Maximally supersymmetric Yang–Mills on the lattice

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arXiv:1505.03135 arXiv:1512.01137 arXiv:1611.06561
& more to come with Simon Catterall, Poul Damgaard and Joel Giedt

Plan

- Motivations for lattice supersymmetry in general
  [focusing on four-dimensional gauge theories]
- Lattice formulation of $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM)
  [new improvement procedure & public code]
- Latest results for static potential and Konishi anomalous dim.
  [confront with perturbation theory, AdS/CFT, bootstrap]
- Prospects and future directions
  [sign problem; lattice superQCD in two & three dimensions]

Motivation: Why lattice supersymmetry

A lot of interesting physics in 4d susy gauge theories:

dualities, holography, confinement, conformality, BSM, ...

Lattice promises non-perturbative insights from first principles

We can brainstorm many potential lattice susy applications:

- Compute Wilson loops, spectrum, scaling dimensions, etc.,
  going beyond perturbation theory, holography, bootstrap, ...
- New non-perturbative tests of conjectured dualities
- Predict low-energy constants from dynamical susy breaking
- Validate or refine AdS/CFT-based models
  for QCD phase diagram, condensed matter systems, ...

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Many ideas probably infeasible; relatively few have been explored

Obstruction: Why not lattice supersymmetry

Recall supersymmetry extends Poincaré symmetry
by spinorial generators Q^{I}_{α} and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ with $\mathrm{I} = 1, \cdots, \mathcal{N}$

The super-Poincaré algebra includes $\left\{ Q^{\mathrm{I}}_{\alpha}, \overline{Q}^{\mathrm{J}}_{\dot{\alpha}} \right\} = 2\delta^{\mathrm{IJ}} \sigma^{\mu}_{\alpha\dot{\alpha}} P_{\mu}$
but infinitesimal translations don't exist in discrete space-time

Consequences for lattice calculations

Explicitly broken supersymmetry $\Longrightarrow$ relevant susy-violating operators

Typically many such operators,
especially with scalar fields from matter multiplets or from $\mathcal{N} > 1$

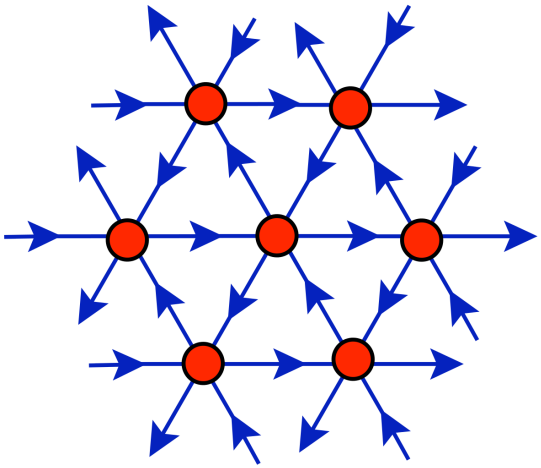
Fine-tuning couplings / counterterms to restore supersymmetry
is generally not practical in numerical lattice calculations

Solution: Exact supersymmetry on the lattice

In certain systems some subset of the susy algebra
can be exactly preserved at non-zero lattice spacing

$\Longrightarrow$ Recover rest of susy in continuum limit with little or no fine tuning

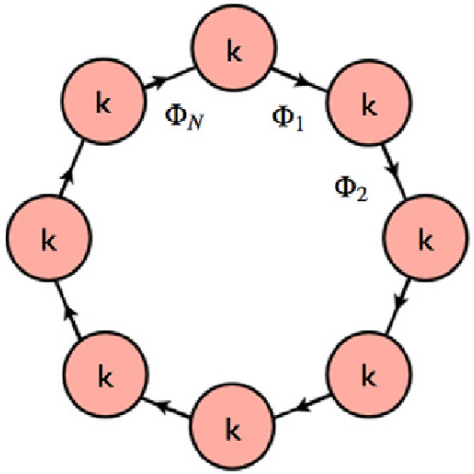
Equivalent constructions obtained from 'topological' twisting
and from orbifolding / dimensional deconstruction



For review see

Catterall, Kaplan & Ünsal

arXiv:0903.4881



In four dimensions these constructions pick out a unique theory:
maximally supersymmetric Yang–Mills ($\mathcal{N} = 4$ SYM)

Basic features of continuum $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM is a particularly interesting theory

Widely used to develop continuum QFT tools & techniques,
from scattering amplitudes to the AdS / CFT correspondence

Arguably simplest non-trivial field theory in four dimensions

- SU(N) gauge theory with four fermions Ψ^{I} and six scalars Φ^{IJ},
all massless and in adjoint rep.
- Action consists of kinetic, Yukawa and four-scalar terms
with coefficients related by symmetries
- Supersymmetric: 16 supercharges Q^{I}_{α} and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ with $\mathrm{I} = 1, \cdots, 4$
Fields and Qs transform under global SU(4) $\simeq$ SO(6) R symmetry
- Conformal: β function is zero for any 't Hooft coupling $\lambda = g^2 N$

Topological twisting for $\mathcal{N} = 4$ SYM

An intuitive picture — expand 4×4 matrix of supercharges

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu}\gamma_\mu\gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}}\gamma_5$$

$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab}\gamma_a\gamma_b$$

with $a, b = 1, \cdots, 5$

'Twisted' supercharges $\mathcal{Q}$ are components of Kähler–Dirac multiplet, transform with **integer spin** under 'twisted rotation group'

$$\mathrm{SO}(4)_{tw} \equiv \mathrm{diag}\left[\mathrm{SO}(4)_{\mathrm{euc}} \otimes \mathrm{SO}(4)_R\right] \qquad \mathrm{SO}(4)_R \subset \mathrm{SO}(6)_R$$

This change of variables gives a susy subalgebra $\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$

This subalgebra can be exactly preserved on the lattice

Susy subalgebra from twisted $\mathcal{N} = 4$ SYM

Fields & $\mathcal{Q}$s transform with **integer spin** under $\mathrm{SO}(4)_{tw}$ — **no spinors**

$$Q_\alpha \text{ and } \overline{Q}_{\dot\alpha} \longrightarrow \mathcal{Q},\ \mathcal{Q}_a \text{ and } \mathcal{Q}_{ab}$$
$$\Psi \text{ and } \overline{\Psi} \longrightarrow \eta,\ \psi_a \text{ and } \chi_{ab}$$
$$A_\mu \text{ and } \Phi^{\mathrm{I}} \longrightarrow \text{complexified gauge field } \mathcal{A}_a \text{ and } \overline{\mathcal{A}}_a$$

Why complexify? [$\Longrightarrow \mathrm{U}(N) = \mathrm{SU}(N) \otimes \mathrm{U}(1)$ gauge theory]

Schematically, under the twisted $\mathrm{SO}(d)_{tw} = \mathrm{diag}\big[\mathrm{SO}(d)_{\mathrm{euc}} \otimes \mathrm{SO}(d)_R\big]$

$$A_\mu \sim \text{vector} \otimes \text{scalar} \longrightarrow \text{vector}$$
$$\Phi^{\mathrm{I}} \sim \text{scalar} \otimes \text{vector} \longrightarrow \text{vector}$$

Easiest to see by dimensionally reducing from 5d,

$$\mathcal{A}_a = A_a + i\Phi_a \longrightarrow (A_\mu, \phi) + i(\Phi_\mu, \overline{\phi})$$

