Lattice studies of maximally supersymmetric Yang–Mills theories

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& more to come with Simon Catterall, Raghav Jha and Toby Wiseman
Overview and plan

Central idea
Preserve (some) susy in discrete space-time
to make lattice investigations practical

Goals
1) Reproduce reliable results in perturbative, holographic, etc. regimes
2) Use lattice to access new domains
Overview and plan

Preserve (some) susy in discrete space-time
Reproduce reliable analytic results
Use lattice to access new domains

Lattice supersymmetry

Lattice $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM)

Selected results as time permits
- Dimensionally reduced (2d) thermodynamics
- Static potential Coulomb coefficient
- Anomalous dimension of Konishi operator

Prospects and future directions
Motivation: Why lattice supersymmetry

Dualities, holography, confinement, conformality, BSM, . . .

Lattice promises non-perturbative insights from first principles

Many potential lattice susy applications. . .
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Many potential lattice susy applications. . .

- Compute Wilson loops, spectrum, scaling dimensions, etc., going beyond perturbation theory, holography, bootstrap
- New non-perturbative tests of conjectured dualities
- Predict low-energy constants from dynamical susy breaking
- Validate or refine holographic models for QCD phase diagram, non-Fermi liquids, etc.
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  going beyond perturbation theory, holography, bootstrap

- New non-perturbative tests of conjectured dualities

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- Validate or refine holographic models
  for QCD phase diagram, non-Fermi liquids, etc.

. . . relatively little exploration
Obstruction: Why not lattice supersymmetry

Supersymmetry extends 4d Poincaré symmetry by $4\mathcal{N}$ spinor generators $Q^I_\alpha$ and $\bar{Q}^I_{\dot{\alpha}}\ (I = 1, \cdots, \mathcal{N})$

Super-Poincaré algebra includes

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\delta^{IJ} \sigma^\mu_{\alpha\dot{\alpha}} P_\mu$$

$\rightarrow$ infinitesimal translations that don’t exist in discrete space-time

Consequences for lattice calculations

Explicitly broken supersymmetry $\implies$ relevant susy-violating operators

Typically many such operators, especially with scalar fields

Fine-tuning to recover supersymmetric continuum limit generally not practical in numerical lattice calculations
Solution: Exact supersymmetry on the lattice

If $2^d$ supersymmetries in $d$ dimensions, can preserve susy sub-algebra at non-zero lattice spacing

⇒ Correct continuum limit with little or no fine tuning

Equivalent constructions

from ‘topological’ twisting and dimensional deconstruction

$d = 4$ picks out maximally supersymmetric Yang–Mills ($\mathcal{N} = 4$ SYM)
\( \mathcal{N} = 4 \) SYM — the fruit fly of QFT

Widely used to develop continuum QFT tools & techniques, from scattering amplitudes to holography

Arguably simplest non-trivial 4d field theory

SU\((\mathcal{N})\) gauge theory with four fermions \( \psi^I \) and six scalars \( \Phi^{IJ} \), all massless and in adjoint rep.

Action consists of kinetic, Yukawa and four-scalar terms with coefficients related by symmetries

Maximal 16 supersymmetries \( Q^I_\alpha \) and \( \bar{Q}^I_{\dot{\alpha}} \) \((I = 1, \cdots, 4)\) transform under global \( SU(4) \sim SO(6) \) R symmetry

Conformal: \( \beta \) function is zero for any 't Hooft coupling \( \lambda = g^2 N \)
Topological twisting for $\mathcal{N} = 4$ SYM

Intuitive picture — expand $4 \times 4$ matrix of supersymmetries

$$
\begin{pmatrix}
Q_1^\alpha & Q_2^\alpha & Q_3^\alpha & Q_4^\alpha \\
\bar{Q}_1^{\dot{\alpha}} & \bar{Q}_2^{\dot{\alpha}} & \bar{Q}_3^{\dot{\alpha}} & \bar{Q}_4^{\dot{\alpha}}
\end{pmatrix}
= Q + Q_\mu \gamma_\mu + Q_{\mu\nu} \gamma_\mu \gamma_\nu + \bar{Q}_\mu \gamma_\mu \gamma_5 + \bar{Q} \gamma_5
$$

$$
\longrightarrow Q + Q_a \gamma a + Q_{ab} \gamma a \gamma b
$$

with $a, b = 1, \ldots, 5$

Kähler–Dirac multiplet of ‘twisted’ supersymmetries $Q$

transform with integer spin under ‘twisted rotation group’

$$
\text{SO}(4)_{tw} \equiv \text{diag}\left[\text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R\right]
$$

$$
\text{SO}(4)_R \subset \text{SO}(6)_R
$$

Change of variables $\longrightarrow$ closed subalgebra $\{Q, Q\} = 2Q^2 = 0$

that can be exactly preserved on the lattice
Susy subalgebra from twisted $\mathcal{N} = 4$ SYM

