under global SU(4) $\sim$ SO(6) R symmetry

Conformal $\longrightarrow$ β function is zero for any 't Hooft coupling $\lambda = g^2 N$

Topological twisting for $\mathcal{N} = 4$ SYM

Intuitive picture — expand 4×4 matrix of supersymmetries

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu}\gamma_\mu\gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}}\gamma_5$$

$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab}\gamma_a\gamma_b$$

with $a, b = 1, \cdots, 5$

R-symmetry index along each row $\times$ Lorentz index along each column

$\Longrightarrow$ $\mathcal{Q}$ transform in reps of 'twisted rotation group'

$$\mathrm{SO}(4)_{tw} \equiv \mathrm{diag}\left[\mathrm{SO}(4)_{\mathrm{euc}} \otimes \mathrm{SO}(4)_R\right] \qquad \mathrm{SO}(4)_R \subset \mathrm{SO}(6)_R$$

Change of variables $\longrightarrow$ $\mathcal{Q}$ transform with integer spin under $\mathrm{SO}(4)_{tw}$

Topological twisting for $\mathcal{N} = 4$ SYM

Intuitive picture — expand 4×4 matrix of supersymmetries

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu}\gamma_\mu\gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}}\gamma_5$$

$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab}\gamma_a\gamma_b$$

with $a, b = 1, \cdots, 5$

'Twisted supersymmetries' $\mathcal{Q}$

transform with integer spin under twisted rotation group

$$\mathrm{SO}(4)_{tw} \equiv \mathrm{diag}\left[\mathrm{SO}(4)_{\mathrm{euc}} \otimes \mathrm{SO}(4)_R\right] \qquad \mathrm{SO}(4)_R \subset \mathrm{SO}(6)_R$$

Can preserve closed subalgebra $\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$ on the lattice

Susy subalgebra from twisted $\mathcal{N} = 4$ SYM

Fields also transform with integer spin under $\mathrm{SO}(4)_{tw}$ — no spinors

$$Q_\alpha \text{ and } \overline{Q}_{\dot\alpha} \longrightarrow \mathcal{Q},\ \mathcal{Q}_a \text{ and } \mathcal{Q}_{ab}$$

$$\Psi \text{ and } \overline{\Psi} \longrightarrow \eta,\ \psi_a \text{ and } \chi_{ab}$$

$$A_\mu \text{ and } \Phi^{\mathrm{I}} \longrightarrow \text{complexified gauge field } \mathcal{A}_a \text{ and } \overline{\mathcal{A}}_a$$

$$\longrightarrow \mathrm{U}(N) = \mathrm{SU}(N) \otimes \mathrm{U}(1) \text{ gauge theory}$$

Twisted-scalar supersymmetry $\mathcal{Q}$

correctly interchanges bosonic $\longleftrightarrow$ fermionic d.o.f. with $\mathcal{Q}^2 = 0$

$$\mathcal{Q}\,\mathcal{A}_a = \psi_a \qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad \mathcal{Q}\,\overline{\mathcal{A}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad \mathcal{Q}\,d = 0$$

$\nwarrow$ bosonic auxiliary field with e.o.m. $d = \overline{\mathcal{D}}_a \mathcal{A}_a$

Lattice $\mathcal{N} = 4$ SYM

Lattice theory looks nearly the same despite breaking $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Covariant derivatives $\longrightarrow$ finite difference operators

Complexified gauge fields $\mathcal{A}_a \longrightarrow$ gauge links $\mathcal{U}_a \in \mathfrak{gl}(N, \mathbb{C})$

$$\mathcal{Q}\,\mathcal{A}_a \longrightarrow \mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad \mathcal{Q}\,\overline{\mathcal{A}}_a \longrightarrow \mathcal{Q}\,\overline{\mathcal{U}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad \mathcal{Q}\,d = 0$$

Geometry: η on sites, ψ_a on links, etc.

Susy lattice action $(\mathcal{Q}S = 0)$ from $\mathcal{Q}^2 \cdot = 0$ and Bianchi identity

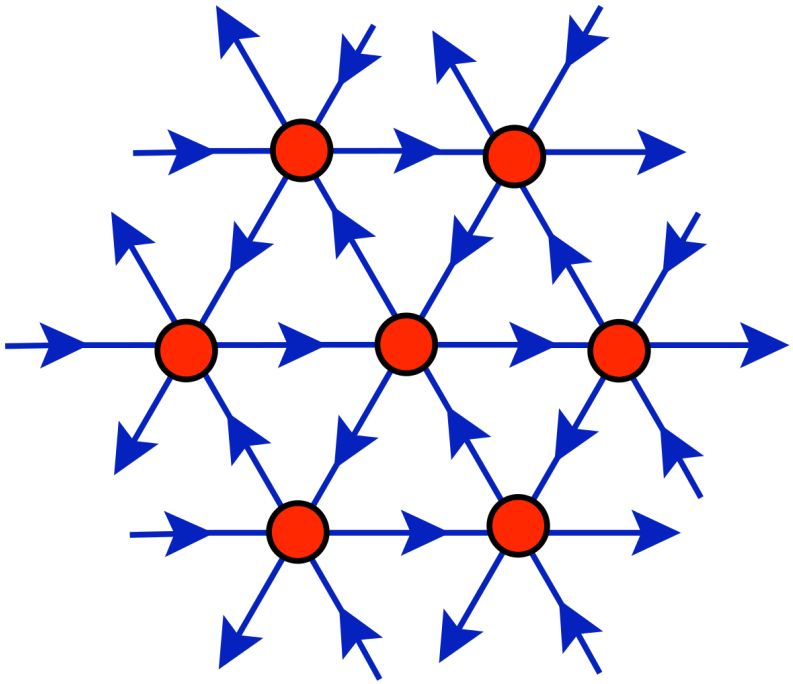
$$S = \frac{N}{4\lambda_{\text{lat}}} \text{Tr}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de}\right]$$

Five links in four dimensions $\longrightarrow$ A_4^* lattice

A_4^* $\sim$ 4d analog of 2d triangular lattice

Basis vectors linearly dependent
and non-orthogonal

Large S_5 point group symmetry



S_5 irreps precisely match onto irreps of twisted SO(4)$_{tw}$

$$\mathbf{5} = \mathbf{4} \oplus \mathbf{1} : \quad \psi_a \longrightarrow \psi_\mu, \ \ \overline{\eta}$$

$$\mathbf{10} = \mathbf{6} \oplus \mathbf{4} : \quad \chi_{ab} \longrightarrow \chi_{\mu\nu}, \ \ \overline{\psi}_\mu$$

$S_5 \longrightarrow$ SO(4)$_{tw}$ in continuum limit restores $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Checkpoint

Analytic results for twisted $\mathcal{N} = 4$ SYM on A_4^* lattice

U(N) gauge invariance + $\mathcal{Q}$ + S_5 lattice symmetries

$\longrightarrow$ Moduli space preserved to all orders

$\longrightarrow$ One-loop lattice β function vanishes

$\longrightarrow$ Only one log. tuning to recover continuum $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

[arXiv:1102.1725, arXiv:1306.3891, arXiv:1408.7067]

Not quite suitable for numerical calculations

Must regulate zero modes and flat directions, especially in U(1) sector

Two deformations in lattice action

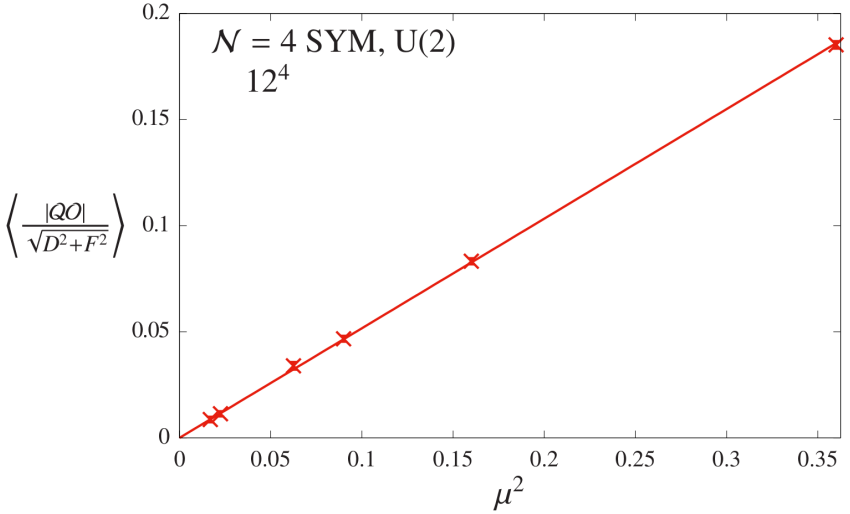
SU(N) scalar potential $\propto \mu^2 \sum_a \left(\mathrm{Tr}\left[\mathcal{U}_a \overline{\mathcal{U}}_a\right] - N\right)^2$

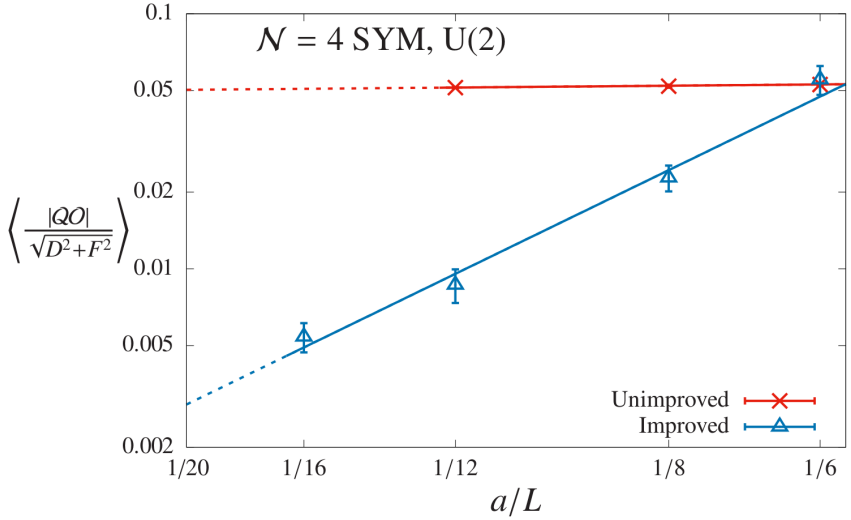
Softly breaks susy $\longrightarrow$ $\mathcal{Q}$-violating operators vanish $\propto \mu^2 \to 0$

U(1) plaquette determinant $\sim G \sum_{a<b} (\det \mathcal{P}_{ab} - 1)$

Implemented supersymmetrically as Fayet–Iliopoulos D-term potential

Test via Ward identity violations: $\mathcal{Q}\left[\eta \mathcal{U}_a \overline{\mathcal{U}}_a\right] \neq 0$





Advertisement: Public code for lattice $\mathcal{N} = 4$ SYM

so that the full improved action becomes

$$
\begin{aligned}
S_{\text{imp}} &= S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \qquad (18)\\
S'_{\text{exact}} &= \frac{N}{4\lambda_{\text{lat}}} \sum_n \text{Tr}\Bigg[-\overline{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi_{b]}(n) - \eta(n)\overline{\mathcal{D}}^{(-)}_{a}\psi_a(n)\\
&\qquad + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_{a}\mathcal{U}_a(n) + G\sum_{a\neq b}\left(\det \mathcal{P}_{ab}(n) - 1\right)\mathbb{I}_N\right)^2\Bigg] - S_{\text{det}}\\
S_{\text{det}} &= \frac{N}{4\lambda_{\text{lat}}} G \sum_n \text{Tr}\left[\eta(n)\right] \sum_{a\neq b}\left[\det \mathcal{P}_{ab}(n)\right] \text{Tr}\left[\mathcal{U}_b^{-1}(n)\psi_b(n) + \mathcal{U}_a^{-1}(n+\widehat{\mu}_b)\psi_a(n+\widehat{\mu}_b)\right]\\
S_{\text{closed}} &= -\frac{N}{16\lambda_{\text{lat}}} \sum_n \text{Tr}\left[\epsilon_{abcde}\ \chi_{de}(n+\widehat{\mu}_a+\widehat{\mu}_b+\widehat{\mu}_c)\overline{\mathcal{D}}^{(-)}_{c}\chi_{ab}(n)\right],\\
S'_{\text{soft}} &= \frac{N}{4\lambda_{\text{lat}}}\mu^2 \sum_n \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a(n)\overline{\mathcal{U}}_a(n)\right] - 1\right)^2
\end{aligned}
$$

$\gtrsim$100 inter-node data transfers in the fermion operator — non-trivial...

