Overview and plan

- **Presentation goal:** Survey of recent work on maximally supersymmetric Yang–Mills (SYM) theories in $d < 4$ dimensions
- **Research goal:** Reproduce known results in perturb., holographic, etc. regimes then use lattice to access new domains

- Review lattice supersymmetry and (4d) twisted lattice $\mathcal{N} = 4$ SYM
- 1d SYM bosonic action (others’ work, 2007 through arXiv:1606.04951)
- 2d $\mathcal{N} = (8,8)$ SYM phase diagram and bosonic action (arXiv:1709.07025)
- Work in progress: 1d supersymmetric mass deformation; max-SYM in 3d and 4d

Quick review of 4d lattice $\mathcal{N} = 4$ SYM

- 16 spinor generators (‘supercharges’) $Q^A_\alpha$ and $\overline{Q}^A_{\dot{\alpha}}$ with $A = 1, \cdots, \mathcal{N}$
  \[
  \left\{ Q^A_\alpha, \overline{Q}^B_{\dot{\alpha}} \right\} = 2\delta^{AB}\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \rightarrow \text{supersymmetry algebra broken on the lattice}
  \]

- Two ways to avoid impractical fine-tuning (will use both):
  1) Work in lower dimensions where theories are super-renormalizable
  2) Preserve closed sub-algebra of supersymmetries via topological twisting

- **Topological twisting:** (introduced for curved manifolds)
  \[
  Q^A_\alpha, \overline{Q}^B_{\dot{\alpha}} \rightarrow Q, Q_\mu, Q_{\mu\nu}, \overline{Q}_\mu, \overline{Q} \text{ in integer-spin reps of “twisted rotation group”}
  \]
  \[
  \text{SO}(4)_{\text{tw}} \equiv \text{diag} \left[ \text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R \right] \quad \text{with} \quad \text{SO}(4)_R \subset \text{SO}(6)_R
  \]

- More generally, for $2 \leq d \leq 5$
  \[
  \text{SO}(d)_{\text{tw}} \equiv \text{diag} \left[ \text{SO}(d)_{\text{euc}} \otimes \text{SO}(d)_R \right]
  \]
  \[
  Q \geq 2^d \text{ supercharges} \rightarrow [Q/2^d] \geq 1 \text{ closed susy subalgebras } Q^2 = 0
  \]

- Reducing 10-dim. $\mathcal{N} = 1$ SYM to $d$ dims. \rightarrow \text{SO}(10 – d) R symmetry
  
  **Fields:** 16 fermions, $d$-component gauge field and $10 – d$ scalars,
  all massless and in adjoint rep

- For $2 \leq d \leq 4$, discretize on $A_d^*$ lattice with $d + 1$ basis vectors
  (familiar triangular lattice for $d = 2$)
“D0 brane” SYM quantum mechanics (de Wit–Hoppe–Nicolai, 1988)

- Reduced to the point that twisting both impossible and unnecessary
  Only temporal component of gauge field remains, plus 9 scalars \( X^A \)
  \[
  S_0 = \frac{N}{2\lambda} \int dt \, \text{Tr} \left[ (D_t X^A)^2 + \Psi^\alpha D_t \Psi^\alpha + \frac{1}{2} [X^A, X^B]^2 + i \Psi^\alpha \gamma^A_{\alpha\beta} [\Psi^\beta, X^A] \right]
  \]
  with \( A, B = 1, \cdots, 9 \) and \( \alpha, \beta = 1, \cdots, 16 \)