Fields & $Q$s transform with integer spin under $\text{SO}(4)_{tw}$ — no spinors

\[
\begin{align*}
Q_\alpha \text{ and } \bar{Q}_{\dot{\alpha}} & \longrightarrow Q, \ Q_a \text{ and } Q_{ab} \\
\psi \text{ and } \bar{\psi} & \longrightarrow \eta, \ \psi_a \text{ and } \chi_{ab} \\
A_\mu \text{ and } \Phi^I & \longrightarrow \text{complexified gauge field } A_a \text{ and } \bar{A}_a \\
& \longrightarrow U(N) = \text{SU}(N) \otimes U(1) \text{ gauge theory}
\end{align*}
\]

Schematically, under $\text{SO}(d)_{tw} = \text{diag}[\text{SO}(d)_{\text{euc}} \otimes \text{SO}(d)_{\text{R}}]$

\[
\begin{align*}
A_\mu & \sim \text{vector} \otimes \text{scalar} \longrightarrow \text{vector} \\
\Phi^I & \sim \text{scalar} \otimes \text{vector} \longrightarrow \text{vector}
\end{align*}
\]

Easiest to see by dimensionally reducing from 5d

\[
A_a = A_a + i\Phi_a \longrightarrow (A_\mu, \phi) + i(\Phi_\mu, \bar{\phi})
\]
Susy subalgebra from twisted $\mathcal{N} = 4$ SYM

Fields & $Q$s transform with integer spin under $SO(4)_{tw} —$ no spinors

$Q_\alpha$ and $\bar{Q}_{\dot{\alpha}} \longrightarrow Q, Q_a$ and $Q_{ab}$

$\psi$ and $\bar{\psi} \longrightarrow \eta, \psi_a$ and $\chi_{ab}$

$A_\mu$ and $\Phi^I \longrightarrow$ complexified gauge field $A_a$ and $\bar{A}_a$

$\longrightarrow U(N) = SU(N) \otimes U(1)$ gauge theory

Twisted-scalar supersymmetry $Q$

correctly interchanges bosonic $\longleftrightarrow$ fermionic d.o.f. with $Q^2 = 0$

$Q A_a = \psi_a$  \hspace{1cm} $Q \psi_a = 0$

$Q \chi_{ab} = -\bar{F}_{ab}$  \hspace{1cm} $Q \bar{A}_a = 0$

$Q \eta = d$  \hspace{1cm} $Q d = 0$

bosonic auxiliary field with e.o.m. $d = \bar{D}_a A_a$
Lattice $\mathcal{N} = 4$ SYM

Lattice theory looks nearly the same despite breaking $Q_a$ and $Q_{ab}$

Covariant derivatives $\longrightarrow$ finite difference operators

Complexified gauge fields $\mathcal{A}_a \longrightarrow$ gauge links $U_a \in \mathfrak{gl}(N, \mathbb{C})$

\[
Q \mathcal{A}_a = Q U_a = \psi_a \\
Q \chi_{ab} = -\overline{F}_{ab} \\
Q \eta = \delta
\]

(geometrically $\eta$ on sites, $\psi_a$ on links, etc.)

Susy lattice action ($QS = 0$) from $Q^2 \cdot = 0$ and Bianchi identity

\[
S = \frac{N}{4\lambda_{\text{lat}}} \operatorname{Tr} \left[ Q \left( \chi_{ab} F_{ab} + \eta \overline{D}_a U_a - \frac{1}{2} \eta \delta \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \overline{D}_c \chi_{de} \right]
\]
Five links in four dimensions $\rightarrow A_4^*$ lattice

Again easiest to dimensionally reduce from 5d, treating all five gauge links $U_a$ symmetrically.

Start with hypercubic lattice in 5d momentum space

**Symmetric** constraint $\sum_a \partial a = 0$

projects to 4d momentum space

Result is $A_4$ lattice

$\rightarrow$ dual $A_4^*$ lattice in real space
Twisted SO(4) symmetry on the $A^*_4$ lattice

Can view $A^*_4$ lattice as 4d analog of 2d triangular lattice

Basis vectors linearly dependent and non-orthogonal $\rightarrow \lambda = \lambda_{\text{lat}} / \sqrt{5}$

Preserves $S_5$ point group symmetry

$S_5$ irreps precisely match onto irreps of twisted SO(4)$_{tw}$

\[
\begin{align*}
5 &= 4 \oplus 1 : \quad \psi_a \rightarrow \psi_\mu, \quad \bar{\eta} \\
10 &= 6 \oplus 4 : \quad \chi_{ab} \rightarrow \chi_{\mu\nu}, \quad \bar{\psi}_\mu \\
\end{align*}
\]