Reduce barriers to entry $\longrightarrow$ public parallel code at

github.com/daschaich/susy

Evolved from MILC lattice QCD code, presented in arXiv:1410.6971

(I) Thermodynamics on a 2-torus

arXiv:1709.07025

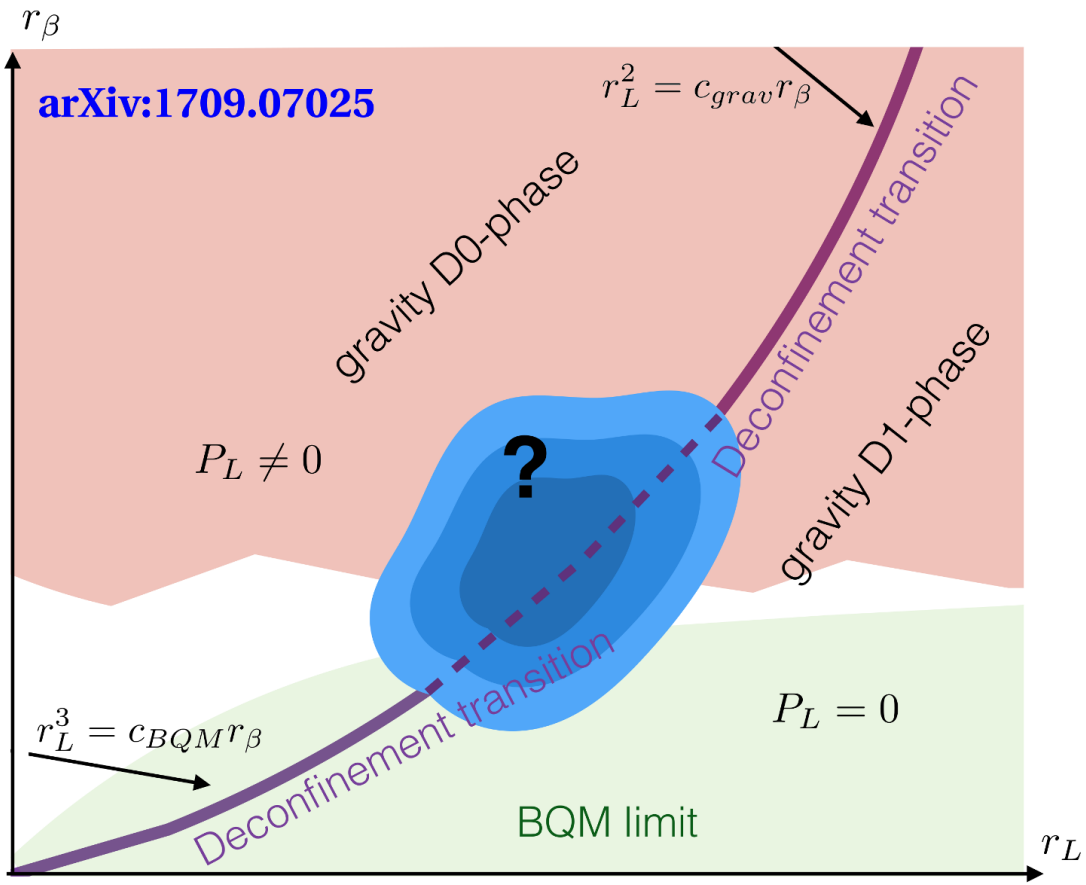
Dimensionally reduce to 2d $\mathcal{N} = (8,8)$ SYM with four scalar $\mathcal{Q}$,
study low temperatures $t = 1/r_\beta \longleftrightarrow$ black holes in dual supergravity

For decreasing r_L **at large *N***

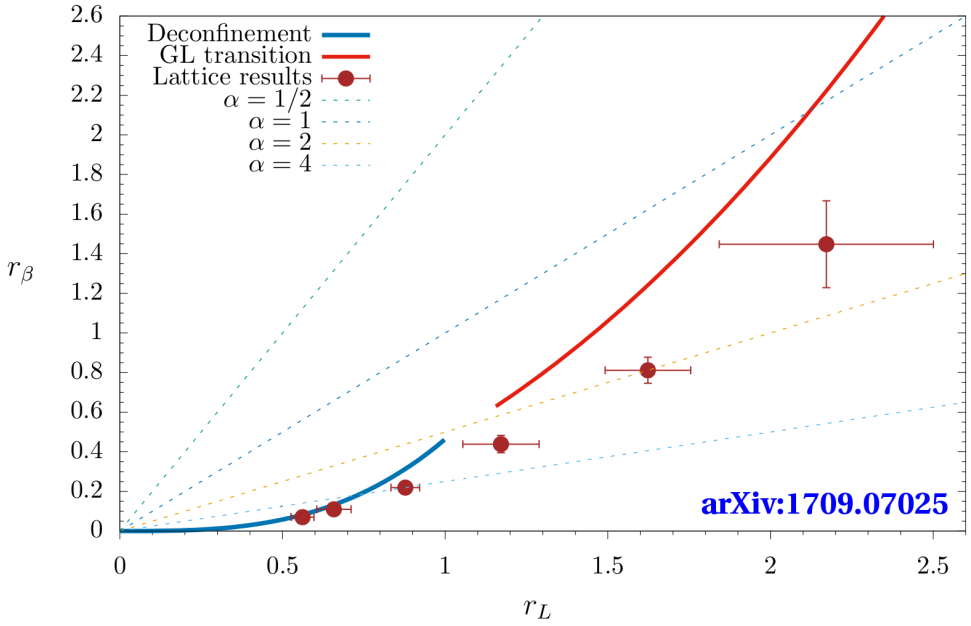
homogeneous black string (D1)
$\longrightarrow$ localized black hole (D0)

$\updownarrow$

"spatial deconfinement"
signalled by Wilson line P_L



$\mathcal{N} = (8,8)$ SYM lattice phase diagram results

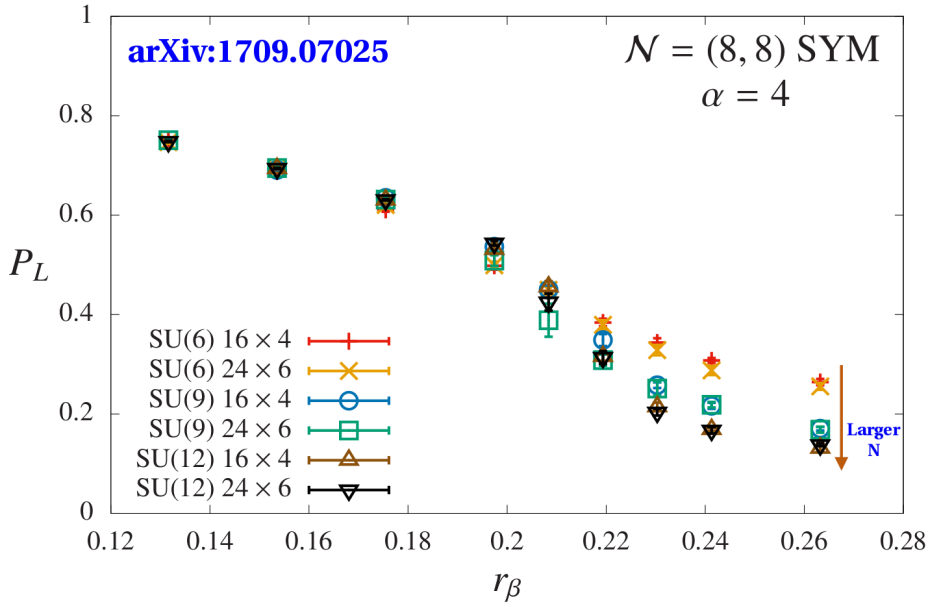


Good agreement
with high-temp. bosonic QM

Consistent with holography
at low temperatures

Example spatial deconfinement
transition in Wilson line

Fixing aspect ratio $\alpha = r_L/r_\beta = 4$,
scanning in $r_\beta = r_L/\alpha$

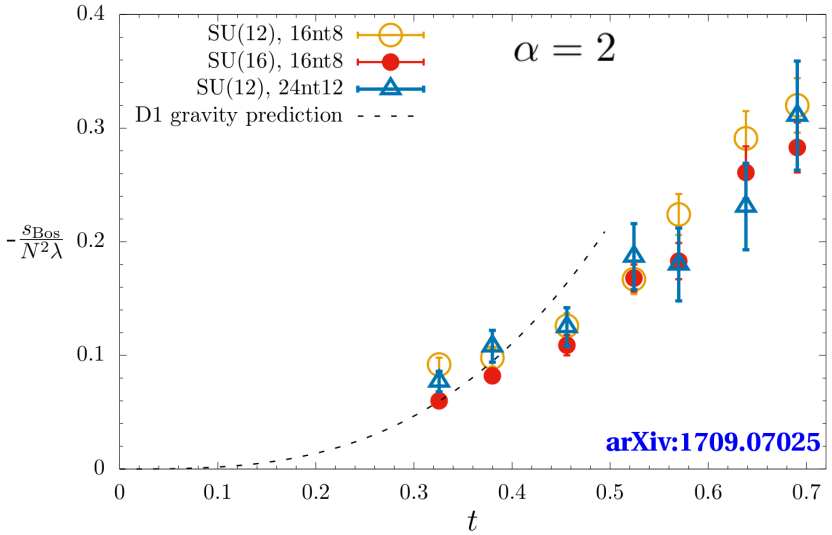


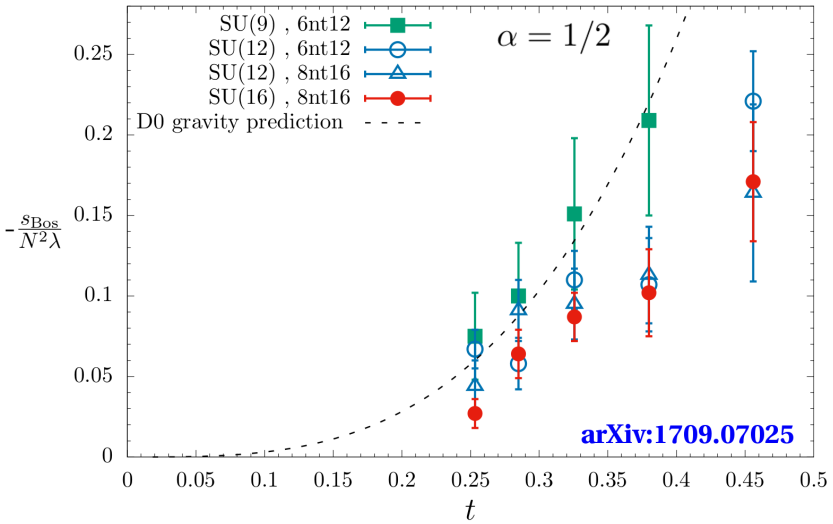
Dual black hole thermodynamics

Holography: bosonic action $\longleftrightarrow$ dual black hole internal energy

$\propto t^3$ for large-r_L D1 phase $\qquad$ $\propto t^{3.2}$ for small-r_L D0 phase

Lattice results consistent with holography for sufficiently low $t \lesssim 0.4$



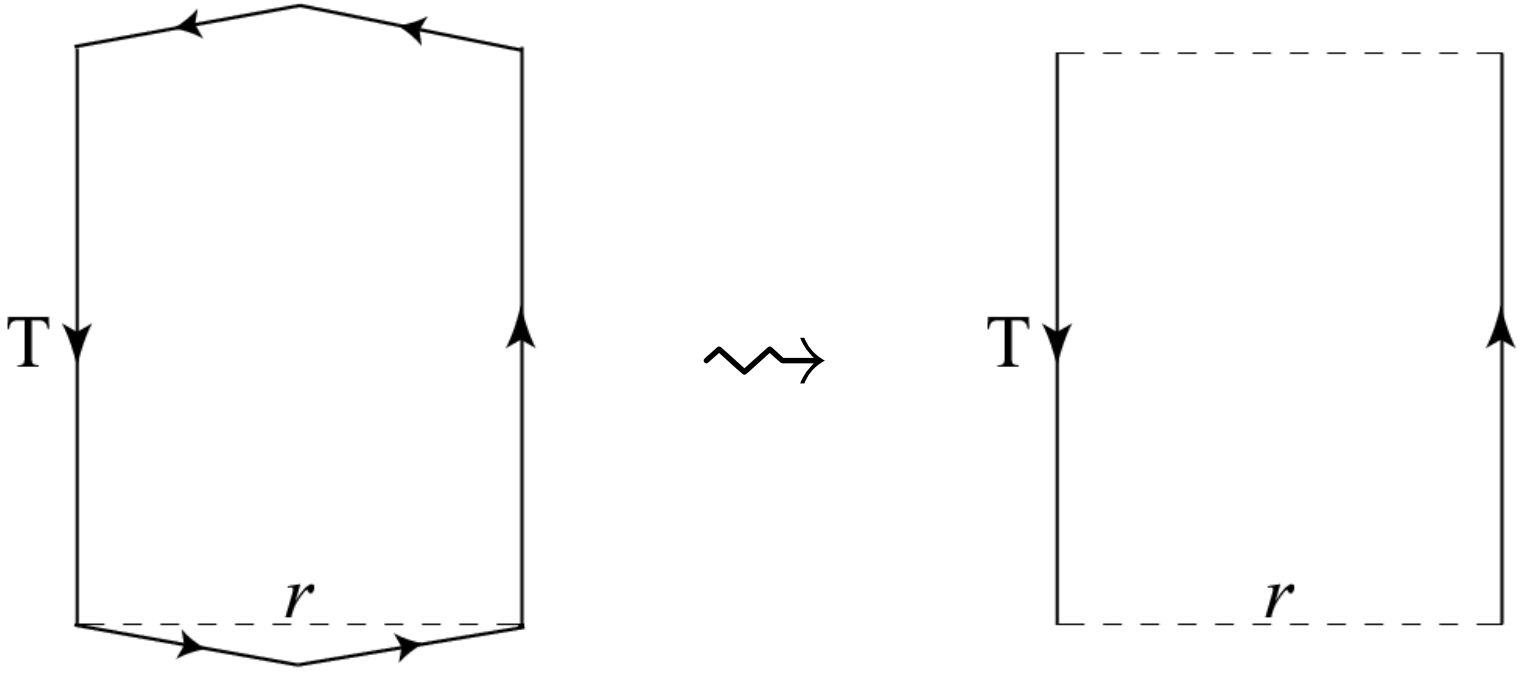


Need larger $N > 16$ to avoid instabilities at lower temperatures

(II) Static potential $V(r)$

Static probes $\longrightarrow$ $r \times T$ Wilson loops $W(r,T) \propto e^{-V(r)\,T}$