Susy subalgebra from twisted $\mathcal{N} = 4$ SYM

Fields & $\mathcal{Q}$s transform with **integer spin** under SO(4)$_{tw}$ — **no spinors**

$$Q_\alpha \text{ and } \overline{Q}_{\dot\alpha} \longrightarrow \mathcal{Q},\ \mathcal{Q}_a \text{ and } \mathcal{Q}_{ab}$$

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$$A_\mu \text{ and } \Phi^{\mathrm{I}} \longrightarrow \text{complexified gauge field } \mathcal{A}_a \text{ and } \overline{\mathcal{A}}_a$$

Now it's easy to check that the twisted-scalar supersymmetry $\mathcal{Q}$ correctly interchanges bosonic $\longleftrightarrow$ fermionic d.o.f. with $\mathcal{Q}^2 = 0$

$$\mathcal{Q}\,\mathcal{A}_a = \psi_a \qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad \mathcal{Q}\,\overline{\mathcal{A}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad \mathcal{Q}\,d = 0$$

bosonic auxiliary field with e.o.m. $d = \overline{\mathcal{D}}_a \mathcal{A}_a$

Lattice $\mathcal{N} = 4$ SYM

The lattice theory is nearly a direct transcription,
despite breaking the 15 $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

- Covariant derivatives $\longrightarrow$ finite difference operators
- Complexified gauge fields $\mathcal{A}_a \longrightarrow$ gauge links $\mathcal{U}_a \in \mathfrak{gl}(N, \mathbb{C})$

$$\mathcal{Q}\,\mathcal{A}_a \longrightarrow \mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{A}}_a \longrightarrow \mathcal{Q}\,\overline{\mathcal{U}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad\qquad \mathcal{Q}\,d = 0$$

Geometry manifest: η and d on sites, $\mathcal{U}_a$ and ψ_a on links, etc.

- Supersymmetric lattice action ($\mathcal{Q}S = 0$)
follows from $\mathcal{Q}^2 \cdot = 0$ and Bianchi identity

$$S = \frac{N}{2\lambda_{\text{lat}}} \text{Tr}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de}\right]$$

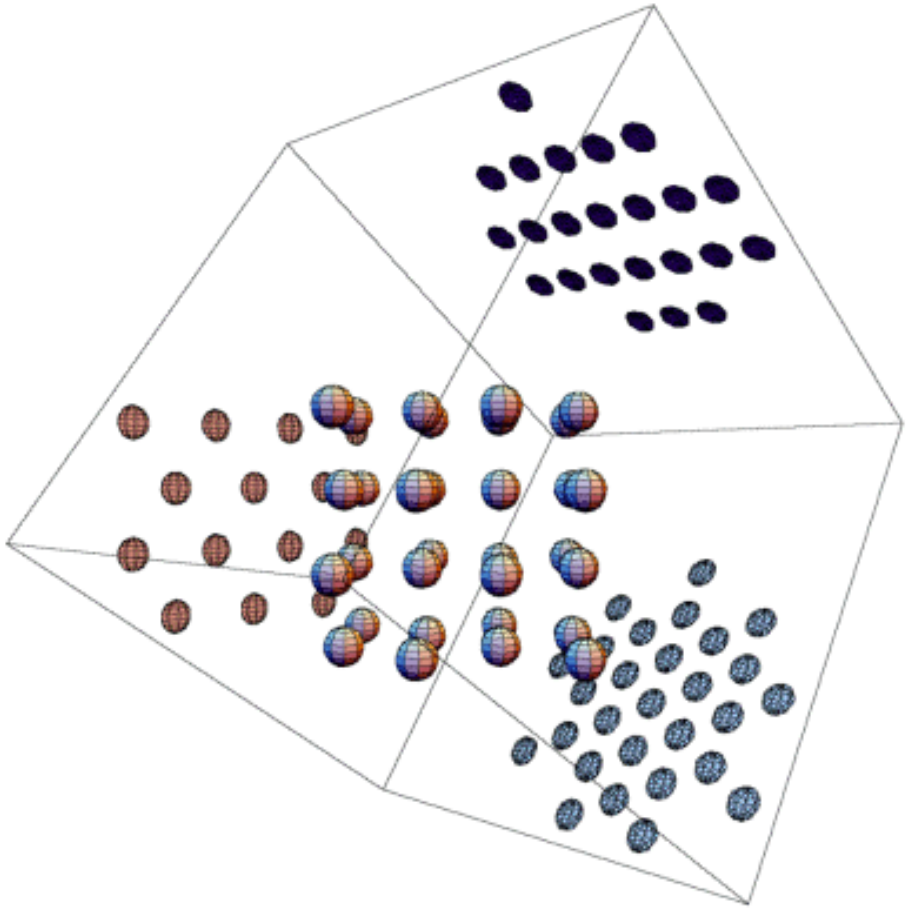
Five links in four dimensions $\longrightarrow A_4^*$ lattice

Again easiest to construct via dimensional reduction from 5d,
treating all five gauge links $\mathcal{U}_a$ symmetrically

—Start with hypercubic lattice
in 5d momentum space

—**Symmetric** constraint $\sum_a \partial_a = 0$
projects to 4d momentum space

—Result is A_4 lattice
$\longrightarrow$ dual A_4^* lattice in real space

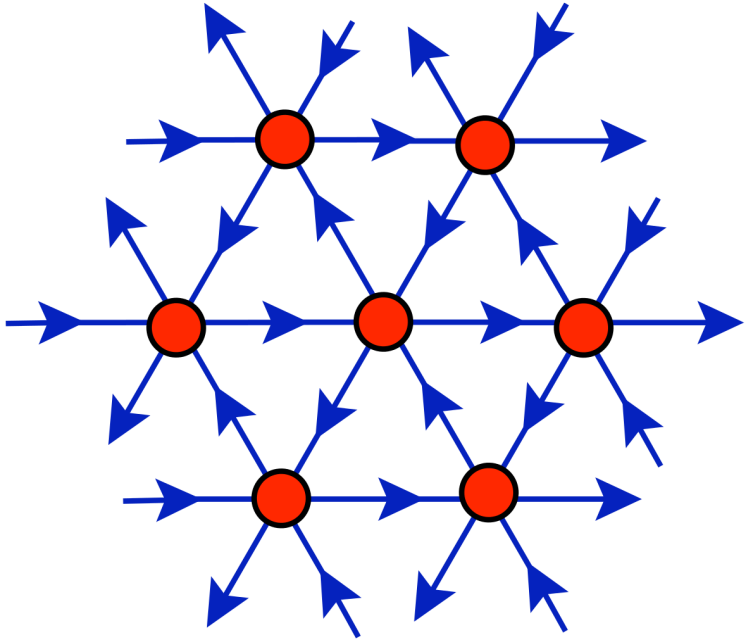


Twisted SO(4) symmetry on the A_4^* lattice

—Simplest to picture A_4^* lattice
as 4d analog of 2d triangular lattice

—Basis vectors are linearly dependent
and non-orthogonal $\longrightarrow \lambda = \lambda_{\rm lat}/\sqrt{5}$