$S_5 \rightarrow$ SO(4)$_{tw}$ in continuum limit restores $Q_a$ and $Q_{ab}$
Summary of twisted $\mathcal{N} = 4$ SYM on the $A_4^*$ lattice

- $U(N)$ gauge invariance + $Q$ + $S_5$ lattice symmetries $\rightarrow$ several significant analytic results

Moduli space preserved to all orders of lattice perturbation theory $\rightarrow$ no scalar potential induced by radiative corrections

- $\beta$ function vanishes at one loop in lattice perturbation theory

Real-space RG blocking transformations preserving $Q$ and $S_5$ $\rightarrow$ no new terms in long-distance effective action

Only one log. tuning to recover continuum $Q_a$ and $Q_{ab}$

Not quite suitable for numerical calculations

Exact zero modes and flat directions must be regulated, especially important in $U(1)$ sector
Regulating SU($N$) flat directions

\[
S = \frac{N}{4\lambda_{\text{lat}}} \left[ Q \left( \chi_{ab} F_{ab} + \eta \overline{D}_a U_a - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \overline{D}_c \chi_{de} + \mu^2 V \right]
\]

Scalar potential \( V = \sum_a \left( \frac{1}{N} \text{Tr} [U_a \overline{U}_a] - 1 \right)^2 \) lifts SU($N$) flat directions and ensures \( U_a = I_N + A_a \) in continuum limit

Softly breaks \( Q \) — all susy violations \( \propto \mu^2 \to 0 \) in continuum limit

Ward identity violations, \( \langle QO \rangle \neq 0 \), show \( Q \) breaking and restoration

Here considering

\[
Q \left[ \eta U_a \overline{U}_a \right] = dU_a \overline{U}_a - \eta \psi_a \overline{U}_a
\]
Full $\mathcal{N} = 4$ SYM lattice action

$$S = \frac{N}{4\lambda_{\text{lat}}} \left[ Q \left( \chi_{ab} F_{ab} + \downward \frac{1}{2} \eta d \right) - \frac{1}{4} \varepsilon_{abcde} \chi_{ab} \overline{D}_c \chi_{de} + \mu^2 V \right]$$

$$\eta \left\{ \overline{D}_a U_a + G \sum_{a<b} \left[ \text{det} \mathcal{P}_{ab} - 1 \right] \mathbb{I}_N \right\}$$

Modify e.o.m. for $d$ to constrain **plaquette determinant**

$\rightarrow$ lifts $U(1)$ zero mode & flat directions without susy breaking

Much better than adding another soft $Q$-breaking term

$$\langle QO \rangle \propto (a/L)^2$$

effective ‘$O(a)$ improvement’, since $Q$ forbids all dim-5 operators
so that the full improved action becomes

\[
S_{\text{imp}} = S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \tag{3.10}
\]

\[
S'_{\text{exact}} = \frac{N}{2\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ -\mathcal{F}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi_{b]}(n) - \eta(n)\mathcal{D}^{(-)}_a\psi_a(n) \right.

\]

\[
\left. + \frac{1}{2} \left( \mathcal{D}^{(-)}_a\mathcal{U}_a(n) + G \sum_{a \neq b} (\det \mathcal{P}_{ab}(n) - 1) I_N \right) \right] - S_{\text{det}}
\]

\[
S_{\text{det}} = \frac{N}{2\lambda_{\text{lat}}} G \sum_n \text{Tr} [\eta(n)] \sum_{a \neq b} [\det \mathcal{P}_{ab}(n)] \text{Tr} [\mathcal{U}^{-1}_b(n)\psi_b(n) + \mathcal{U}^{-1}_a(n + \hat{\mu}_b)\psi_a(n + \hat{\mu}_b)]
\]

\[
S_{\text{closed}} = -\frac{N}{8\lambda_{\text{lat}}} \sum_n \text{Tr} [\epsilon_{abcde} \chi_{de}(n + \hat{\mu}_a + \hat{\mu}_b + \hat{\mu}_c)\mathcal{D}^{(-)}_a\chi_{ab}(n)]
\]

\[
S'_{\text{soft}} = \frac{N}{2\lambda_{\text{lat}}} \mu^2 \sum_n \sum_a \left( \frac{1}{N} \text{Tr} [\mathcal{U}_a(n)\mathcal{U}_a(n)] - 1 \right)^2
\]

The full \( \mathcal{N} = 4 \) SYM lattice action is somewhat complicated

(\( \gtrsim 100 \) inter-node data transfers in the fermion operator)

To reduce barriers to entry our parallel code is publicly developed at

[github.com/daschaich/susy](https://github.com/daschaich/susy)