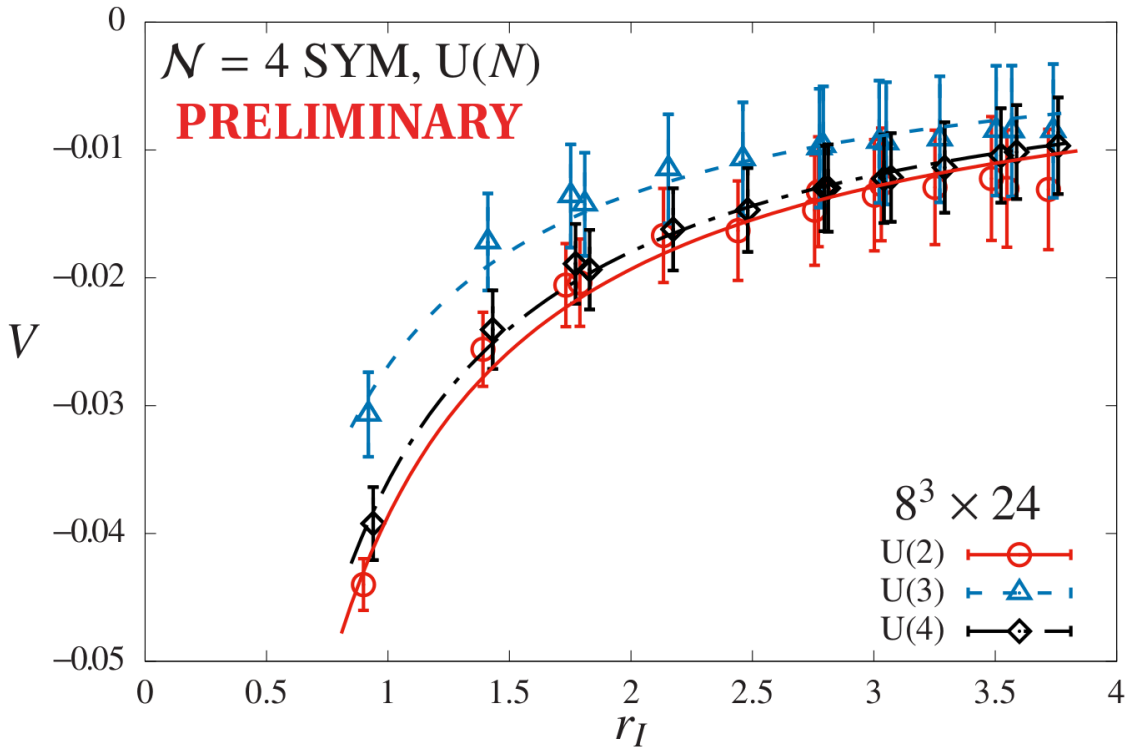
Coulomb gauge trick reduces A_4^* lattice complications



Static potential is Coulombic at all λ

Fits to confining $V(r) = A - C/r + \sigma r \longrightarrow$ vanishing string tension σ

$\Longrightarrow$ Fit to just $V(r) = A - C/r$ to extract Coulomb coefficient $C(\lambda)$

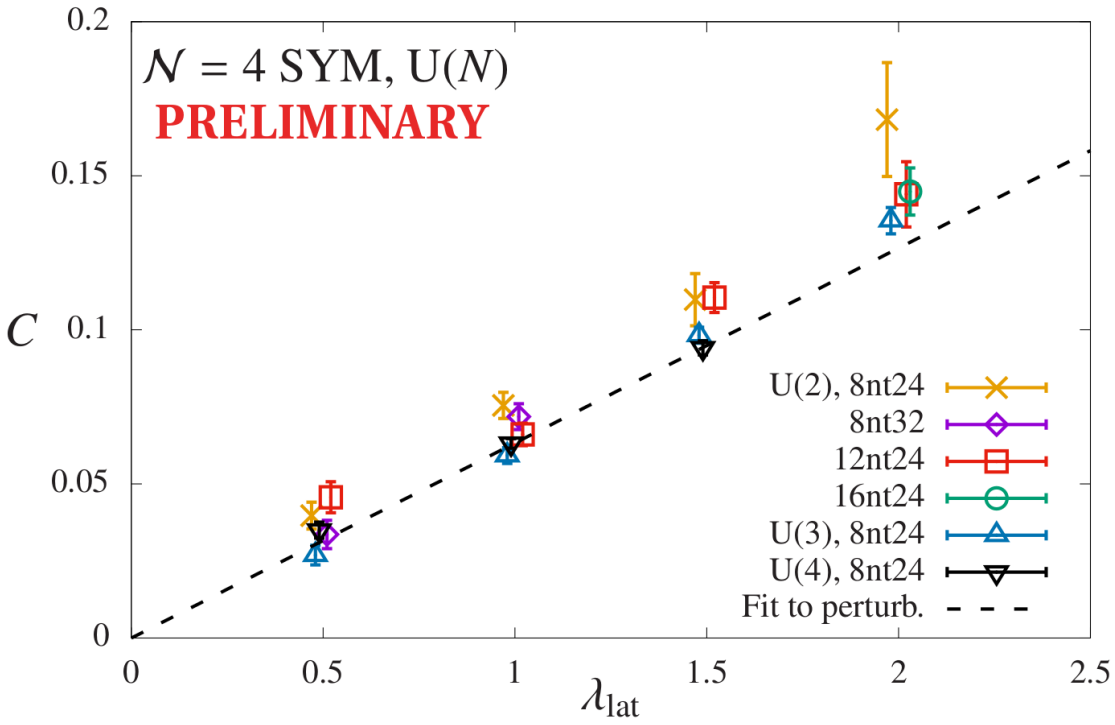


Discretization artifacts reduced by tree-level improved analysis

Coupling dependence of Coulomb coefficient

Continuum perturbation theory $\longrightarrow$ $C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

Holography $\longrightarrow$ $C(\lambda) \propto \sqrt{\lambda}$ for $N \to \infty$ and $\lambda \to \infty$ with $\lambda \ll N$



Consistent with leading-order perturbation theory for $\lambda_{\rm lat} \leq 2$

(III) Konishi operator scaling dimension

$\mathcal{O}_K(x) = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}(x)\Phi^{\mathrm{I}}(x)\right]$ is simplest conformal primary operator

Scaling dimension $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$ investigated through

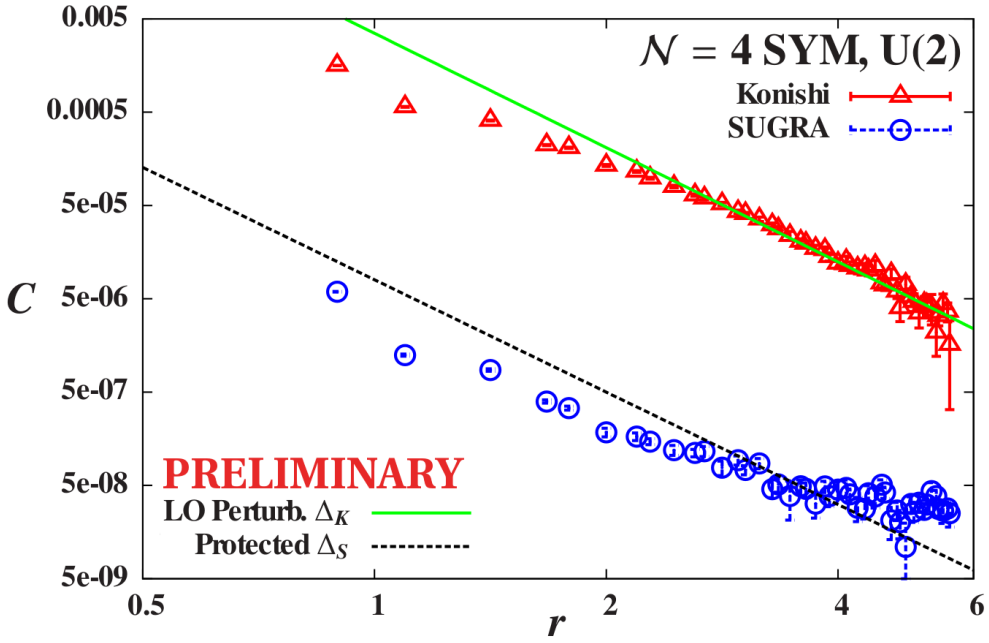
perturbation theory (& S duality), holography, conformal bootstrap

$C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$

'SUGRA' is 20′ op., $\Delta_S = 2$

Will compare:

- Direct power-law decay
- Finite-size scaling
- Monte Carlo RG



(III) Konishi operator scaling dimension

Lattice scalars $\varphi(n)$ from polar decomposition of complexified links

$$\mathcal{U}_a(n) \longrightarrow e^{\varphi_a(n)} U_a(n) \qquad \mathcal{O}_K^{\text{lat}}(n) = \sum_a \text{Tr}\left[\varphi_a(n)\varphi_a(n)\right] - \text{vev}$$

$$\mathcal{O}_S^{\text{lat}}(n) \sim \text{Tr}\left[\varphi_a(n)\varphi_b(n)\right]$$

$$C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

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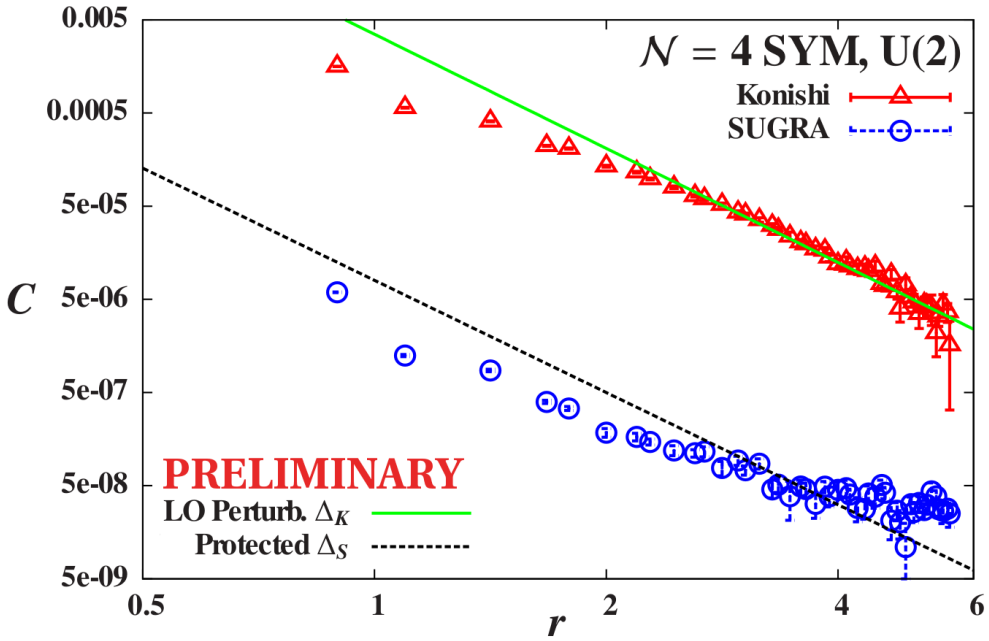


Scaling dimensions from MCRG stability matrix

Lattice system: $H = \sum_i c_i \, \mathcal{O}_i$ (infinite sum)

Couplings flow under RG blocking $\longrightarrow$ $H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Fixed point $\longrightarrow$ $H^\star = R_b H^\star$ with couplings $c_i^\star$

Linear expansion around fixed point $\longrightarrow$ stability matrix $T_{ik}^\star$

$$c_i^{(n)} - c_i^\star = \sum_k \left. \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \right|_{H^\star} \left(c_k^{(n-1)} - c_k^\star \right) \equiv \sum_k T_{ik}^\star \left(c_k^{(n-1)} - c_k^\star \right)$$

Correlators of $\mathcal{O}_i$, $\mathcal{O}_k$ $\longrightarrow$ elements of stability matrix [Swendsen, 1979]

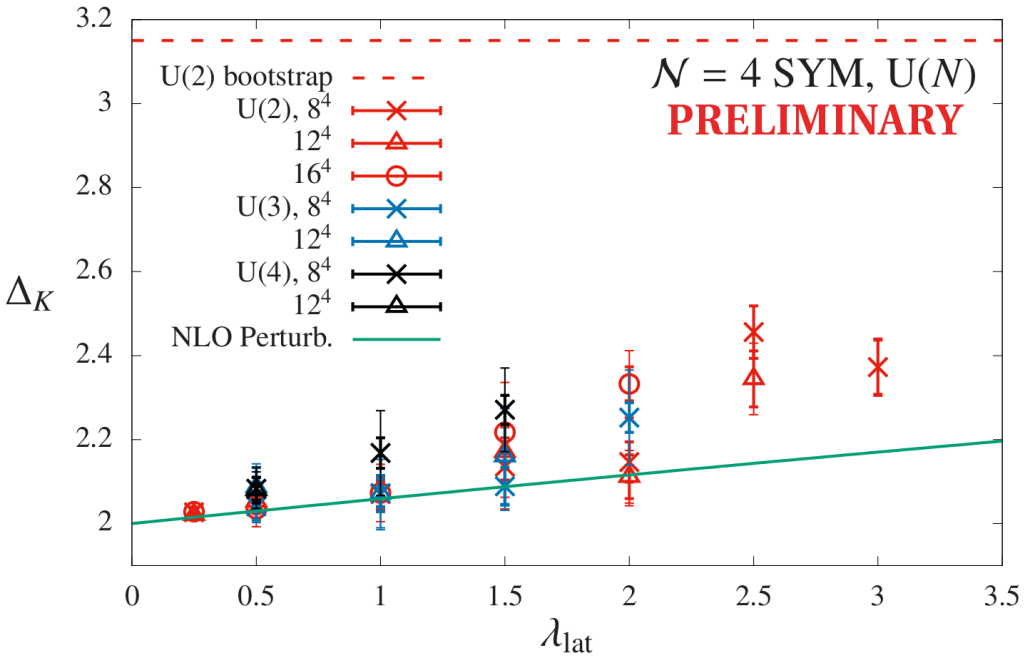
Eigenvalues of $T_{ik}^\star$ $\longrightarrow$ scaling dimensions of corresponding operators

Preliminary Δ_K results from Monte Carlo RG

Analyzing both $\mathcal{O}_K^{\text{lat}}$ and $\mathcal{O}_S^{\text{lat}}$

Imposing protected $\Delta_S = 2$
$\longrightarrow \Delta_K(\lambda)$ looks perturbative

Systematic uncertainties from different amounts of smearing



Complication: Twisting involves only $\text{SO}(4)_R \subset \text{SO}(6)_R$

$\Longrightarrow$ Lattice Konishi op. mixes with $\text{SO}(4)_R$-singlet part of $\text{SO}(6)_R$-nonsinglet SUGRA op.