—Preserves S_5 point group symmetry



S_5 irreps precisely match onto irreps of twisted SO(4)$_{tw}$

$$\mathbf{5} = \mathbf{4} \oplus \mathbf{1}: \quad \psi_a \longrightarrow \psi_\mu, \ \ \overline{\eta}$$

$$\mathbf{10} = \mathbf{6} \oplus \mathbf{4}: \quad \chi_{ab} \longrightarrow \chi_{\mu\nu}, \ \ \overline{\psi}_\mu$$

$S_5 \longrightarrow$ SO(4)$_{tw}$ in continuum limit restores the rest of $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Summary of twisted $\mathcal{N} = 4$ SYM on the A_4^* lattice

U(N) gauge invariance + $\mathcal{Q}$ + S_5 lattice symmetries
allow several significant analytic results:

- Moduli space preserved to all orders of lattice perturbation theory
  $\longrightarrow$ no scalar potential induced by radiative corrections
- β function vanishes at one loop in lattice perturbation theory
- Real-space RG blocking transformations preserve $\mathcal{Q}$ and S_5
  $\longrightarrow$ no new terms in long-distance effective action
- Only one log. tuning to recover $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ in the continuum

Not quite suitable for numerical calculations

Exact zero modes and flat directions must be regulated,
especially important in U(1) sector

Regulating SU(N) flat directions

$$S = \frac{N}{2\lambda_{\text{lat}}}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de} + \mu^2 V\right]$$

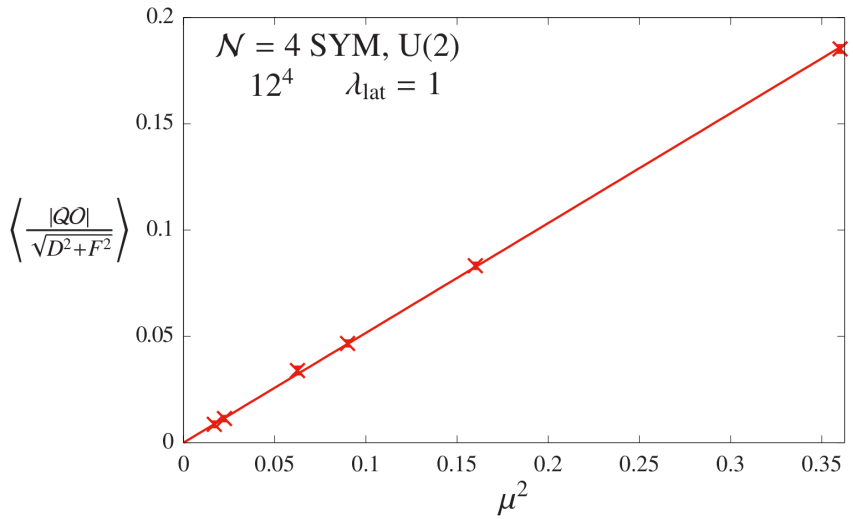
Scalar potential $V = \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2$ lifts SU(N) flat directions

and ensures $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$ in continuum limit

Breaks $\mathcal{Q}$ **softly** — susy breaking automatically vanishes as $\mu^2 \to 0$

Ward identity violations, $\langle \mathcal{Q}\mathcal{O}\rangle \neq 0$,

show $\mathcal{Q}$ breaking and restoration



Here considering

$$\mathcal{Q}\left[\eta\mathcal{U}_a\overline{\mathcal{U}}_a\right] = d\mathcal{U}_a\overline{\mathcal{U}}_a - \eta\psi_a\overline{\mathcal{U}}_a$$

Full $\mathcal{N} = 4$ SYM lattice action

arXiv:1505.03135

$$S = \frac{N}{2\lambda_{\text{lat}}}\left[\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \quad \downarrow \quad - \frac{1}{2}\eta d\right) - \frac{1}{4}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de} + \mu^2 V\right]$$

$$\eta\left(\overline{\mathcal{D}}_a\mathcal{U}_a + G\sum_{a<b}\left[\det\mathcal{P}_{ab} - 1\right]\mathbb{I}_N\right)$$

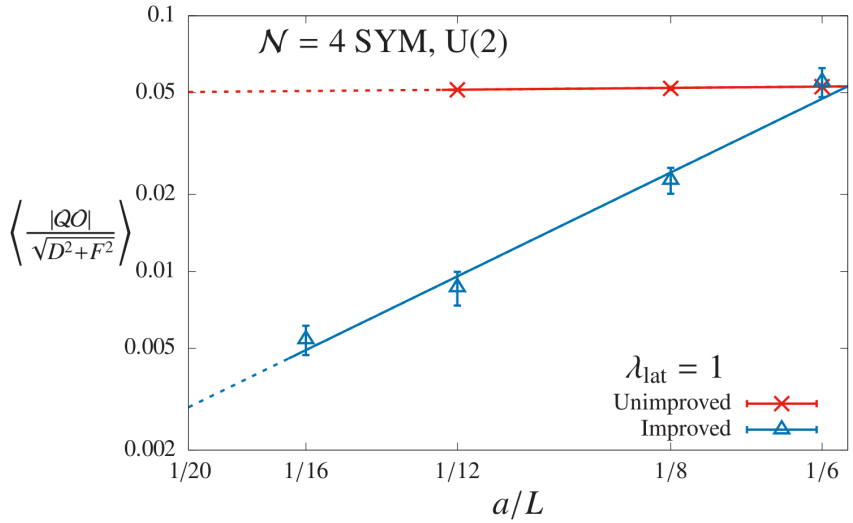
Constraint on plaquette det. lifts U(1) zero mode & flat directions

Imposed supersymmetrically as new moduli space condition

Leads to $\langle\mathcal{Q}\mathcal{O}\rangle \propto (a/L)^2$

Much better than adding $(\det\mathcal{P} - 1)$
as another soft $\mathcal{Q}$-breaking term

Effective $\mathcal{O}(a)$ improvement
since $\mathcal{Q}$ forbids all dim-5 operators



Advertisement: Public code for lattice $\mathcal{N} = 4$ SYM

so that the full improved action becomes

$$
\begin{aligned}
S_{\text{imp}} &= S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \qquad (3.10)\\
S'_{\text{exact}} &= \frac{N}{2\lambda_{\text{lat}}} \sum_n \text{Tr}\Bigg[-\overline{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi_{b]}(n) - \eta(n)\overline{\mathcal{D}}^{(-)}_{a}\psi_a(n)\\
&\qquad + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_{a}\mathcal{U}_a(n) + G\sum_{a\neq b}\left(\det \mathcal{P}_{ab}(n) - 1\right)\mathbb{I}_N\right)^2\Bigg] - S_{\text{det}}\\
S_{\text{det}} &= \frac{N}{2\lambda_{\text{lat}}} G \sum_n \text{Tr}\left[\eta(n)\right] \sum_{a\neq b}\left[\det \mathcal{P}_{ab}(n)\right] \text{Tr}\left[\mathcal{U}_b^{-1}(n)\psi_b(n) + \mathcal{U}_a^{-1}(n+\widehat{\mu}_b)\psi_a(n+\widehat{\mu}_b)\right]\\
S_{\text{closed}} &= -\frac{N}{8\lambda_{\text{lat}}} \sum_n \text{Tr}\left[\epsilon_{abcde}\ \chi_{de}(n+\widehat{\mu}_a+\widehat{\mu}_b+\widehat{\mu}_c)\overline{\mathcal{D}}^{(-)}_{c}\chi_{ab}(n)\right],\\
S'_{\text{soft}} &= \frac{N}{2\lambda_{\text{lat}}}\mu^2 \sum_n \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a(n)\overline{\mathcal{U}}_a(n)\right] - 1\right)^2
\end{aligned}
$$