Application I: Thermodynamics on a 2-torus

Improve arXiv:1008.4964 with new parallel code

Dimensionally reduce to 2d $\mathcal{N} = (8, 8)$ SYM with four scalar $Q$, study low temperatures $t = 1/r_\beta \leftrightarrow$ black holes in dual supergravity

For decreasing $r_L$ at large $N$

homogeneous black string (D1) $\longrightarrow$ localized black hole (D0)

“spatial deconfinement” signalled by Wilson line $P_L$
$\mathcal{N} = (8, 8)$ SYM lattice phase diagram results

Fix aspect ratio $\alpha = r_L/r_\beta$, scan in $r_\beta = r_L/\alpha = \beta\sqrt{\lambda}$

Clear transition in Wilson line and its susceptibility

Lower-temperature transitions at smaller $\alpha < 1 \rightarrow$ larger errors

Results consistent with holography and high-temp. bosonic QM
Dual black hole thermodynamics

Holography predicts bosonic action for corresponding dual black holes

\[ \propto t^3 \text{ for large-} r_L \text{ D1 phase} \]

\[ \propto t^{3.2} \text{ for small-} r_L \text{ D0 phase} \]

Lattice results consistent with holography for sufficiently low \( t \lesssim 0.4 \)

Need larger \( N > 16 \) to avoid instabilities at lower temperatures
Application II: Static potential $V(r)$

Static probes $\rightarrow r \times T$ Wilson loops $W(r, T) \propto e^{-V(r)T}$

Coulomb gauge trick reduces $A_4^*$ lattice complications
Static potential is Coulombic at all $\lambda$

Fits to confining $V(r) = A - C/r + \sigma r \rightarrow$ vanishing string tension $\sigma$

$\rightarrow$ Fit to just $V(r) = A - C/r$ to extract Coulomb coefficient $C(\lambda)$

Recent progress: Incorporating tree-level improvement into analysis
Coupling dependence of Coulomb coefficient

Continuum perturbation theory predicts \[ C(\lambda) = \frac{\lambda}{4\pi} + O(\lambda^2) \]

Holography predicts \[ C(\lambda) \propto \sqrt{\lambda} \] for \( N \to \infty \) and \( \lambda \to \infty \) with \( \lambda \ll N \)

Surprisingly good agreement with perturbation theory for \( \lambda_{\text{lat}} \leq 4 \)
Application III: Konishi operator scaling dimension

Conformality $\longrightarrow$ spectrum of scaling dimensions $\Delta(\lambda)$

govern power-law decays of correlation functions

Konishi is simplest conformal primary operator

\[ \mathcal{O}_K(x) = \sum_I \text{Tr} [\Phi^I(x)\Phi^I(x)] \]

\[ C_K(r) \equiv \mathcal{O}_K(x + r)\mathcal{O}_K(x) \propto r^{-2\Delta_K} \]

Predictions for Konishi scaling dimension $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$

- From weak-coupling perturbation theory, related to strong coupling by $\frac{4\pi N}{\lambda} \longleftrightarrow \frac{\lambda}{4\pi N}$ S duality
- From holography for $N \to \infty$ and $\lambda \to \infty$ with $\lambda \ll N$
- Upper bounds from conformal bootstrap

Only lattice gauge theory can access nonperturbative $\lambda$ at moderate $N$
Konishi operator on the lattice

Scalar fields $\varphi(n)$ from polar decomposition of complexified links

$$U_a(n) \rightarrow e^{\varphi_a(n)} U_a(n) \quad \mathcal{O}_{K}^{\text{lat}}(n) = \sum_a \text{Tr} [\varphi_a(n) \varphi_a(n)] - \text{vev}$$

Also looking at ‘SUGRA’ (20’)

$$\mathcal{O}_S \sim \varphi_a \varphi_b \text{ with protected } \Delta_S = 2$$

Challenging systematics from directly fitting power-law decay

Better lattice tools to find $\Delta$:

Finite-size scaling

Monte Carlo RG

Need lattice RG blocking transformation to carry out MCRG…
Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve $Q$ and $S_5$ symmetries $\leftrightarrow$ geometric structure

Simple transformation constructed in arXiv:1408.7067

$$
\mathcal{U}'_a(n') = \xi \mathcal{U}_a(n) \mathcal{U}_a(n + \hat{\mu}_a) \quad \quad \eta'(n') = \eta(n)
$$

$$
\psi'_a(n') = \xi [\psi_a(n) \mathcal{U}_a(n + \hat{\mu}_a) + \mathcal{U}_a(n) \psi_a(n + \hat{\mu}_a)]
$$

Doubles lattice spacing $a \rightarrow a' = 2a$, with $\xi$ a tunable rescaling factor

Scalar fields from polar decomposition $\mathcal{U}(n) = e^{\varphi(n)} U(n)$

are shifted, $\varphi \rightarrow \varphi + \log \xi$, since blocked $U$ must remain unitary