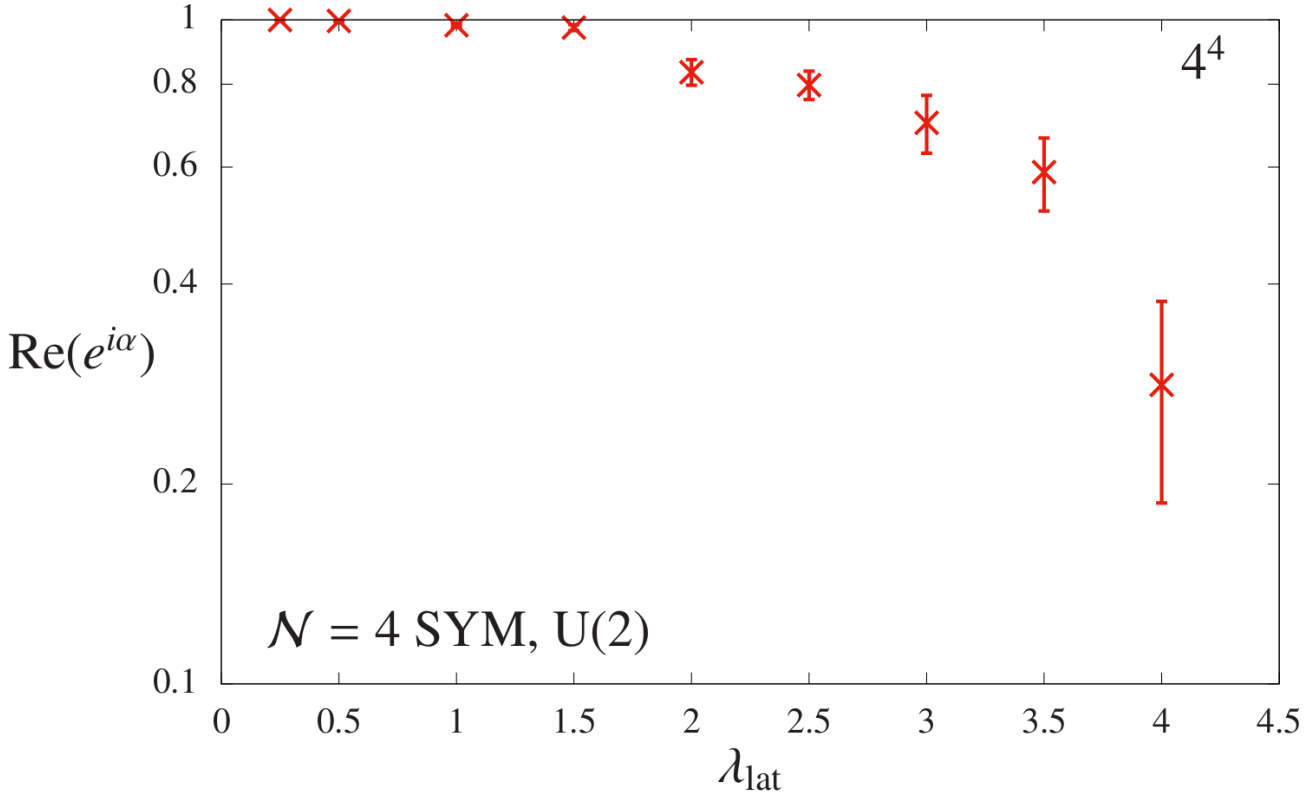
Working on variational analyses to disentangle operators

Future: Pushing $\mathcal{N} = 4$ SYM to stronger coupling

✓ Reproduce reliable (4d) results in perturbative regime

$\longrightarrow$ Check holographic predictions and access new domains

Sign problem seems to become obstruction

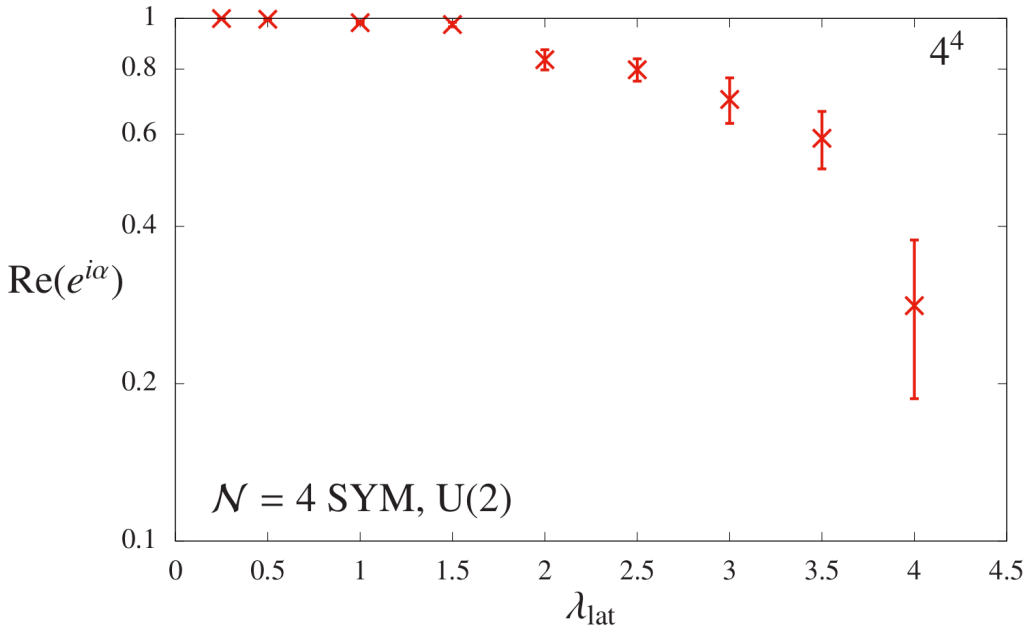


Quick review of sign problem

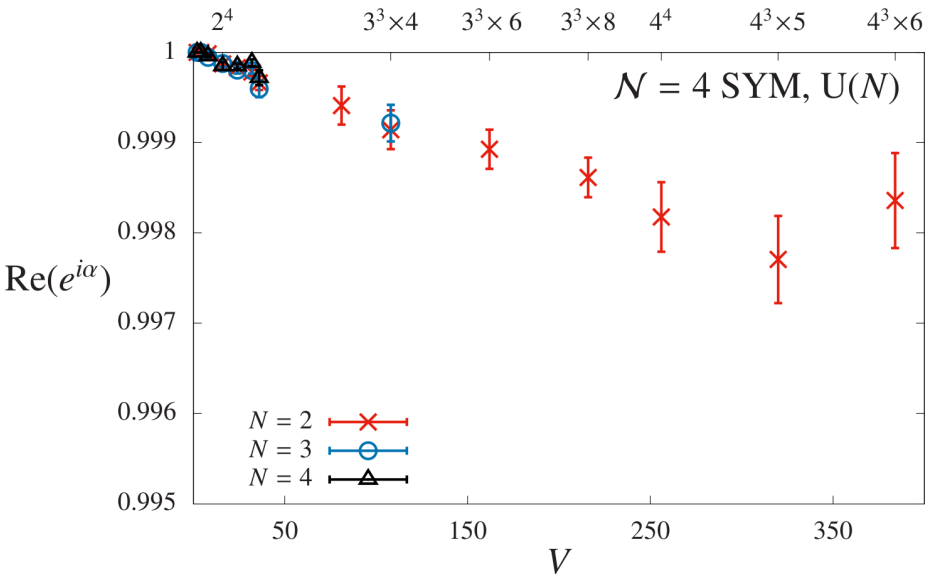
$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [d\mathcal{U}][d\overline{\mathcal{U}}] \;\; \mathcal{O} \;\; e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]} \;\; \mathrm{pf}\, \mathcal{D}[\mathcal{U},\overline{\mathcal{U}}]$$

Complex pfaffian $\mathrm{pf}\,\mathcal{D} = |\mathrm{pf}\,\mathcal{D}| e^{i\alpha}$ complicates importance sampling

We phase quench, $\mathrm{pf}\,\mathcal{D} \longrightarrow |\mathrm{pf}\,\mathcal{D}|$, need to reweight $\langle \mathcal{O} \rangle = \dfrac{\langle \mathcal{O} e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}}$



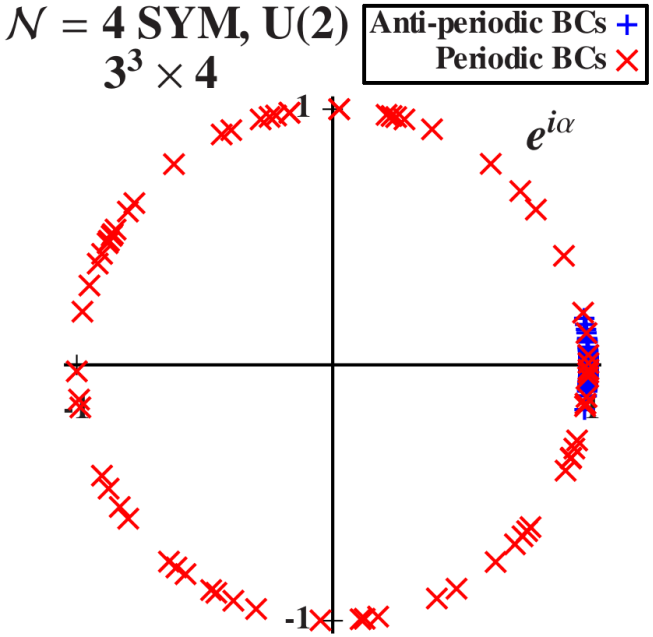
$\mathcal{N} = 4$ SYM sign problem puzzles



Pfaffian nearly real positive
for all accessible volumes
(at fixed $\lambda_{\text{lat}} = 0.5$)

$\langle e^{i\alpha} \rangle_{pq}$ extremely sensitive
to boundary conditions

But other $\langle \mathcal{O} \rangle_{pq}$ are not!



Future: Lattice superQCD (in 2d & 3d)

Preserve twisted supersymmetry sub-algebra on the lattice

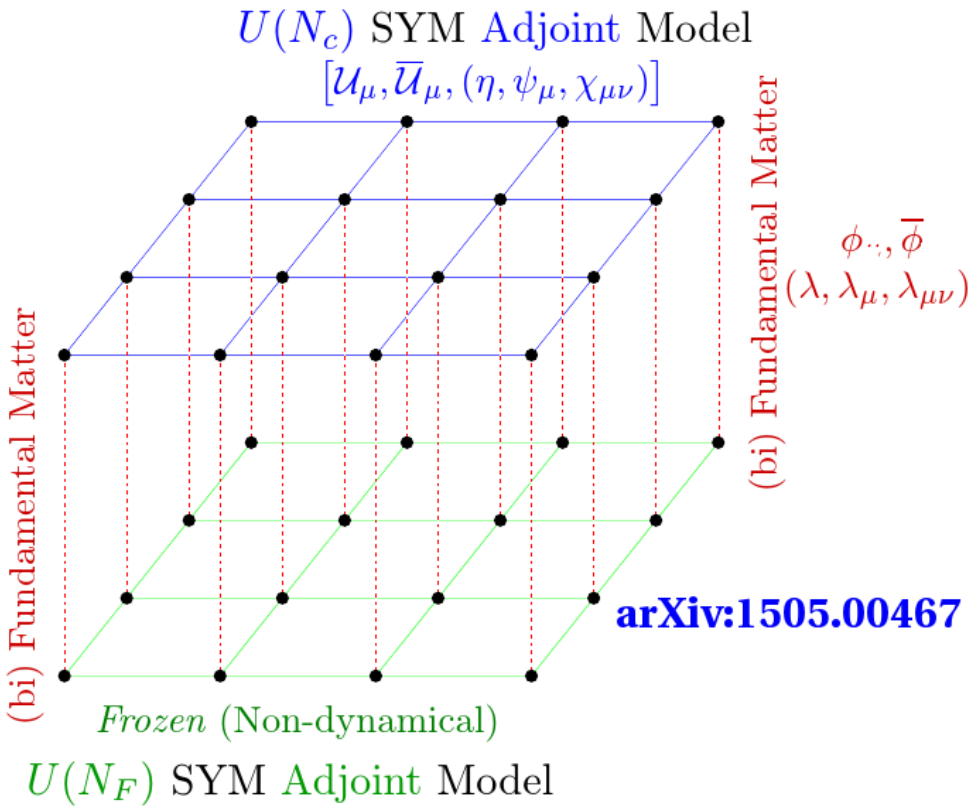
Proposed by Matsuura [0805.4491] and Sugino [0807.2683],
first numerical study by Catterall & Veernala [1505.00467]

2-slice lattice SYM
with U(N) $\times$ U(F) gauge group

Adj. fields on each slice

Bi-fundamental in between

Decouple U(F) slice
$\longrightarrow$ U(N) SQCD in $d-1$ dims.
with F fund. hypermultiplets



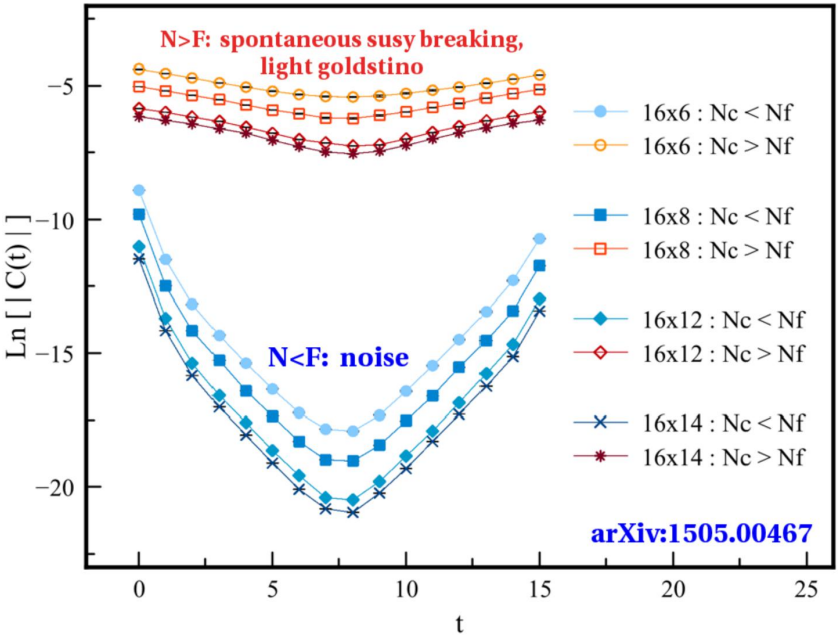
Dynamical susy breaking in 2d lattice superQCD