- Finite-temperature system holographically dual to stringy black hole geometry
- Temperature and dimension-3 \( \text{ motives coupling} \rightarrow \text{ dim'less } T = T_{\text{dim}}/\lambda^{1/3} \equiv 1/r_\beta \)
- Low \( T \ll 1 \) and large number of colors \( N \rightarrow \text{ classical supergravity (SUGRA)} \)
  Large \( N \) suppresses string quantum \( (g_s) \) corrections
  Low temperatures (large \( \lambda \)) suppress \( \alpha' \) corrections (string size \( \propto \sqrt{\alpha'} \))
- **Numerical state of the art:** gauge groups SU(16)–SU(32) with \( L \) up to 32
- Investigate dual black hole internal energy \( \leftrightarrow \) SYM bosonic action
- **Fits to SUGRA prediction** \[ E/N^2 = a_0 T^{2.8} + a_1 T^{4.6} + a_2 T^{5.8} + \ldots, \text{ with } a_0 = 7.41 \]
  reproduce \( a_0 = 7.4(5) \) and predict unknown \( a_1 = -10.0(4), \quad a_2 = 5.8(5) \)
Two-dimensional $\mathcal{N} = (8,8)$ SYM

- Naive dimensional reduction of twisted 4d $\mathcal{N} = 4$ SYM

$$S = \frac{N}{4\lambda} \mathcal{Q} \int d^4x \ Tr \left[ \chi_{ab} F_{ab} + \eta [\mathcal{D}_a, \mathcal{D}_d] - \frac{1}{2} \eta d + \epsilon_{abcde} \mathcal{D}_c \chi_{ab} \right]$$

$a, b = 1, \cdots, 5$ now include ‘flavor’ and three $A_2^*$ basis vectors

- Space-time is torus with $r_\beta = 1/t = \beta \sqrt{\lambda}$, $r_L = L \sqrt{\lambda}$ and aspect ratio $\alpha = r_L/r_\beta$ → more complicated phase diagram

- **Solid expectations** for phase transitions at both high and low temperatures

  Same sort of transition in each limit, with different dependence on $r_\beta$ vs. $r_L$

  (All phases still thermally deconfined $\leftrightarrow$ dual stringy black holes)

- **High temperatures**: Bosonic quantum mechanics transition

  Wilson line order parameter $W_L = \frac{1}{N} \left| \langle \text{Tr} \left[ \mathcal{P} e^{i \oint L A} \right] \rangle \right|$ around spatial circle

  $W_L = 0$ at large $r_L$ (‘spatial confinement’) $\rightarrow$ $W_L \neq 0$ (‘deconf.’) at small $r_L$

  (Order of transition debated:
   first-order vs. strong second-order plus nearby Gross–Witten–Wadia)

- **Low temperatures**: Large-$N$ classical SUGRA transition

  Large-$r_L$ homogeneous D1 ‘black strings’ with horizon $\mathbb{R} \times S^7$

  $\rightarrow$ small-$r_L$ D0 black holes with horizon $S^8$ localized on spatial circle

  (Radial direction $U$ and time fill out 10 dimensions in total)

  Type IIB SUGRA has winding mode instability at small $r_L \lesssim c_{GL} r_\beta^2$,

  related to Type IIA classical Gregory–Laflamme transition by T duality

- Non-orthogonal basis vectors of triangular lattice

  $\rightarrow$ **skewed** tori, “generalized” thermal ensemble

- **Restricted** $\text{SL}(2,\mathbb{Z})$ modular transformations describe same torus geometry

  despite sometimes changing skewed $\rightarrow$ rectangular

  $$\left( \begin{array}{c} \bar{L}' \\ \bar{\beta}' \end{array} \right) = M \cdot \left( \begin{array}{c} \bar{L} \\ \bar{\beta} \end{array} \right), \quad M = \left( \begin{array}{cc} a & 2n \\ c & 2m - 1 \end{array} \right) \in \text{SL}(2,\mathbb{Z})$$

  with $n, m, c \in \mathbb{Z}$ $\rightarrow$ $a \in 2\mathbb{Z} - 1$

- **Numerical results**: Horizon $\leftrightarrow$ distribution of Wilson line eigenvalue phases