$Q$-preserving RG blocking needed

to show only one log. tuning to recover continuum $Q_a$ and $Q_{ab}$
Scaling dimensions from MCRG stability matrix

System as (infinite) sum of operators

\[ H = \sum_i c_i \mathcal{O}_i \]

Couplings \( c_i \) flow under RG blocking \( R_b \)

\( n \)-times-blocked system

\[ H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)} \]

Fixed point defined by \( H^* = R_b H^* \) with couplings \( c_i^* \)

Linear expansion around fixed point defines stability matrix \( T_{ij}^* \)

\[
c_i^{(n)} - c_i^* = \sum_k \left. \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \right|_{H^*} \left( c_k^{(n-1)} - c_k^* \right) \equiv \sum_j T_{ik}^* \left( c_k^{(n-1)} - c_k^* \right)
\]

Correlators of \( \mathcal{O}_i, \mathcal{O}_k \rightarrow \) elements of stability matrix \[ \text{[Swendsen, 1979]} \]

Eigenvalues of \( T_{ik}^* \rightarrow \) scaling dimensions of corresponding operators
Preliminary $\Delta_K$ results from Monte Carlo RG

MCRG stability matrix includes both $O_{K}^{\text{lat}}$ and $O_{S}^{\text{lat}}$

Impose protected $\Delta_S = 2$

Systematic uncertainties from different amounts of smearing

Complication: Twisted $\text{SO}(4)_{tw}$ involves only $\text{SO}(4)_R \subset \text{SO}(6)_R$

$\implies$ Lattice Konishi operator mixes with $\text{SO}(4)_R$-singlet part of the $\text{SO}(6)_R$-nonsinglet SUGRA operator

Working on variational analyses to disentangle operators
Recapitulation and outlook

- Lattice promises non-perturbative insights from first principles
- Lattice $\mathcal{N} = 4$ SYM is practical thanks to exact $Q$ susy
- Public code to reduce barriers to entry

Significant progress toward goals of lattice investigations

- 2d $\mathcal{N} = (8, 8)$ SYM thermodynamics consistent with holography
- Static potential Coulomb coefficient $C(\lambda)$ at weak coupling
- Preliminary conformal scaling dimension of Konishi operator

Many more directions are being — or can be — pursued

- Understanding the (absence of a) sign problem
- Systems with less supersymmetry, in lower dimensions, including matter fields, exhibiting spontaneous susy breaking, ...
Thank you!

Collaborators
Simon Catterall, Raghav Jha, Toby Wiseman
also Georg Bergner, Poul Damgaard, Joel Giedt, Anosh Joseph

Funding and computing resources
Add fundamental matter multiplets without breaking $Q^2 = 0$

Proposed by Matsuura [arXiv:0805.4491] and Sugino [arXiv:0807.2683],
first numerical study by Catterall & Veernala [arXiv:1505.00467]

2-slice lattice SYM
with $U(N) \times U(F)$ gauge group
Adj. fields on each slice
Bi-fundamental in between

Set $U(F)$ coupling to zero
$\rightarrow U(N)$ SQCD in $d - 1$ dims.
with $F$ fund. hypermultiplets

$U(N_c)$ SYM Adjoint Model
$[U_\mu, \mathcal{U}_\mu, (\eta, \psi_\mu, \chi_{\mu\nu})]$

Frozen (Non-dynamical)
$U(N_F)$ SYM Adjoint Model

$\phi^., \bar{\phi}$
$(\lambda, \lambda_\mu, \lambda_{\mu\nu})$

David Schaich (Bern)
Lattice MSYM
CERN, 7 June 2018 29 / 33
Dynamical susy breaking in 2d quiver superQCD

Auxiliary field e.o.m. $\rightarrow$ Fayet–Iliopoulos $D$-term potential

$$d = \overline{D}_a U_a + \sum_{i=1}^{F} \phi_i \bar{\phi}_i + r I_N \rightarrow S_D \propto \sum_{i=1}^{F} \left( \text{Tr} \left[ \phi_i \bar{\phi}_i + r I_N \right] \right)^2$$

Zero out $N$ diagonal elements via $F$ scalar vevs

or else susy breaking, $\langle Q \eta \rangle = \langle d \rangle \neq 0 \leftrightarrow \langle 0 | H | 0 \rangle > 0$

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Graphs showing $\lambda = 1.0 ; \mu = 0.3$ with data points for different lattice sizes and gauge couplings, indicating spontaneous susy breaking and noise.