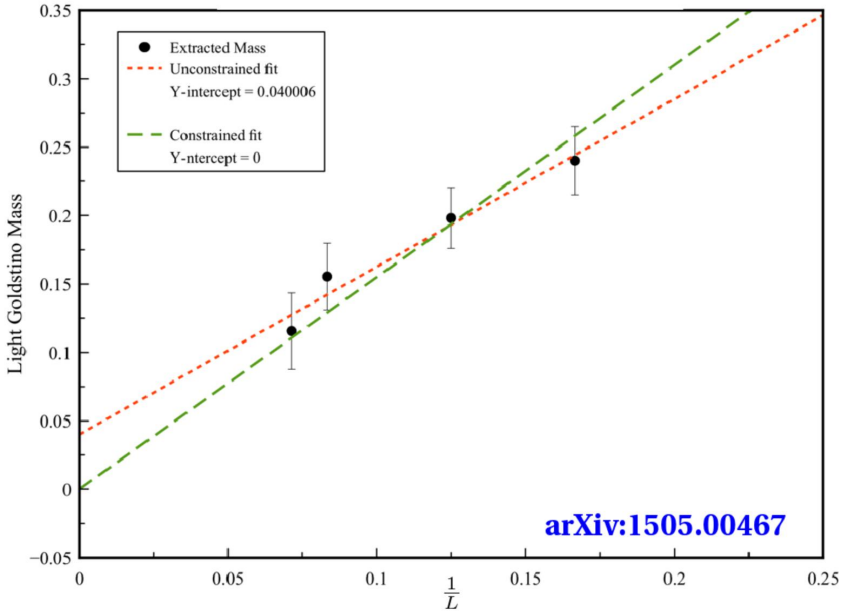
Auxiliary field e.o.m. $\longrightarrow$ Fayet–Iliopoulos D-term potential

$$d = \overline{\mathcal{D}}_a \mathcal{U}_a + \sum_{i=1}^{F} \phi_i \overline{\phi}_i + r\mathbb{I}_N \quad \longrightarrow \quad S_D \propto \sum_{i=1}^{F} \left(\mathrm{Tr}\left[\phi_i \overline{\phi}_i + r\mathbb{I}_N \right] \right)^2$$

Zero out N diagonal elements via F scalar vevs

or else susy breaking, $\langle \mathcal{Q}\eta \rangle = \langle d \rangle \neq 0 \longleftrightarrow \langle 0 \,|H|\, 0 \rangle > 0$





Recap: An exciting time for lattice supersymmetry

✓ Preserve (some) susy in discrete space-time

$\longrightarrow$ practical lattice $\mathcal{N} = 4$ SYM, public code available

Reproduce reliable analytic results

✓ 2d $\mathcal{N} = (8, 8)$ SYM thermodynamics consistent with holography

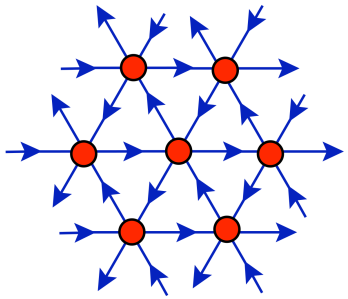
✓ Perturbative static potential Coulomb coefficient $C(\lambda)$

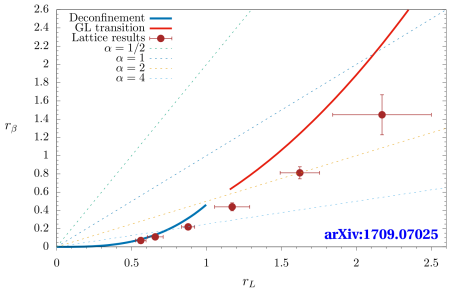
and Konishi operator conformal scaling dimension $\Delta_K(\lambda)$

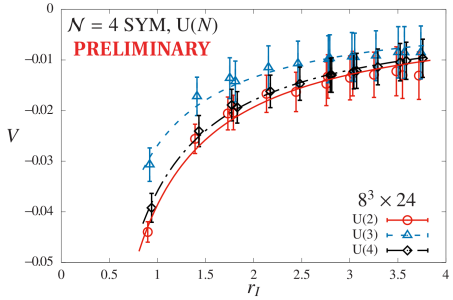
Access new domains

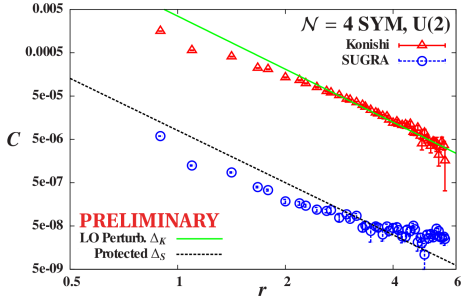
$\longrightarrow$ Understanding the sign problem at stronger couplings

$\longrightarrow$ Lower-dimensional superQCD and more...









Thank you!

Collaborators

Simon Catterall, Raghav Jha, Toby Wiseman

also Georg Bergner, Poul Damgaard, Joel Giedt, Anosh Joseph

Funding and computing resources

Backup: Breakdown of Leibniz rule on the lattice

$\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$ is problematic

$\Longrightarrow$ try finite difference $\partial\phi(x) \longrightarrow \Delta\phi(x) = \frac{1}{a}\left[\phi(x+a) - \phi(x)\right]$

Crucial difference between ∂ and Δ

$$\Delta\left[\phi\eta\right] = a^{-1}\left[\phi(x+a)\eta(x+a) - \phi(x)\eta(x)\right]$$
$$= \left[\Delta\phi\right]\eta + \phi\Delta\eta + a\left[\Delta\phi\right]\Delta\eta$$

Only recover Leibniz rule $\partial\left[\phi\eta\right] = \left[\partial\phi\right]\eta + \phi\partial\eta$ when $a \to 0$

$\Longrightarrow$ "discrete supersymmetry" breaks down on the lattice

Backup: Complexified gauge field from twisting

Why combine A_μ and $\Phi^{\mathrm{I}} \longrightarrow$ complexified gauge field $\mathcal{A}_a$ and $\overline{\mathcal{A}}_a$?

This is source of $\mathrm{U}(N) = \mathrm{SU}(N) \otimes \mathrm{U}(1)$ that complicates lattice action

Schematically, under $\mathrm{SO}(d)_{tw} = \mathrm{diag}\big[\mathrm{SO}(d)_{\mathrm{euc}} \otimes \mathrm{SO}(d)_R\big]$

$$\begin{aligned} A_\mu &\sim \text{vector} \otimes \text{scalar} \longrightarrow \text{vector} \\ \Phi^{\mathrm{I}} &\sim \text{scalar} \otimes \text{vector} \longrightarrow \text{vector} \end{aligned}$$

Easiest to see by dimensionally reducing from 5d

$$\mathcal{A}_a = A_a + i\Phi_a \longrightarrow (A_\mu, \phi) + i(\Phi_\mu, \overline{\phi})$$

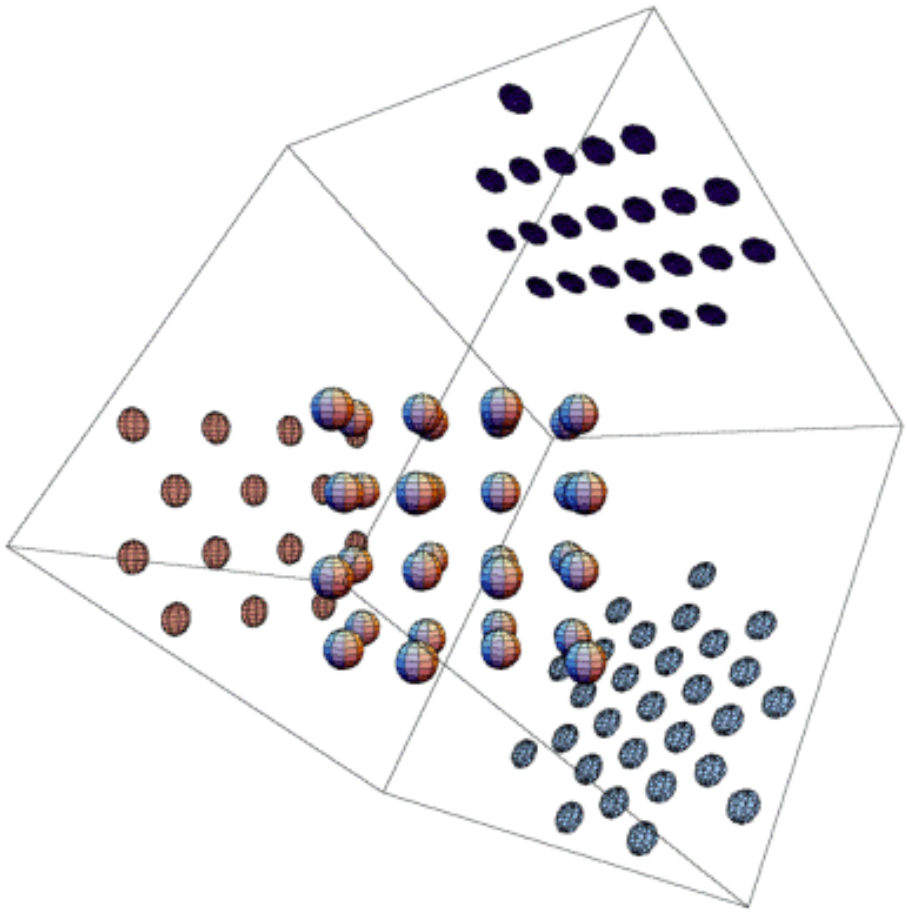
Backup: A_4^* lattice from five dimensions

Again dimensionally reduce, treating all five gauge links symmetrically

Start with hypercubic lattice
in 5d momentum space

Symmetric constraint $\sum_a \partial_a = 0$
projects to 4d momentum space

Result is A_4 lattice
$\longrightarrow$ dual A_4^* lattice in real space

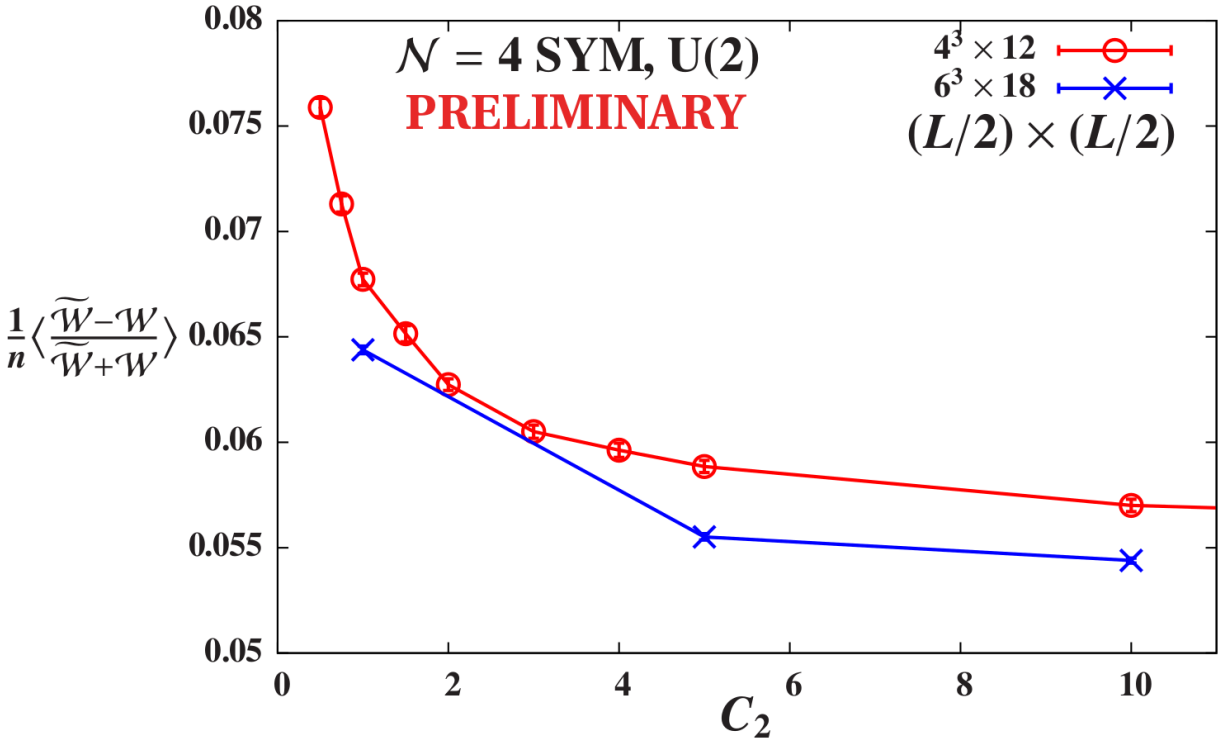


Backup: Restoration of $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supersymmetries

"$\mathcal{Q}$ + discrete $R_a \subset \mathrm{SO}(4)_{tw} = \mathcal{Q}_a$ and $\mathcal{Q}_{ab}$"

Test R_a on Wilson loops $\widetilde{\mathcal{W}}_{ab} \equiv R_a \mathcal{W}_{ab}$,
tune coeff. c_2 of d^2 term to ensure restoration in continuum