The full $\mathcal{N} = 4$ SYM lattice action is somewhat complicated

(For experts: $\gtrsim$100 inter-node data transfers in the fermion operator)

To reduce barriers to entry our parallel code is publicly developed at

github.com/daschaich/susy

Evolved from MILC lattice QCD code, presented in arXiv:1410.6971

Application: Static potential

Static potential $V(r)$ from $r \times T$ Wilson loops: $W(r,T) \propto e^{-V(r)\,T}$

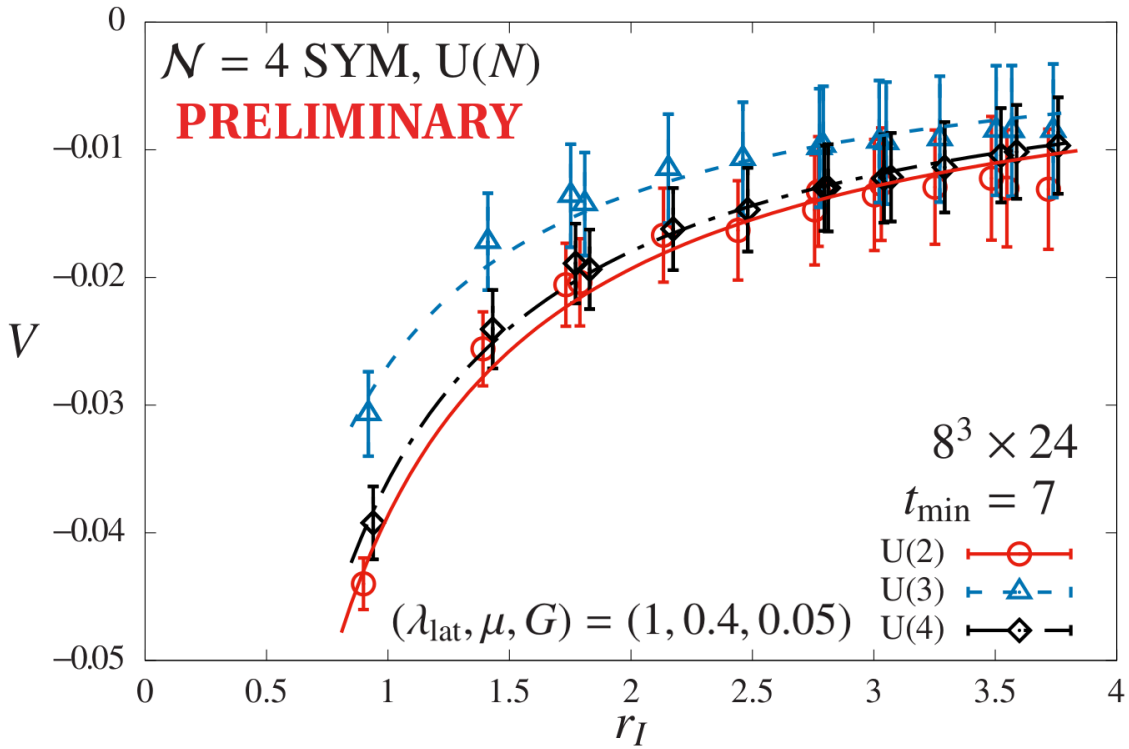
Fit $V(r)$ to Coulombic or confining form

$$V(r) = A - C/r$$

$$V(r) = A - C/r + \sigma r$$

C is Coulomb coefficient

σ is string tension



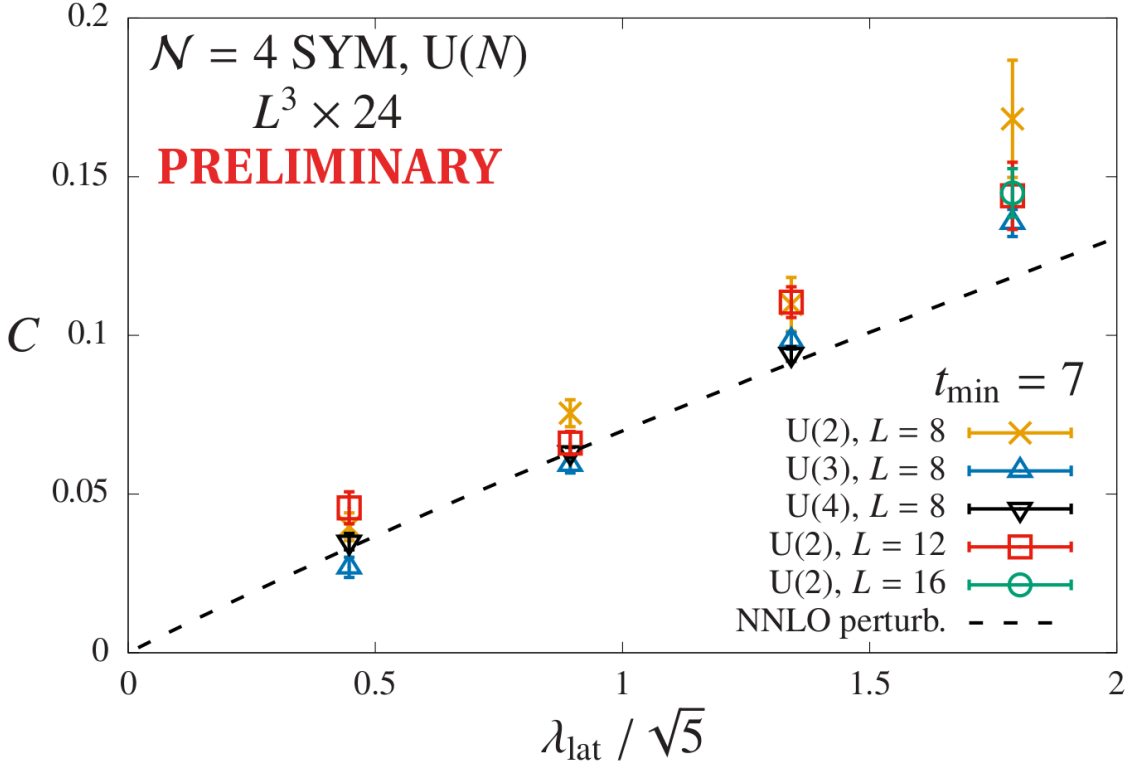
$V(r)$ is Coulombic at all λ: fits to confining form produce vanishing σ

New tree-level improved analysis reduces discretization artifacts

Coupling dependence of Coulomb coefficient

Continuum perturbation theory predicts $C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

AdS/CFT predicts $C(\lambda) \propto \sqrt{\lambda}$ for $N \to \infty,\ \lambda \to \infty,\ \lambda \ll N$



Results consistent with perturbation theory

for these relatively weak couplings $\lambda_{\rm lat} \leq 4$

Application: Konishi operator scaling dimension

$\mathcal{N} = 4$ SYM is conformal at all $\lambda \longrightarrow$ spectrum of scaling dimensions Δ that govern power-law decay of correlation functions

The Konishi operator is the simplest conformal primary operator

$$\mathcal{O}_K(x) = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}(x)\Phi^{\mathrm{I}}(x)\right] \qquad C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