  In D0 / spatially deconfined phase, distribution more localized as $N$ increases,

  $$E/(N^2 \lambda) \propto t^{3.2}$$ from leading-order SUGRA

  In D1 / spatially confined phase, distribution more uniform as $N$ increases,

  $$E/(N^2 \lambda) \propto t^3$$ from leading-order SUGRA

- Large-$N$ continuum extrapolations remain to be done in this case
Deformed quantum mechanics (Berenstein–Maldacena–Nastase, ’02)

- Dim’l reduction of 10d plane-wave background preserving all 16 supersymmetries
  \[ S = S_0 - \delta S \]
  \[ \delta S = N 2\frac{\mu}{\lambda} \int dt \text{Tr} \left[ \frac{\mu^2}{3}(X^i)^2 + \frac{\mu^2}{6}(X^a)^2 + \frac{\mu}{24} \Psi^\alpha \epsilon_{ijk} (\gamma^i \gamma^j \gamma^k)_{\alpha\beta} \Psi^\beta + i \frac{2\mu}{3} \epsilon_{ijk} X^i X^j X^k \right] \]
  with \( i, j, k = 1, 2, 3 \) and \( a = 4, \cdots, 9 \)
  \( \rightarrow \) dim’ful \( \mu \neq 0 \) breaks SO(9) R symmetry to SO(3)×SO(6)

- Deformation lifts moduli space \( \rightarrow \) discrete set of vacua
  Can also regulate low-\( t \) instability, though this may need small \( \mu \sim 1/N \)

- Now have non-trivial phase instability in plane of \( T/\mu \) vs. dim’less \( g = \lambda/\mu^3 \)
  Can consider strong coupling at both large and small \( T/\mu \)

- Phase diagram / transition “qualitatively similar” to 2d \( \mathcal{N} = (8, 8) \), [arXiv:1411.5541]
  although lose thermal deconfinement \( \rightarrow \) energy scales \( \propto N^0 \) rather than \( N^2 \)
  (no dual black holes?)

- First-order Hagedorn transition at \( g = 0 \)
  Hawking–Page-like transition as \( g \to \infty \)

Higher dimensions, \( d = 3 \) and 4

- Empirically, larger \( d \) allow low-temperature stability with smaller \( N \)
  Compare \( 16 \leq N \) for quantum mechanics vs. \( 6 \leq N \leq 16 \) for 2d \( \mathcal{N} = (8, 8) \) SYM
  Also expect small corrections \( \propto 1/N^2 \) for adjoint fermions in 4d gauge theories

- Interpret as trading d.o.f. between space-time volume and internal large \( N \)?

- Work in progress: 3d 16-supercharge SYM in uniform D2 phase, \( 4 \leq N \leq 6 \)
  Preliminary consistency with leading SUGRA prediction \( E/(N^2 \lambda^3) \propto t^{10/3} \)

- For the future: “3/4 problem” in four-dimensional \( \mathcal{N} = 4 \) (16-supercharge) SYM
  Perturbative energy \( 1 \times cN^2T^4 \) for small \( \lambda \to 0 \)
  Holographic energy \( \frac{3}{4} \times cN^2T^4 \) for large \( 1 \ll \lambda \ll N \)
This work considers gauge groups SU($N$) with $16 \leq N \leq 32$ on lattices with up to $L = 32$ sites. These two plots fix the dimensionless temperature $T = T_{\text{dim}}/\lambda^{1/3} = 0.5$ and show both individual and combined extrapolations to the limits $N^2 \to \infty$ and $L \to \infty$. The latter is the continuum limit in which the lattice UV cutoff is removed while the large-$N$ limit suppresses string quantum ($g_s$) corrections in the dual gravitational calculation.

These fits correctly reproduce the leading $a_0 = 7.41$ predicted by SUGRA, and provide lattice predictions for the unknown coefficients $a_i$ for $i \geq 1$. The two colored curves without error bands are results from earlier studies with smaller $N$ and $L$. 