Results from arXiv:1411.0166 to be revisited with improved action



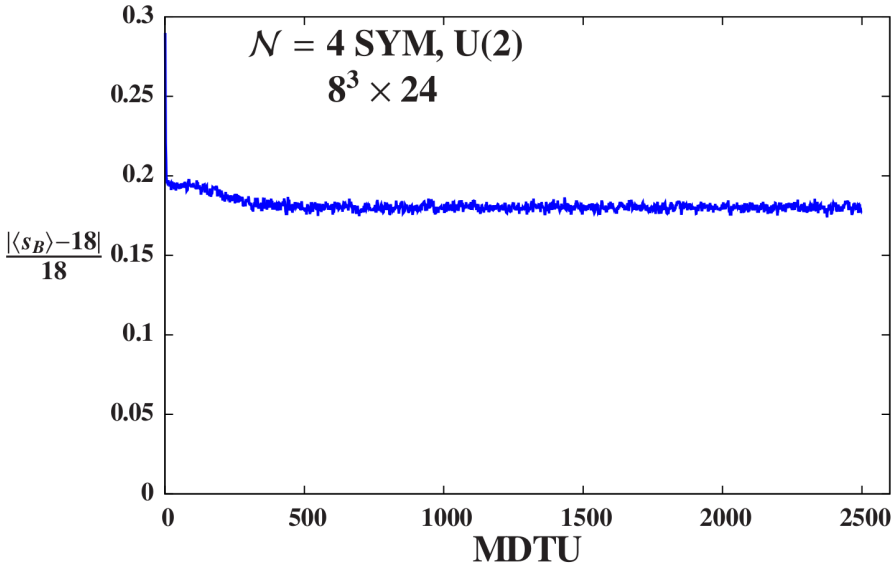
Backup: Problem with SU(N) flat directions

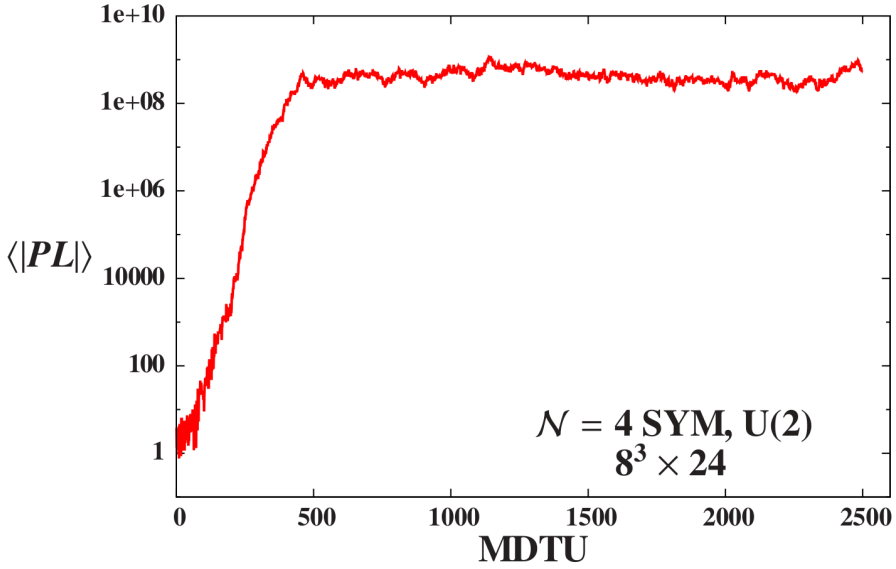
$\mu^2/\lambda_{\text{lat}}$ too small $\longrightarrow$ $\mathcal{U}_a$ can move far from continuum form $\mathbb{I}_N + \mathcal{A}_a$

Example: $\mu = 0.2$ and $\lambda_{\text{lat}} = 2.5$ on $8^3 \times 24$ volume

Left: Bosonic action stable $\sim$18% off its supersymmetric value

Right: (Complexified) Polyakov loop wanders off to $\sim 10^9$

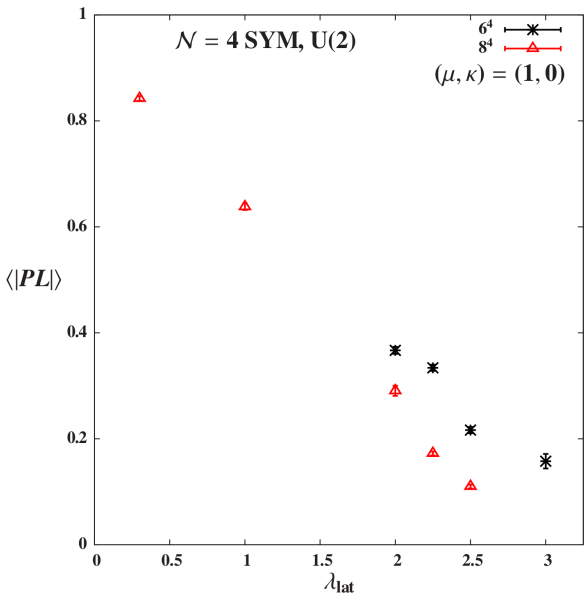


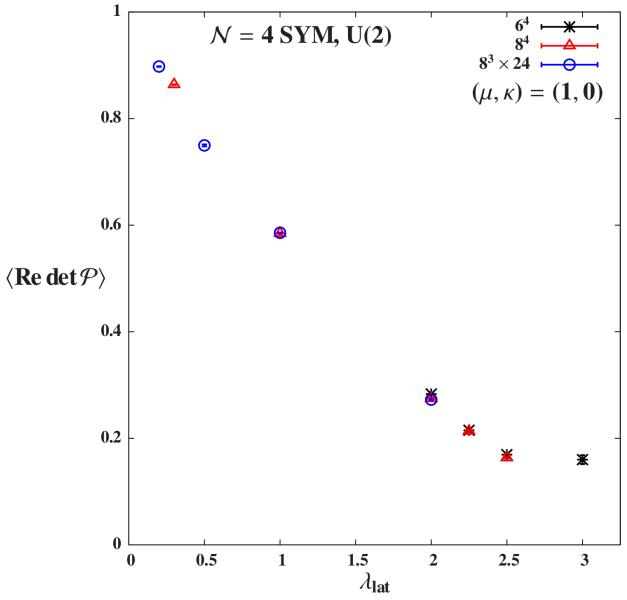


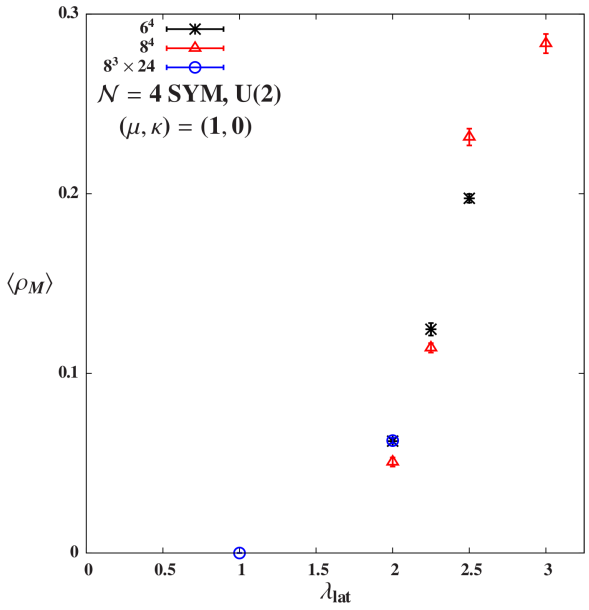
Backup: Problem with U(1) flat directions

Monopole condensation $\longrightarrow$ confined lattice phase

not present in continuum $\mathcal{N} = 4$ SYM







Around the same $2\lambda_{\rm lat} \approx 2\ldots$

Left: Polyakov loop falls towards zero

Center: Plaquette determinant falls towards zero

Right: Density of U(1) monopole world lines becomes non-zero

Backup: Regulating SU(N) flat directions

Add soft $\mathcal{Q}$-breaking scalar potential to lattice action

$$S = \frac{N}{4\lambda_{\text{lat}}}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de} + \mu^2 V\right]$$

$V = \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2$ lifts SU(N) flat directions,

ensures $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$ in continuum limit

Correct continuum limit requires $\mu^2 \to 0$

to restore $\mathcal{Q}$ and recover physical flat directions

Typically scale $\mu \propto 1/L$ in $L \to \infty$ continuum extrapolation

Backup: Poorly regulating U(1) flat directions

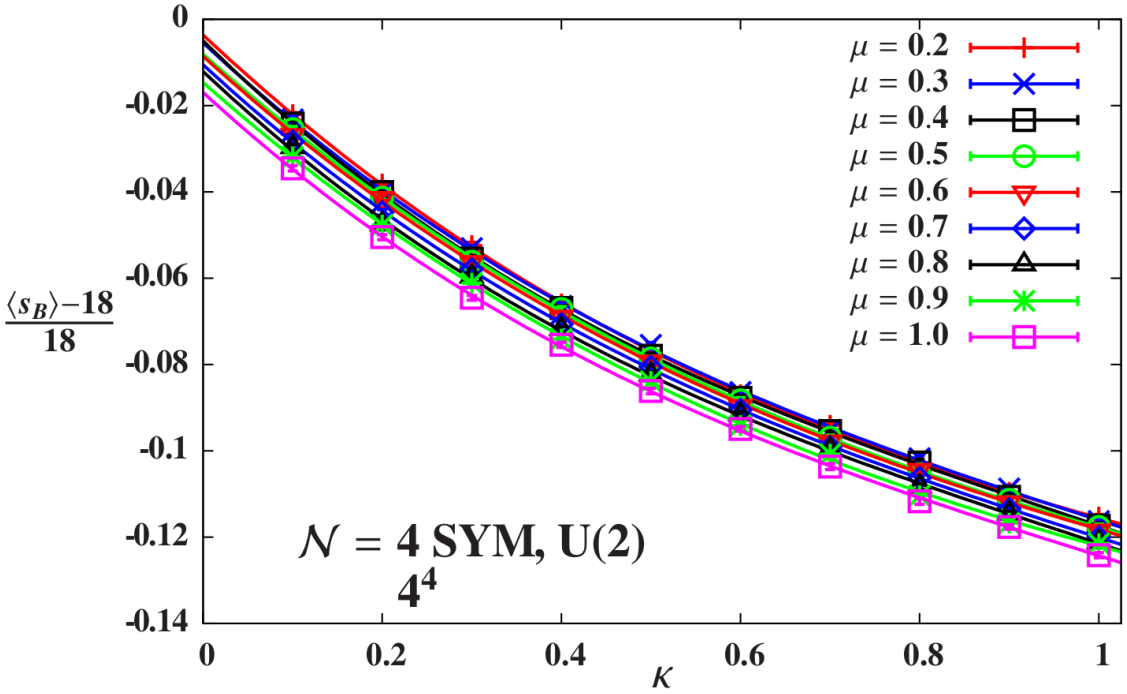
Until 2015 we added another soft $\mathcal{Q}$-breaking term

$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}}\mu^2 \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2 + \kappa \sum_{a<b} |\det \mathcal{P}_{ab} - 1|^2$$

More sensitivity to κ than to μ^2

Showing $\mathcal{Q}$ Ward identity from bosonic action

$$\langle s_B \rangle = 9N^2/2$$



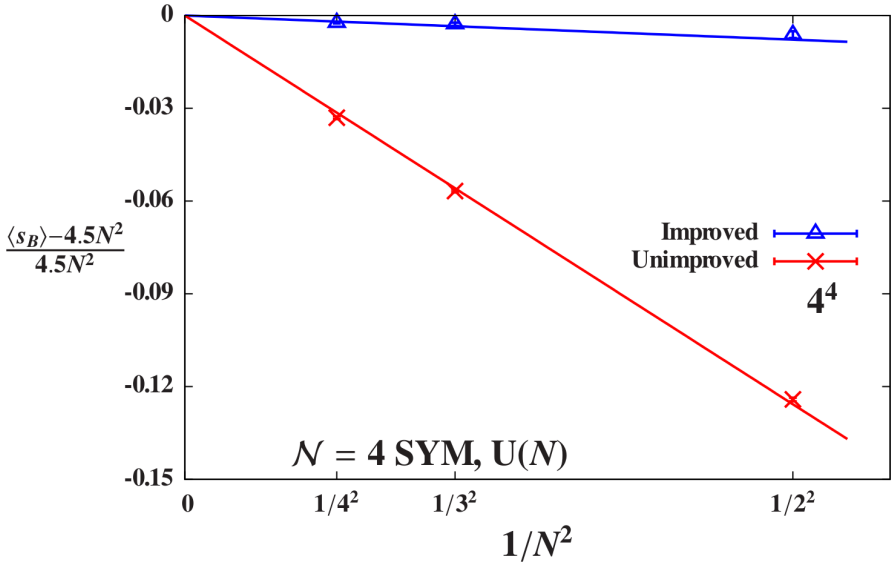
Backup: Better regulating U(1) flat directions

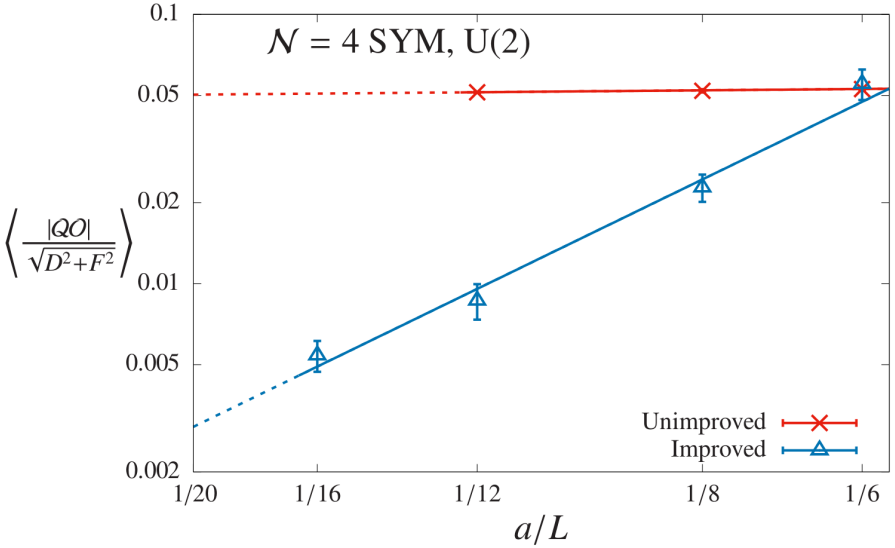
$$S = \frac{N}{4\lambda_{\text{lat}}}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \quad \downarrow \quad - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de} + \mu^2 V\right]$$

$$\eta\left\{\overline{\mathcal{D}}_a\mathcal{U}_a + G\sum_{a<b}\left[\det\mathcal{P}_{ab} - 1\right]\mathbb{I}_N\right\}$$