There are many predictions for its scaling dim. $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$

- From weak-coupling perturbation theory, related to strong coupling by $\frac{4\pi N}{\lambda} \longleftrightarrow \frac{\lambda}{4\pi N}$ S duality
- From holography for $N \to \infty$ and $\lambda \to \infty$ but $\lambda \ll N$
- Upper bounds from the conformal bootstrap program

Only lattice gauge theory can access nonperturbative λ at moderate N

Konishi operator on the lattice

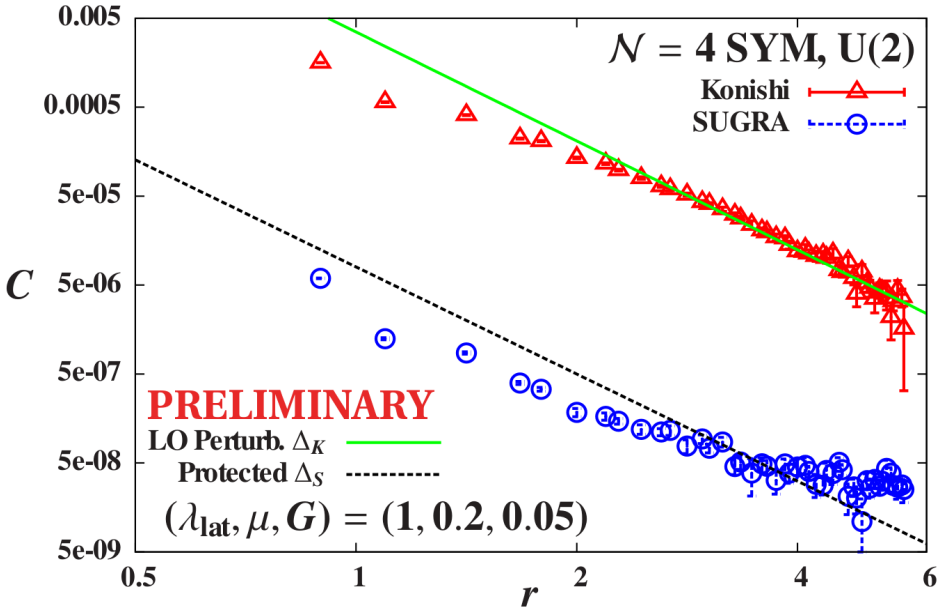
Extract scalar fields from polar decomposition of complexified links

$$\mathcal{U}_a(n) \longrightarrow e^{\varphi_a(n)} U_a(n) \qquad \mathcal{O}_K^{\text{lat}}(n) = \sum_a \text{Tr}\left[\varphi_a(n)\varphi_a(n)\right] - \text{vev}$$

Also looking at (20′) ‘SUGRA’
$\mathcal{O}_S \sim \varphi_a \varphi_b$ with protected $\Delta_S = 2$

Challenging systematics from
directly fitting power-law decay

Better lattice tools to find Δ:
—Finite-size scaling
—Monte Carlo RG



Need lattice RG blocking transformation to carry out MCRG. . .

Real-space RG for lattice $\mathcal{N} = 4$ SYM

Lattice RG blocking transformation must preserve symmetries

$\mathcal{Q}$ and $S_5 \longleftrightarrow$ geometric structure of the system

Simple scheme constructed in arXiv:1408.7067

$$\mathcal{U}'_a(n') = \xi\, \mathcal{U}_a(n)\mathcal{U}_a(n+\widehat{\mu}_a) \qquad \eta'(n') = \eta(n)$$

$$\psi'_a(n') = \xi\, [\psi_a(n)\mathcal{U}_a(n+\widehat{\mu}_a) + \mathcal{U}_a(n)\psi_a(n+\widehat{\mu}_a)] \qquad \text{etc.}$$

Doubles lattice spacing $a \longrightarrow a' = 2a$, with ξ a tunable rescaling factor

Scalar fields from polar decomposition $\mathcal{U}(n) = e^{\varphi(n)} U(n)$

are shifted $\varphi \longrightarrow \varphi + \log \xi$, since blocked U must remain unitary

$\mathcal{Q}$-preserving RG blocking is necessary ingredient in derivation that only one log. tuning may be needed to recover continuum $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

Scaling dimensions from MCRG stability matrix

Write system as (infinite) sum of operators, $H = \sum_i c_i \, \mathcal{O}_i$
with couplings c_i that flow under RG blocking transformation R_b

n-times-blocked system is $H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Fixed point defined by $H^\star = R_b H^\star$ with couplings $c_i^\star$

Linear expansion around fixed point defines stability matrix $T_{ij}^\star$

$$c_i^{(n)} - c_i^\star = \sum_k \left. \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \right|_{H^\star} \left(c_k^{(n-1)} - c_k^\star \right) \equiv \sum_j T_{ik}^\star \left(c_k^{(n-1)} - c_k^\star \right)$$

Correlators of $\mathcal{O}_i$, $\mathcal{O}_k \longrightarrow$ elements of stability matrix (Swendsen, 1979)

Eigenvalues of $T_{ik}^\star \longrightarrow$ scaling dimensions of corresponding operators

Preliminary Δ_K results from Monte Carlo RG

One more complication for lattice analyses

Recall twisted $SO(4)_{tw}$ involves only $SO(4)_R \subset SO(6)_R$

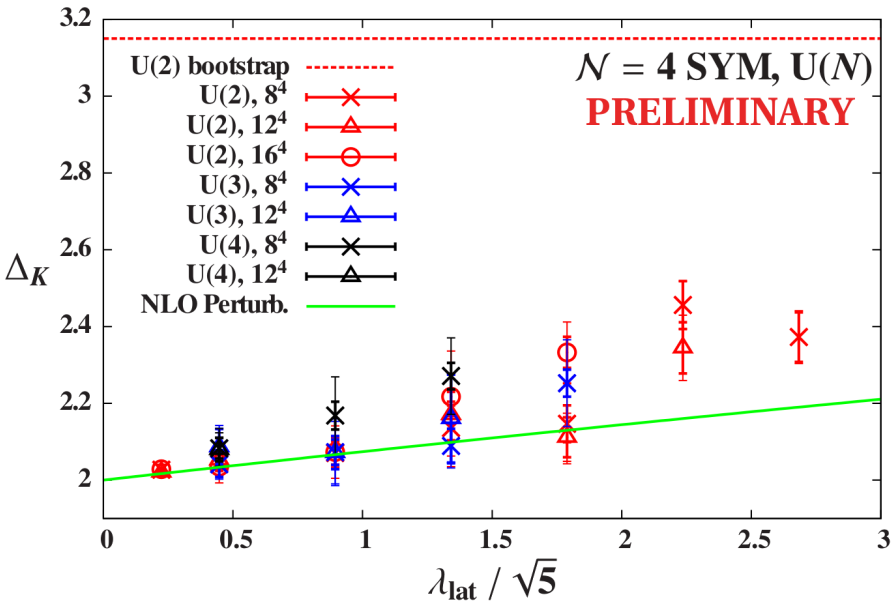
$\Longrightarrow$ The lattice Konishi operator mixes with the $SO(4)_R$-singlet part of the $SO(6)_R$-nonsinglet SUGRA operator

Currently working on variational analyses to disentangle operators

Konishi scaling dimension from MCRG stability matrix including both $\mathcal{O}_K^{\text{lat}}$ and $\mathcal{O}_S^{\text{lat}}$