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Data from arXiv:1505.00467
Supplement: Potential sign problem

Observables: \[ \langle O \rangle = \frac{1}{Z} \int [dU][d\overline{U}] \ O \ e^{-S_B[U,\overline{U}]} \ \text{pf} \ D[U,\overline{U}] \]

Pfaffian can be complex for lattice \( \mathcal{N} = 4 \) SYM, \( \text{pf} \ D = |\text{pf} \ D| e^{i\alpha} \)

Complicates interpretation of \( \{ e^{-S_B} \ \text{pf} \ D \} \) as Boltzmann weight

RHMC uses phase quenching, \( \text{pf} \ D \longrightarrow |\text{pf} \ D| \), needs reweighting

\[ \langle O \rangle = \frac{\langle O e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}} \quad \text{with} \quad \langle O e^{i\alpha} \rangle_{pq} = \frac{1}{Z_{pq}} \int [dU][d\overline{U}] \ O e^{i\alpha} e^{-S_B} |\text{pf} \ D| \]

\( \rightarrow \) Monitor \( \langle e^{i\alpha} \rangle_{pq} \) as function of volume, coupling, \( N \)
Pfaffian phase dependence on volume and coupling

**Left:** $1 - \langle \cos(\alpha) \rangle_{pq} \ll 1$ independent of volume and $N$ at $\lambda_{\text{lat}} = 1$

**Right:** Larger $\lambda_{\text{lat}} \geq 4 \rightarrow$ much larger phase fluctuations

**To do:** Analyze more volumes and $N$ with improved action

Extremely expensive $O(n^3)$ computation

$\sim 50$ hours $\times$ 16 cores for single U(2) $4^4$ measurement
Two puzzles posed by the sign problem

Periodic temporal boundary conditions for the fermions

\[ \langle e^{i\alpha} \rangle_{pq} \approx 0 \]

Anti-periodic BCs \[ e^{i\alpha} \approx 1 \], phase reweighting negligible

Why such sensitivity to the BCs?

Other \( pq \) observables are nearly identical for these two ensembles

Why doesn’t the sign problem affect other observables?
Evaluate observables from functional integral via importance sampling Monte Carlo

\[ \langle O \rangle = \frac{1}{Z} \int \mathcal{D}U \ O(U) \ e^{-S[U]} \]

\[ \rightarrow \frac{1}{N} \sum_{i=1}^{N} O(U_i) \text{ with uncert. } \propto \sqrt{\frac{1}{N}} \]

\( U \) are field configurations in discretized euclidean space-time, sampled with probability \( \propto e^{-S} \)

\( S[U] \) is lattice action, should be real and positive \( \rightarrow \frac{1}{Z} e^{-S} \) as probability distribution
Backup: More features of lattice calculations

Spacing “a” between lattice sites

$\rightarrow$ UV cutoff scale $1/a$

Removing cutoff: $a \rightarrow 0$ (with $L/a \rightarrow \infty$)

Lattice cutoff preserves hypercubic subgroup

$\rightarrow$ restore Poincaré in continuum limit

Lattice action $S$ defined by bare lagrangian at the UV cutoff $1/a$

After generating and saving ensembles $\{U_n\}$ distributed $\propto e^{-S}$

often quick and easy to measure many observables $\langle O \rangle$

Changing the action (generally) requires generating new ensembles
Backup: Hybrid Monte Carlo (HMC) algorithm

Goal: Sample field configurations $U$ with probability $\frac{1}{Z} \, e^{-S[U]}$

HMC is Markov process based on Metropolis–Rosenbluth–Teller

Fermions $\rightarrow$ extensive action computation

$\rightarrow$ Global updates using fictitious molecular dynamics

1. Introduce fictitious “MD time” $\tau$
   and stochastic canonical momenta for fields

2. Inexact MD evolution along trajectory in $\tau$ $\rightarrow$ new configuration

3. Accept/reject test on MD discretization error
Backup: Failure of Leibnitz rule in discrete space-time

\[
\{ Q_\alpha, \overline{Q}_{\dot{\alpha}} \} = 2\sigma^{\mu}_{\alpha\dot{\alpha}} P_\mu = 2i\sigma^{\mu}_{\alpha\dot{\alpha}} \partial_\mu \quad \text{is problematic}
\]

\[\rightarrow \text{try} \quad \{ Q_\alpha, \overline{Q}_{\dot{\alpha}} \} = 2i\sigma^{\mu}_{\alpha\dot{\alpha}} \nabla_\mu \quad \text{for a discrete translation}\]

\[\nabla_\mu \phi(x) = \frac{1}{a} [\phi(x + a\hat{\mu}) - \phi(x)] = \partial_\mu \phi(x) + \frac{a}{2} \partial^2_\mu \phi(x) + O(a^2)\]

**Essential difference between $\partial_\mu$ and $\nabla_\mu$ on the lattice, $a > 0$**

\[
\nabla_\mu [\phi(x)\eta(x)] = a^{-1} [\phi(x + a\hat{\mu})\eta(x + a\hat{\mu}) - \phi(x)\eta(x)]
\]

\[= [\nabla_\mu \phi(x)] \eta(x) + \phi(x)\nabla_\mu \eta(x) + a [\nabla_\mu \phi(x)] \nabla_\mu \eta(x)\]