$\mathcal{Q}$ Ward identity violations scale $\propto 1/N^2$ (**left**) and $\propto (a/L)^2$ (**right**)

$\sim$ effective '$O(a)$ improvement' since $\mathcal{Q}$ forbids all dim-5 operators





Backup: Supersymmetric moduli space modification

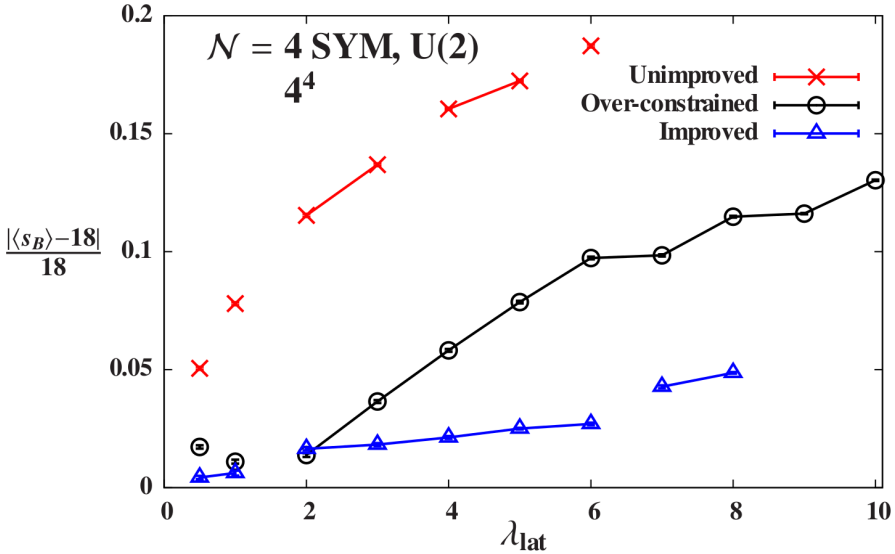
Method to impose $\mathcal{Q}$-invariant constraints

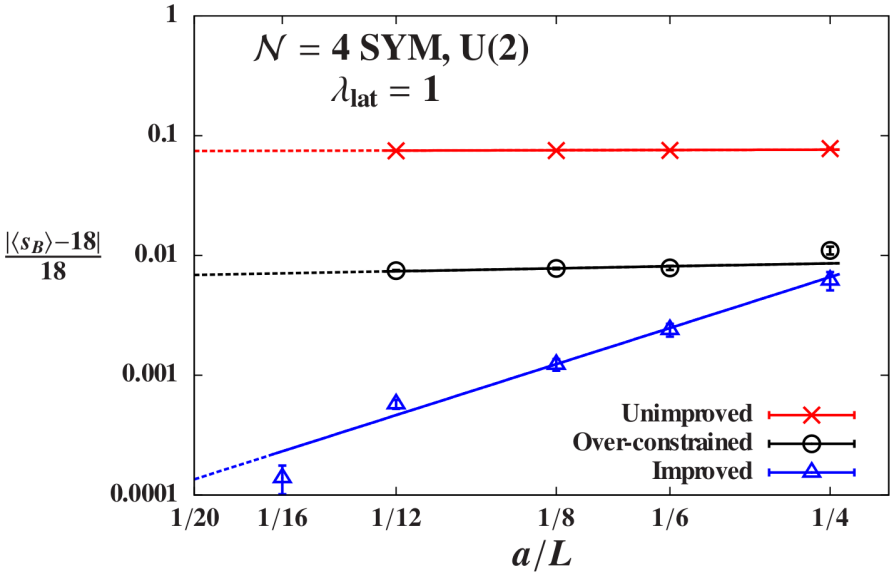
applicable to generic site operator $\mathcal{O}(n)$ [arXiv:1505.03135]

Modify auxiliary field equations of motion $\longrightarrow$ moduli space

$$d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) \qquad \longrightarrow \qquad d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) + G\mathcal{O}(n)\mathbb{I}_N$$

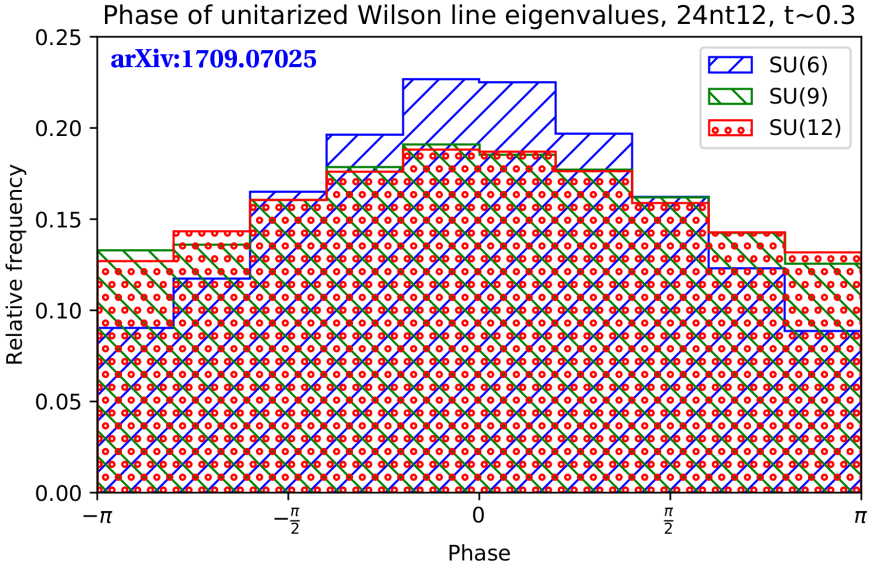
However, both U(1) and SU(N) $\in \mathcal{O}(n)$ over-constrains system

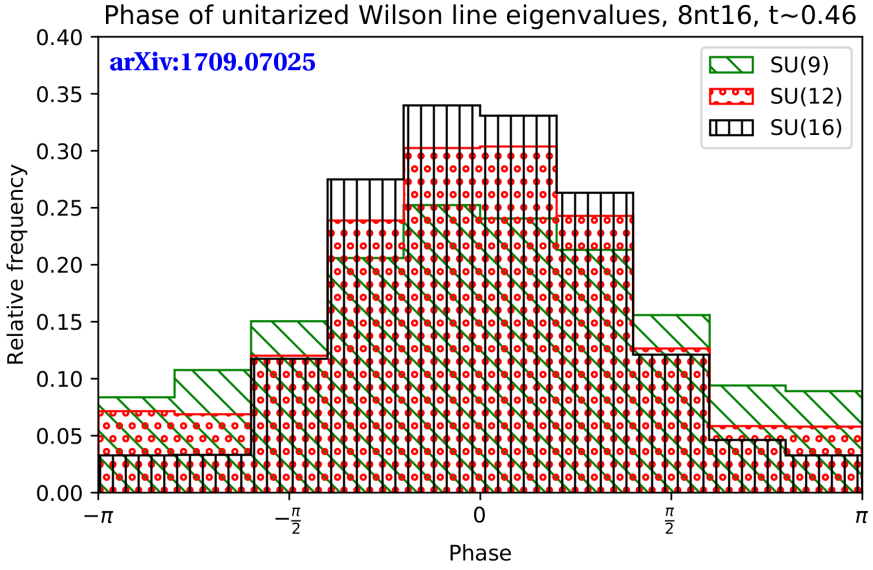




Backup: $\mathcal{N} = (8, 8)$ SYM Wilson line eigenvalues

Check 'spatial deconfinement' through Wilson line eigenvalue phases





Left: $\alpha = 2$ distributions more extended as N increases

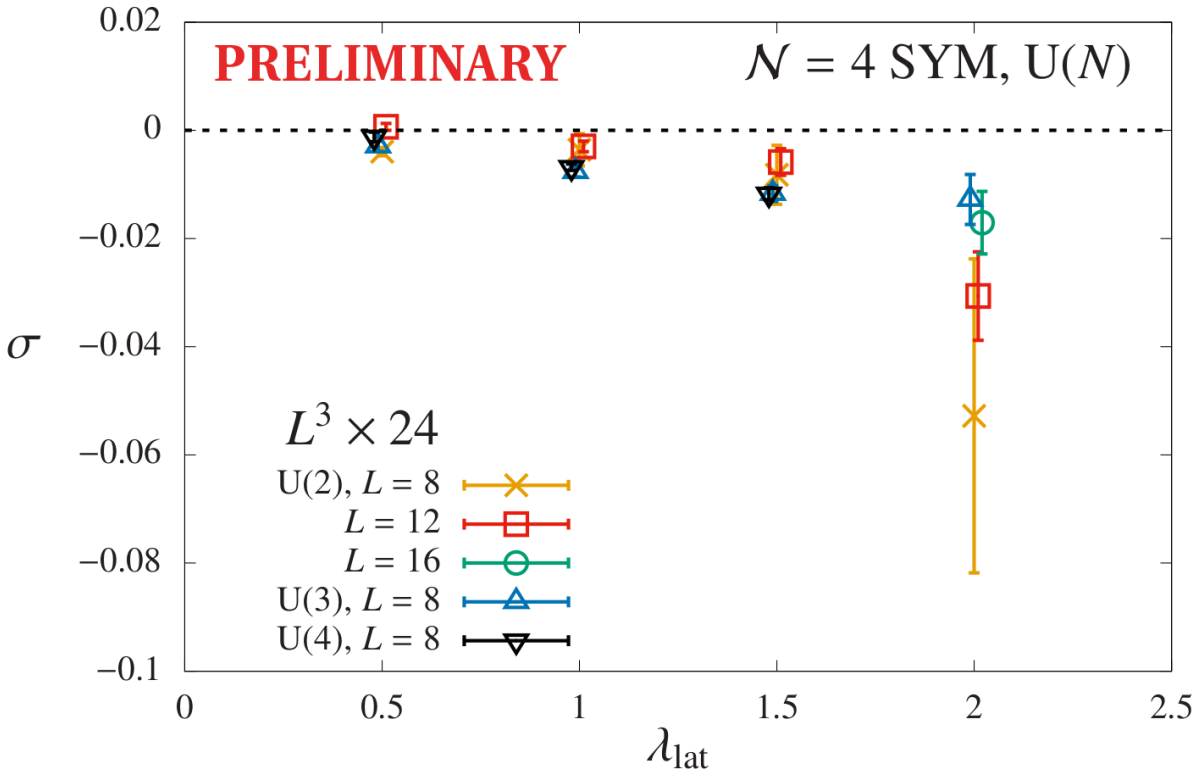
$\longrightarrow$ dual gravity describes homogeneous black string (D1 phase)

Right: $\alpha = 1/2$ distributions more compact as N increases

$\longrightarrow$ dual gravity describes localized black hole (D0 phase)

Backup: Static potential is Coulombic at all λ

String tension σ from fits to confining form $V(r) = A - C/r + \sigma r$



Slightly negative values flatten $V(r_l)$ for $r_l \lesssim L/2$

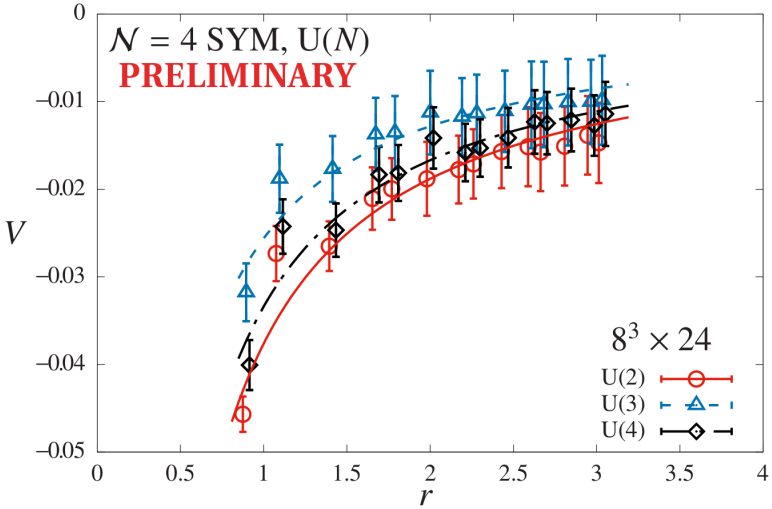
$\sigma \to 0$ as accessible range of r_l increases on larger volumes

Backup: Discretization artifacts in static potential

Discretization artifacts visible at short distances

where Coulomb term in $V(r) = A - C/r$ is most significant

Right: Fluctuations around Coulomb fit highlight artifacts



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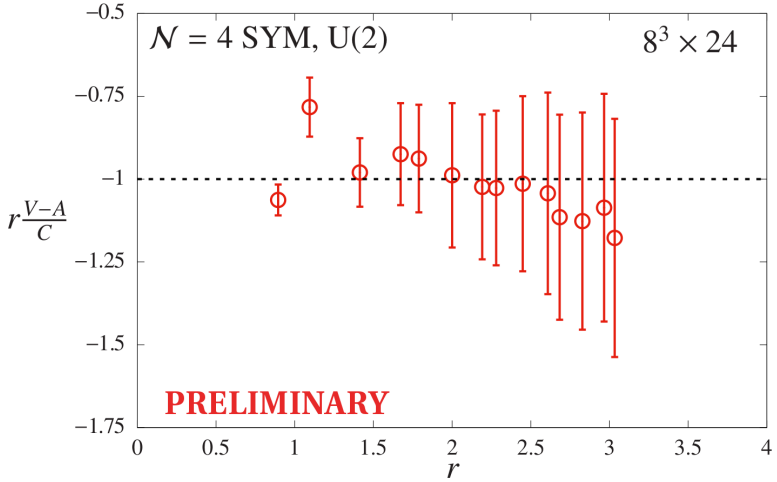