Impose protected $\Delta_S = 2$

Systematic uncertainties from different amounts of smearing



Practical question: Potential sign problem

In lattice gauge theory we compute operator expectation values

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [d\mathcal{U}][d\overline{\mathcal{U}}] \;\; \mathcal{O} \;\; e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]} \;\; \mathrm{pf}\, \mathcal{D}[\mathcal{U},\overline{\mathcal{U}}]$$

Pfaffian can be complex for lattice $\mathcal{N} = 4$ SYM, $\mathrm{pf}\, \mathcal{D} = |\mathrm{pf}\, \mathcal{D}| e^{i\alpha}$

Complicates interpretation of $\left\{ e^{-S_B}\, \mathrm{pf}\, \mathcal{D} \right\}$ as Boltzmann weight

We carry out phase-quenched calculations with $\mathrm{pf}\, \mathcal{D} \longrightarrow |\mathrm{pf}\, \mathcal{D}|$

In principle need to reweight phase-quenched (pq) observables:

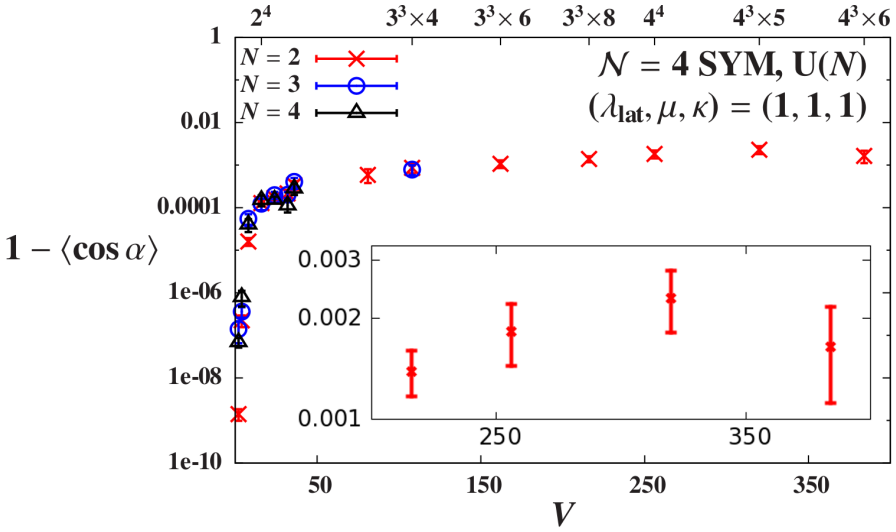
$$\langle \mathcal{O} \rangle = \frac{\left\langle \mathcal{O} e^{i\alpha} \right\rangle_{pq}}{\left\langle e^{i\alpha} \right\rangle_{pq}} \qquad \text{with } \left\langle \mathcal{O} e^{i\alpha} \right\rangle_{pq} = \frac{1}{\mathcal{Z}_{pq}} \int [d\mathcal{U}][d\overline{\mathcal{U}}]\, \mathcal{O} e^{i\alpha}\, e^{-S_B}\, |\mathrm{pf}\, \mathcal{D}|$$

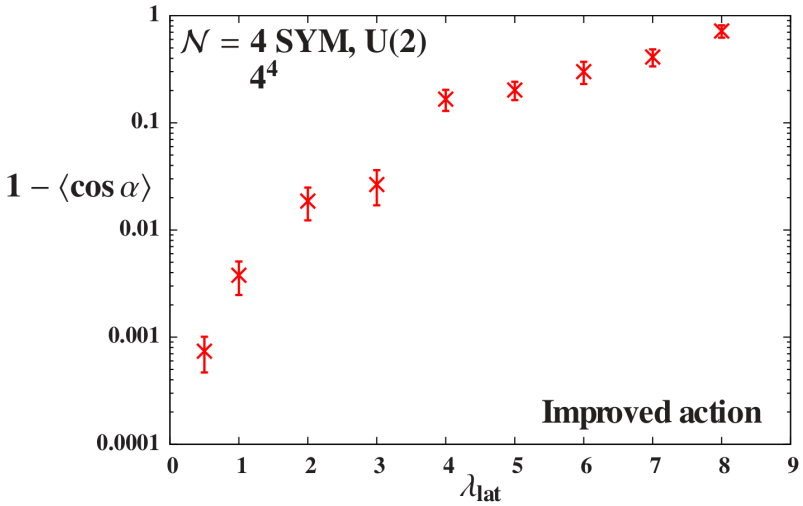
$\Longrightarrow$ Monitor $\left\langle e^{i\alpha} \right\rangle_{pq}$ as function of volume, coupling, N

Pfaffian phase dependence on volume and coupling

Left: $1 - \langle \cos(\alpha) \rangle_{pq} \ll 1$ independent of volume and N at $\lambda_{\text{lat}} = 1$

Right: New 4^4 results at $4 \leq \lambda_{\text{lat}} \leq 8$ show much larger fluctuations





May be interesting to check more volumes and N for improved action

Extremely expensive computation despite parallelization:

$\mathcal{O}(n^3)$ scaling $\longrightarrow$ ~50 hours for single U(2) 4^4 measurement

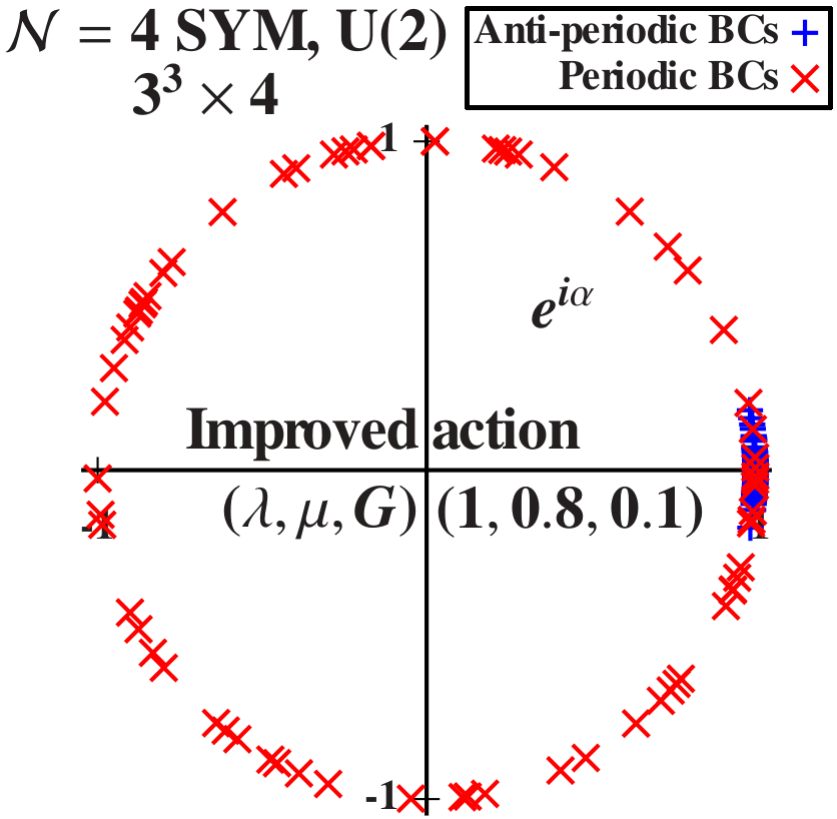
Two puzzles posed by the sign problem

- With periodic temporal boundary conditions for the fermions
  we have an obvious sign problem, $\langle e^{i\alpha} \rangle_{pq}$ consistent with zero
- With anti-periodic BCs and all else the same $e^{i\alpha} \approx 1$,
  phase reweighting has negligible effect

Why such sensitivity to the BCs?