Figure 2: Extrapolated lattice results for the dual black hole internal energy of D0 brane quantum mechanics, from arXiv:1606.04951, using combined $N^2 \to \infty$ and $L \to \infty$ extrapolations like the one shown in Fig. 1 for $T = 0.5$. The results are consistently below the leading-order SUGRA prediction (solid black line), but can be fit to expressions including subleading corrections (three colored curves with error bands).
Figure 3: Schematic phase diagram for two-dimensional $\mathcal{N} = (8,8)$ SYM on an $r_\beta \times r_L$ torus, from arXiv:1709.07025, showing the two limits where first-order transitions are expected. At high temperatures (small $r_\beta = 1/t$) the system reduces to a simple one-dimensional bosonic quantum mechanics (BQM) with a first-order deconfinement transition at small $r_L$. A similar first-order deconfinement transition is predicted by holography at low temperatures (in the large-$N$ limit), with the large-$r_L$ homogeneous black string (D1) phase becoming unstable and collapsing to a localized black hole (D0) phase as $r_L$ decreases.
Figure 4: The complex modular parameters $\tau = \alpha \gamma + i \alpha \sqrt{1 - \gamma^2}$ for skewed tori with skewing parameter $\gamma = -1/2$ and different aspect ratios $\alpha$ given by the labels on the red points, from arXiv:1709.07025. When $\tau$ falls outside the shaded fundamental domain, a restricted SL(2, $\mathbb{Z}$) modular transformation gives the equivalent $\tau'$ in the fundamental domain (blue points). This reveals a few cases ($\alpha = 1/2, 4$ and 8) for which the fundamental representation of the torus geometry is rectangular, $\text{Re}(\tau') = 0$. 
Figure 5: Numerical lattice results for two-dimensional $\mathcal{N} = (8,8)$ SYM, from arXiv:1709.07025. **Top:** Representative signals for the ‘spatial deconfinement’ transition in the spatial Wilson line (left) and its susceptibility (right), for fixed aspect ratio $\alpha = L/N_t = 4$. In the deconfined small-$r_L$ phase at small $r_\beta = r_L/\alpha = 1/t$, the Wilson line is large and independent of $N$, while it vanishes in the large-$N$ limit at large $r$, with a clear peak in the susceptibility at the transition between these two phases. **Center:** The resulting predictions for phase transitions for various $\alpha$, compared to the expected asymptotic behavior from Fig. 3. There is good agreement at high temperatures, and reasonable consistency at lower temperatures. **Bottom:** The dual black hole internal energy, as in Fig. 2 but without large-$N$ or continuum extrapolations. The $\alpha = 1/2$ data are consistent with the leading gravitational prediction $E/(N^2\lambda) \propto t^{3/2}$ for the D0 phase (left), while those for $\alpha = 2$ are consistent with $E/(N^2\lambda) \propto t^3$ for the D1 phase (right).
Figure 6: The distributions of Wilson line eigenvalue phases for two-dimensional $\mathcal{N} = (8,8)$ SYM (from arXiv:1709.07025) indicate which side of the transition we are on for a given temperature $t = 1/r_\beta$ and aspect ratio $\alpha = L/N_t$. For $\alpha = 1/2$ and $t \approx 0.46$ (left), the distributions become more localized as $N$ increases, corresponding to the D0 phase (and the highest $\otimes$ in the central plot of Fig. 5). For $\alpha = 2$ and $t \approx 0.3$, the distributions become more uniform as $N$ increases, corresponding to the D1 phase (far beyond the right edge of the central plot in Fig. 5).

Figure 7: Preliminary lattice results for the dual black hole internal energy of three-dimensional 16-supercharge SYM in the uniform D2 phase, as in the bottom plots of Fig. 5. Despite the smaller $N \leq 6$ the results are quite close to the leading gravitational prediction $E/(N^2\lambda^3) \propto t^{10/3}$.