Danger of distorting Coulomb coefficient C

Backup: Tree-level improvement

Classic trick to reduce discretization artifacts in static potential
(Lang & Rebbi '82; Sommer '93; Necco '03)

Associate $V(r)$ data with r from Fourier transform of gluon propagator

Recall $\frac{1}{4\pi^2 r^2} = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{e^{ir\cdot k}}{k^2}$ where $\frac{1}{k^2} = G(k)$ in continuum

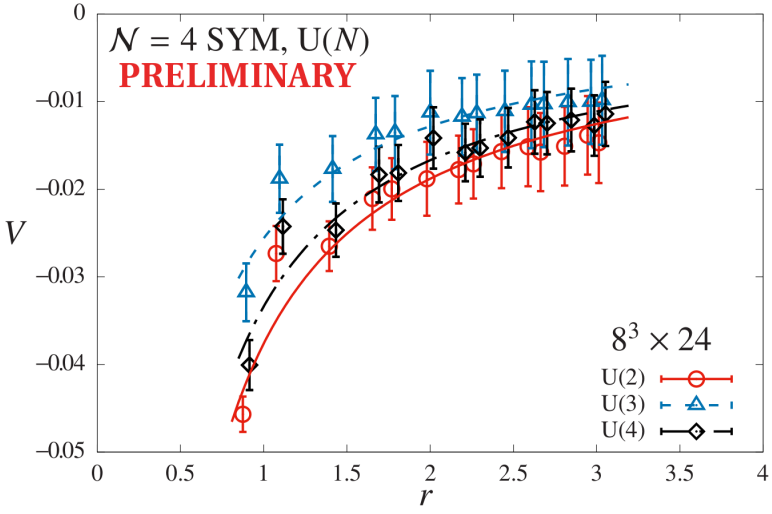
$$A_4^* \text{ lattice} \longrightarrow \frac{1}{r_l^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4\widehat{k}}{(2\pi)^4} \frac{\cos\left(ir_l \cdot \widehat{k}\right)}{4\sum_{\mu=1}^{4} \sin^2\left(\widehat{k}\cdot\widehat{e}_\mu \,/\, 2\right)}$$

Tree-level lattice propagator from arXiv:1102.1725

$\widehat{e}_\mu$ are A_4^* lattice basis vectors;
momenta $\widehat{k} = \frac{2\pi}{L}\sum_{\mu=1}^{4} n_\mu \widehat{g}_\mu$ depend on dual basis vectors

Backup: Tree-level-improved static potential

Tree-level improvement significantly reduces discretization artifacts



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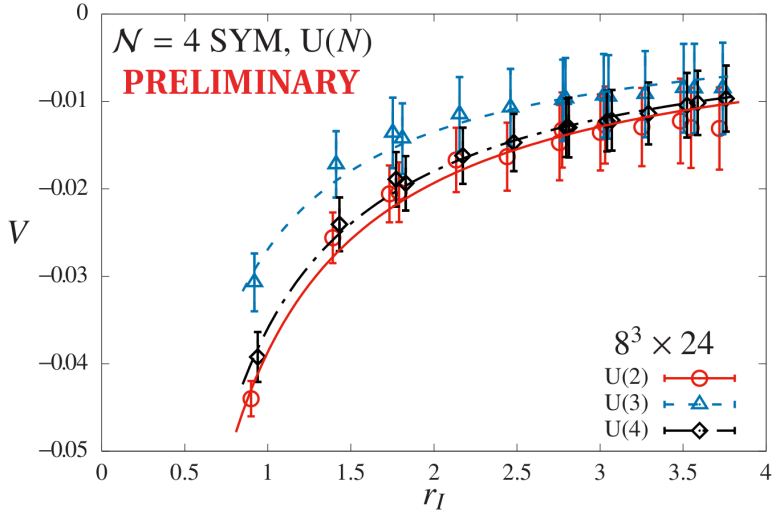

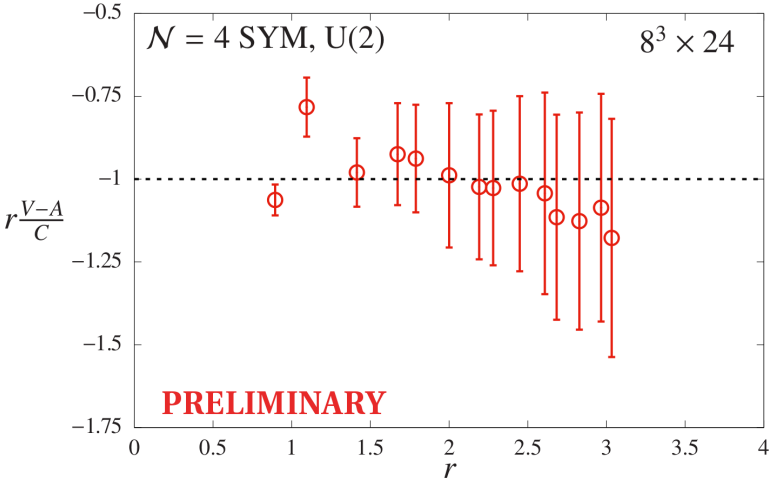

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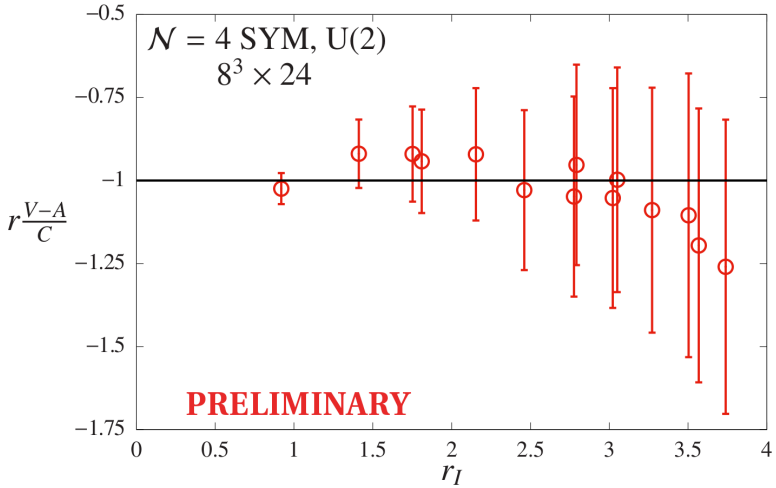


Backup: Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve $\mathcal{Q}$ and S_5 symmetries $\longleftrightarrow$ geometric structure

Simple transformation constructed in arXiv:1408.7067

$$\mathcal{U}'_a(n') = \xi\, \mathcal{U}_a(n)\mathcal{U}_a(n+\widehat{\mu}_a) \qquad \eta'(n') = \eta(n)$$

$$\psi'_a(n') = \xi\, [\psi_a(n)\mathcal{U}_a(n+\widehat{\mu}_a) + \mathcal{U}_a(n)\psi_a(n+\widehat{\mu}_a)] \qquad \text{etc.}$$

Doubles lattice spacing $a \longrightarrow a' = 2a$, with tunable rescaling factor ξ

Scalar fields from polar decomposition $\mathcal{U}(n) = e^{\varphi(n)} U(n)$

$\longrightarrow$ shift $\varphi \longrightarrow \varphi + \log \xi$ to keep blocked U unitary

This $\mathcal{Q}$-preserving RG transformation needed
to show only one log. tuning to recover continuum $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Backup: Smearing for Konishi analyses

Smear to enlarge (MCRG or variational) operator basis

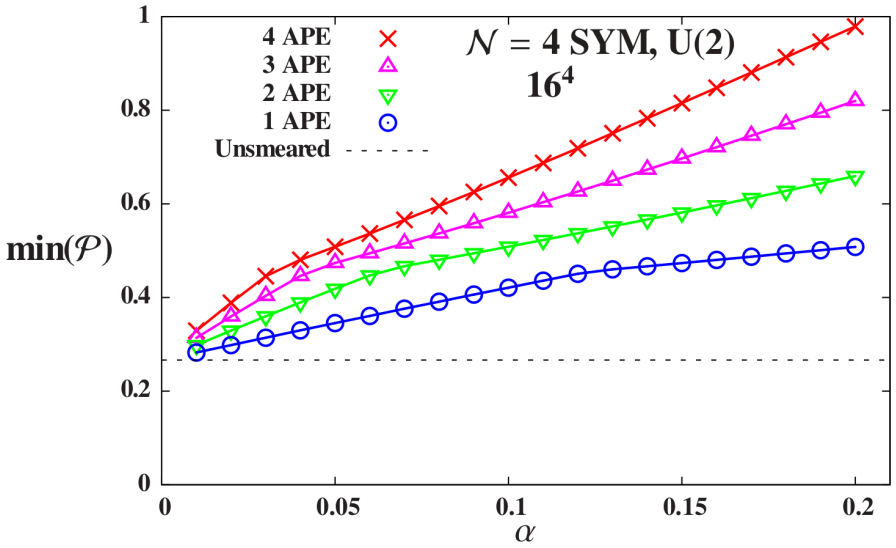
APE-like smearing: $\text{—} \longrightarrow (1-\alpha)\text{—} + \frac{\alpha}{8}\sum \sqcap$,

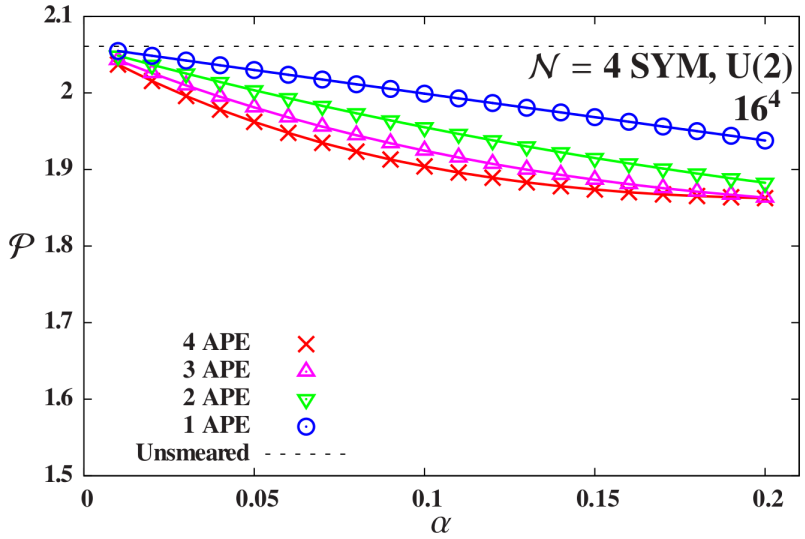
staples built from unitary parts of links but no final unitarization

(unitarized smearing — e.g. stout — doesn't affect Konishi)

Average plaquette stable upon smearing (**right**),

minimum plaquette steadily increases (**left**)





Backup: Dimensional reduction to $\mathcal{N} = (8,8)$ SYM

Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_x = N_y = 1$

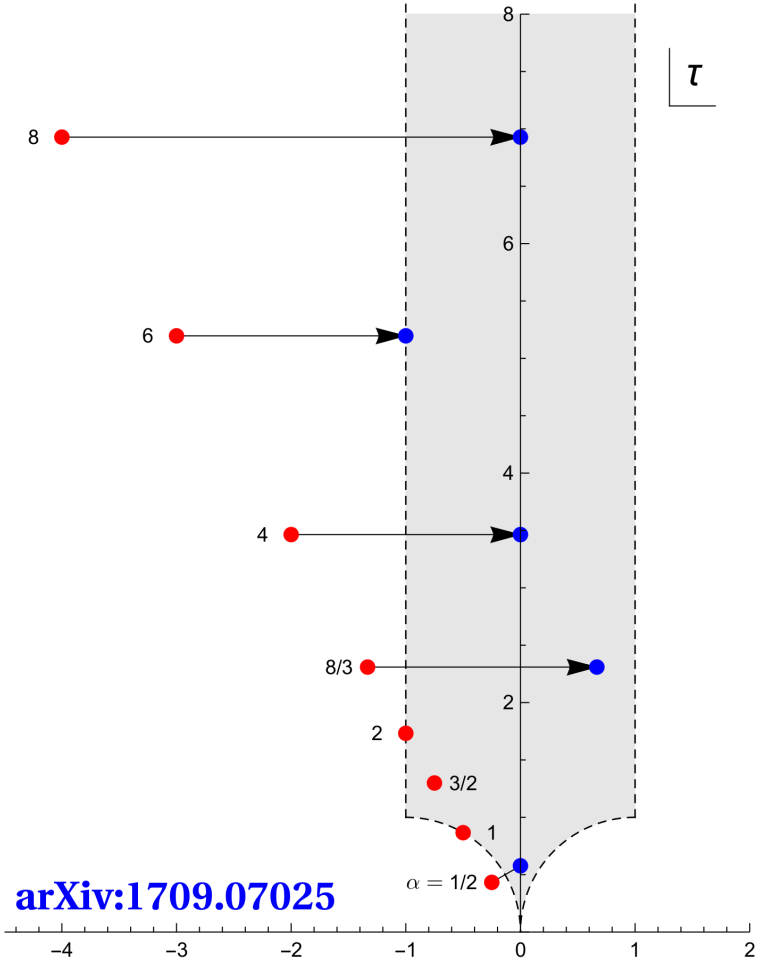
$A_4^* \longrightarrow A_2^*$ (triangular) lattice

Torus **skewed** depending on $\alpha = N_t/L$

Modular trans. into fund. domain

$\longrightarrow$ some skewed tori actually rectangular

Also need to stabilize compactified links
to ensure broken center symmetries



arXiv:1709.07025