Also, other pq observables
are nearly identical
for these two ensembles

Why doesn't the sign problem
affect other observables?



Preview: $(d-1)$-dimensional lattice superQCD

Method to add fundamental matter multiplets without breaking $\mathcal{Q}^2 = 0$

—Proposed by Matsuura (arXiv:0805.4491), Sugino (arXiv:0807.2683)

—First numerical study by Catterall & Veernala, arXiv:1505.00467

Consider 2-slice lattice
with U(N)×U(F) gauge group:

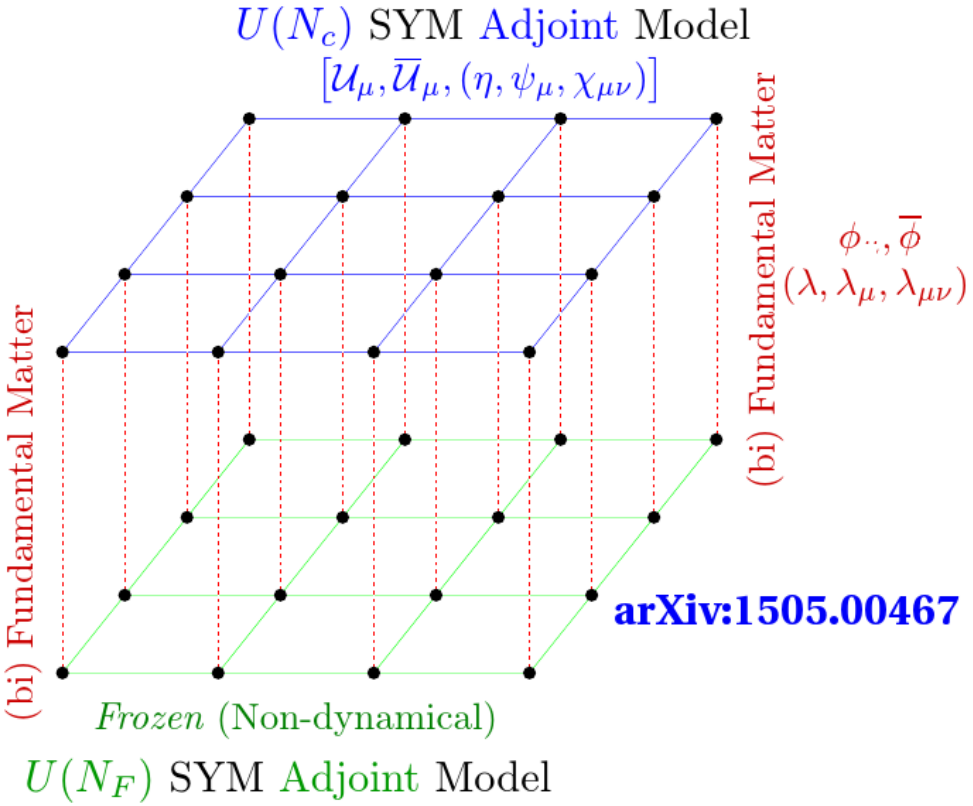
—(Adj, 1) fields on one slice

—(1, Adj) fields on the other

—Bi-fundamental in between

Set U(F) gauge coupling to zero

$\longrightarrow$ U(N) in $d-1$ dims.
with F fund. hypermultiplets



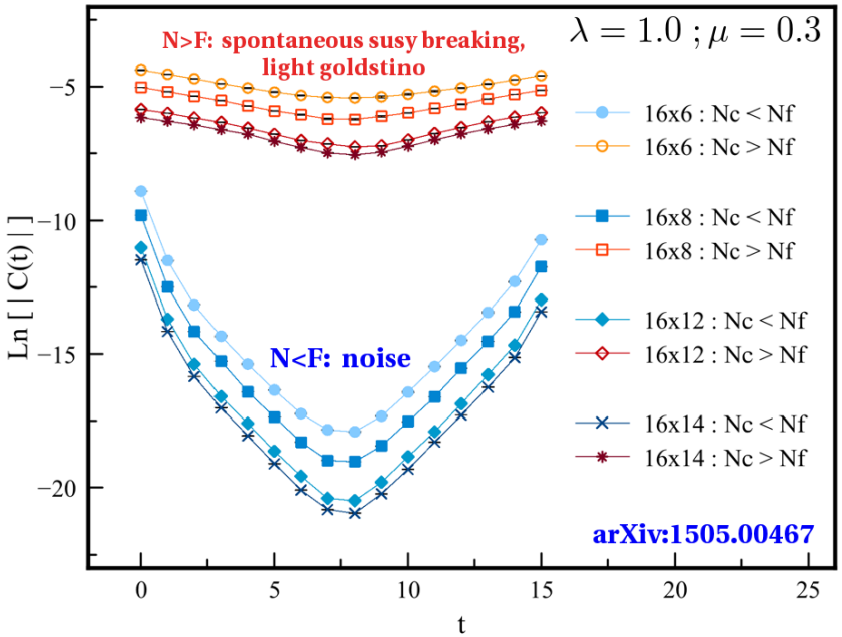
Spontaneous supersymmetry breaking

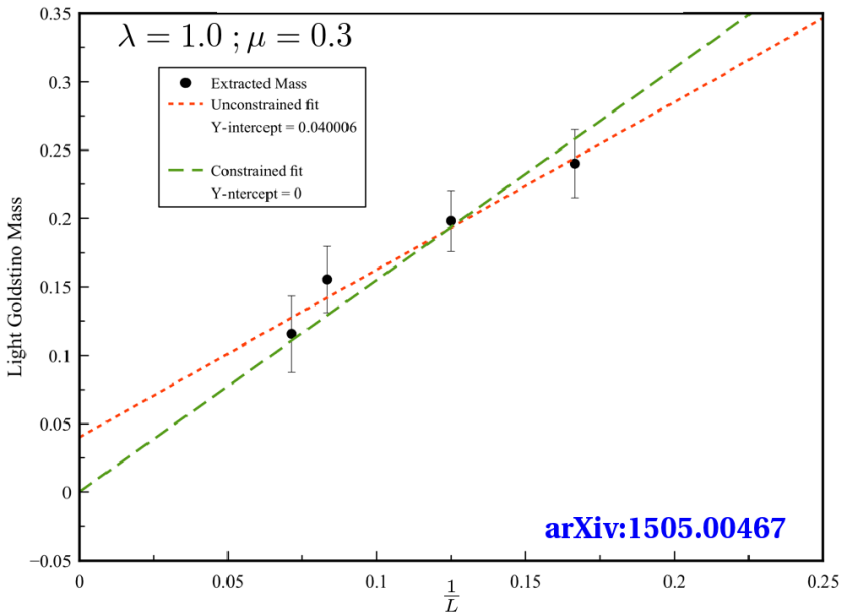
Auxiliary field e.o.m. produce Fayet–Iliopoulos D-term potential

$$d = \overline{\mathcal{D}}_a \mathcal{A}_a + \sum_{i=1}^{F} \phi_i \overline{\phi}_i + r\mathbb{I}_N \quad \longrightarrow \quad S_D \propto \sum_{i=1}^{F} \mathrm{Tr}\left[\phi_i \overline{\phi}_i + r\mathbb{I}_N\right]^2$$

$\langle \mathcal{Q}\eta \rangle = \langle d \rangle \neq 0 \Longrightarrow \langle 0 \,|H|\, 0 \rangle > 0$ (spontaneous susy breaking)

Effectively $N \times N$ conditions imposed on $N \times F$ degrees of freedom...