Only recover Leibnitz rule $\partial_\mu (fg) = (\partial_\mu f)g + f\partial_\mu g$ when $a \rightarrow 0$

\[\rightarrow \text{“Discrete supersymmetry” breaks down on the lattice}\]

(Dondi & Nicolai, “Lattice Supersymmetry”, 1977)
Backup: Twisting $\leftrightarrow$ Kähler–Dirac fermions

Kähler–Dirac representation related to spinor $Q^I_\alpha$, $\overline{Q}^I_{\dot{\alpha}}$ by

$$
\begin{pmatrix}
Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\
\overline{Q}^1_{\dot{\alpha}} & \overline{Q}^2_{\dot{\alpha}} & \overline{Q}^3_{\dot{\alpha}} & \overline{Q}^4_{\dot{\alpha}}
\end{pmatrix} = Q + Q_\mu \gamma_\mu + Q_{\mu \nu} \gamma_\mu \gamma_\nu + \overline{Q}_\mu \gamma_\mu \gamma_5 + \overline{Q} \gamma_5 \\
\rightarrow Q + Q_a \gamma_a + Q_{ab} \gamma_a \gamma_b
$$

with $a, b = 1, \cdots, 5$

The $4 \times 4$ matrix involves R symmetry transformations along each row, (euclidean) Lorentz transformations along each column

$\implies$ Kähler–Dirac components transform under "twisted rotation group"

$$
SO(4)_{tw} \equiv \text{diag}
\begin{bmatrix}
SO(4)_{\text{euc}} \otimes SO(4)_R
\end{bmatrix}
$$

only $SO(4)_R \subset SO(6)_R$
In the code it is very convenient to represent the $A_4^*$ lattice as a hypercube plus one backwards diagonal link.
Backup: Restoration of $Q_a$ and $Q_{ab}$ supersymmetries

$Q_a$ and $Q_{ab}$ from restoration of R symmetry (motivation for $A_4^*$ lattice)

Modified Wilson loops test R symmetries at non-zero lattice spacing

Parameter $c_2$ may need logarithmic tuning in continuum limit

Results from arXiv:1411.0166 to be revisited with improved action
Backup: More on flat directions

Complexified links $\longrightarrow$ $U(N) = SU(N) \otimes U(1)$ gauge invariance

Supersymmetry transformation $Q U_a = \psi_a$

$\longrightarrow$ links must be in algebra with continuum limit $U_a = \mathbb{I}_N + A_a$

Flat directions in $SU(N)$ sector are physical,

those in $U(1)$ sector decouple only in continuum limit

Both must be regulated in calculations $\longrightarrow$ two deformations

Scalar potential $\propto \mu^2 \sum_a (\text{Tr} [U_a \overline{U}_a] - N)^2$ for $SU(N)$ sector

Plaquette determinant $\sim G \sum_{a<b} (\det P_{ab} - 1)$ for $U(1)$ sector

Scalar potential **softly** breaks $Q$ supersymmetry

susy-violating operators vanish as $\mu^2 \rightarrow 0$

Plaquette determinant can be made $Q$-invariant $\longrightarrow$ improved action
Backup: Problem with SU($N$) flat directions

$\mu^2/\lambda_{\text{lat}}$ too small $\rightarrow U_a$ can move far from continuum form $\mathbb{I}_N + A_a$

Example: $\mu = 0.2$ and $\lambda_{\text{lat}} = 5$ on $8^3 \times 24$ volume

Left: Bosonic action stable $\sim 18\%$ off its supersymmetric value

Right: Complexified Polyakov (‘Maldacena’) loop wanders off to $\sim 10^9$
Backup: Problem with U(1) flat directions

Monopole condensation $\rightarrow$ confined lattice phase

not present in continuum $\mathcal{N} = 4$ SYM

Around the same $\lambda_{\text{lat}} \approx 2\ldots$

**Left:** Polyakov loop falls towards zero

**Center:** Plaquette determinant falls towards zero

**Right:** Density of U(1) monopole world lines becomes non-zero
Backup: More on soft supersymmetry breaking

Until 2015 $(\det P - 1)$ was another soft susy-breaking term

$$S_{soft} = \frac{N}{4\lambda_{lat}} \mu^2 \sum_a \left( \frac{1}{N} \text{Tr} [U_a \bar{U}_a] - 1 \right)^2 + \kappa \sum_{a<b} |\det P_{ab} - 1|^2$$

Much larger $Q$-breaking effects than scalar potential

**Left:** $Q$ Ward identity from bosonic action $\langle s_B \rangle = 9N^2/2$

**Right:** Soft susy breaking suppressed $\sim 1/N^2$
Backup: Supersymmetric moduli space modification

arXiv:1505.03135 introduces method to impose $Q$-invariant constraints

Modify auxiliary field equations of motion $\longrightarrow$ moduli space

\[
d(n) = \overline{D}^a_a(n) U_a(n) \quad \longrightarrow \quad d(n) = \overline{D}^a_a(n) U_a(n) + G O(n) \Pi^N
\]

Including both plaquette determinant and scalar potential in $O(n)$ over-constrains system $\longrightarrow$ sub-optimal Ward identity violations
Backup: Code performance—weak and strong scaling

Results from arXiv:1410.6971 to be revisited with improved action

**Left:** Strong scaling for $U(2)$ and $U(3)$ $16^3 \times 32$ RHMC

**Right:** Weak scaling for $O(n^3)$ pfaffian calculation (fixed local volume)

$n \equiv 16N^2V$ is number of fermion degrees of freedom

Dashed lines are optimal scaling

Solid line is power-law fit
Backup: Numerical costs for $N = 2, 3$ and $4$ colors

**Red:** Original RHMC cost scaling $\sim N^5$ now improved to $\sim N^{3.5}$
Plot from arXiv:1410.6971 to be updated

**Blue:** Pfaffian cost scaling consistent with expected $N^6$
Backup: Dimensional reduction to $\mathcal{N} = (8, 8)$ SYM

Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_x = N_y = 1$

$A_4^*$ lattice $\longrightarrow A_2^*$ (triangular) lattice

$\longrightarrow$ Torus **skewed** depending on $\alpha$

Modular trans. into fund. domain
can make skewed torus rectangular

Also need to stabilize compactified links
to ensure broken center symmetries

arXiv:1709.07025
Backup: $\mathcal{N} = (8, 8)$ SYM Wilson line eigenvalues

Check ‘spatial deconfinement’ through histograms of Wilson line eigenvalue phases

**Left:** $\alpha = 2$ distributions more extended as $N$ increases
$\rightarrow$ dual gravity describes homogeneous black string (D1 phase)

**Right:** $\alpha = 1/2$ distributions more compact as $N$ increases
$\rightarrow$ dual gravity describes localized black hole (D0 phase)
Backup: Static potential is Coulombic at all $\lambda$

String tension $\sigma$ from fits to confining form $V(r) = A - C/r + \sigma r$

Slightly negative values flatten $V(r_I)$ for $r_I \lesssim L/2$

$\sigma \to 0$ as accessible range of $r_I$ increases on larger volumes
Backup: Discretization artifacts in static potential

Discretization artifacts visible at short distances where Coulomb term in $V(r) = A - C/r$ is most significant

Right: Highlight artifacts by extracting fluctuations around Coulomb fit

Danger of potential contamination in results for Coulomb coefficient $C$
Backup: Tree-level improvement

Classic trick to reduce discretization artifacts in static potential

(Lang & Rebbi ’82; Sommer ’93; Necco ’03)

Associate $V(r)$ data with $r$ from Fourier transform of gluon propagator

Recall

$$\frac{1}{4\pi^2 r^2} = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{e^{i r \cdot k}}{k^2}$$

where $\frac{1}{k^2} = G(k)$ in continuum

On $A^*_4$ lattice

$$\frac{1}{r^2_I} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \frac{\cos (i r_I \cdot \hat{k})}{4 \sum_{\mu=1}^{4} \sin^2 \left( \hat{k} \cdot \hat{e}_\mu / 2 \right)}$$

Tree-level perturbative lattice propagator from arXiv:1102.1725

$\hat{e}_\mu$ are $A^*_4$ lattice basis vectors

while momenta $\hat{k} = \frac{2\pi}{L} \sum_{\mu=1}^{4} n_\mu \hat{g}_\mu$ depend on dual basis vectors
Backup: Tree-level-improved static potential

Tree-level improvement significantly reduces discretization artifacts
Backup: More $\mathcal{N} = 4$ SYM static potential tests

**Left:** Projecting Wilson loops from $U(N) \rightarrow SU(N) \rightarrow$ factor of $\frac{N^2 - 1}{N^2}$

**Right:** Unitarizing links removes scalars $\rightarrow$ factor of $1/2$

Several ratios end up above expected values

Cause not clear — seems insensitive to lattice volume and $\mu$
Backup: Smearing for Konishi analyses

As for glueballs, smear to enlarge operator basis

APE-like smearing: \[ \rightarrow (1 - \alpha) \rightarrow + \frac{\alpha}{8} \sum \prod, \]

staples built from unitary parts of links but no final unitarization

(unitarized smearing — e.g. stout — doesn’t affect Konishi)

Average plaquette stable upon smearing (right)
while minimum plaquette steadily increases (left)