Recapitulation and outlook

Rapid recent progress in lattice supersymmetry

- Lattice promises non-perturbative insights from first principles
- Lattice $\mathcal{N} = 4$ SYM is practical thanks to exact $\mathcal{Q}$ susy
- Public code to reduce barriers to entry

Latest results from ongoing calculations

- Static potential is Coulombic at all couplings,
  $C(\lambda)$ checked against perturbation theory at weak coupling
- Progress toward conformal scaling dimension of Konishi operator

Many more directions are being — or can be — pursued

- Understanding the (absence of a) sign problem
- Systems with less supersymmetry, in lower dimensions,
  including matter fields, exhibiting spontaneous susy breaking, ...

Thank you!

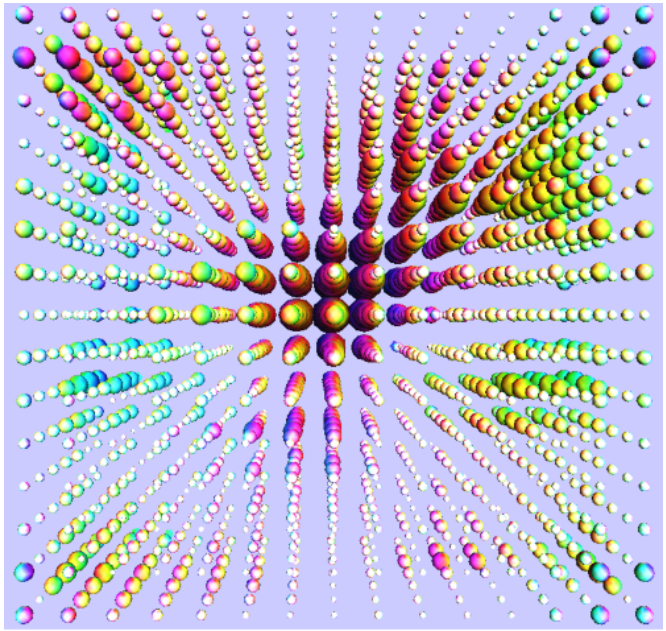
Thank you!

Collaborators

Simon Catterall, Poul Damgaard and Joel Giedt

Funding and computing resources

Backup: Essence of numerical lattice calculations



(Image credit: Claudio Rebbi)

Evaluate observables from functional integral via importance sampling Monte Carlo

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}U \;\; \mathcal{O}(U) \;\; e^{-S[U]}$$

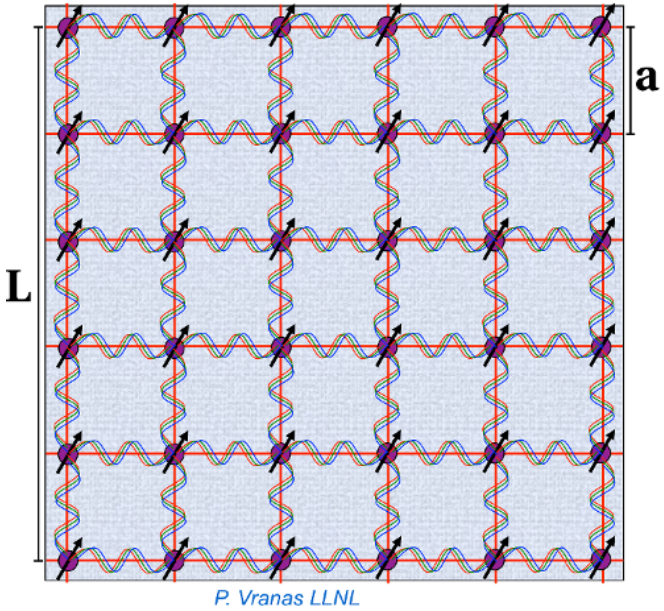
$$\longrightarrow \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(U_i) \;\; \text{with uncert.} \;\; \propto \sqrt{\frac{1}{N}}$$

U are field configurations in discretized euclidean spacetime

$S[U]$ is the lattice action, which should be real and non-negative so that $\frac{1}{\mathcal{Z}} e^{-S}$ can be treated as a probability distribution

The hybrid Monte Carlo algorithm samples U with probability $\propto e^{-S}$

Backup: More features of lattice calculations



Spacing between lattice sites ("a")
introduces UV cutoff scale $1/a$

Lattice cutoff preserves hypercubic subgroup
of full Poincaré symmetry

Remove cutoff by taking
continuum limit $a \to 0$ (with $L/a \to \infty$)

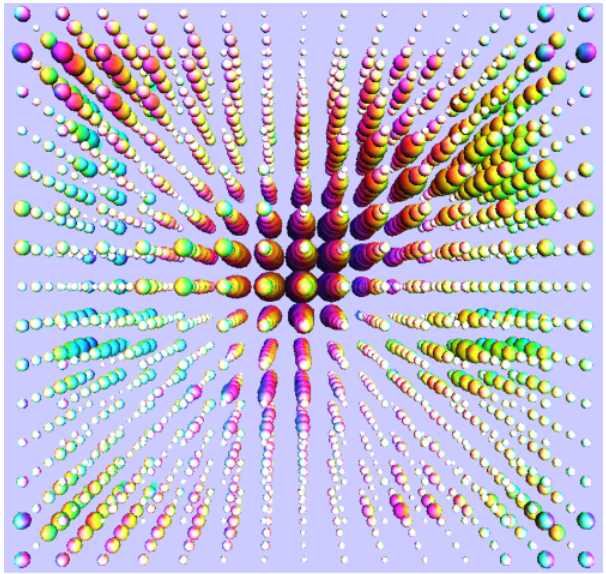
The lattice action S is defined by the bare lagrangian
at the UV cutoff set by the lattice spacing

After generating and saving an ensemble $\{U_n\}$ distributed $\propto e^{-S}$
it is usually quick and easy to measure many observables $\langle \mathcal{O} \rangle$

Changing the action (generally) requires generating a new ensemble

Backup: Hybrid Monte Carlo (HMC) algorithm

Goal: Sample field configurations U_i with probability $\frac{1}{\mathcal{Z}} e^{-S[U_i]}$



(Image credit: Claudio Rebbi)

HMC is a Markov process, based on Metropolis–Rosenbluth–Teller (MRT)

Fermions $\longrightarrow$ extensive action computation, so best to update entire system at once

Use fictitious molecular dynamics evolution

1. Introduce a fictitious fifth dimension ("MD time" τ) and stochastic canonical momenta for all field variables
2. Run inexact MD evolution along a trajectory in τ to generate a new four-dimensional field configuration
3. Apply MRT accept/reject test to MD discretization error

Backup: Failure of Leibnitz rule in discrete space-time

Given that $\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$ is problematic,

why not try $\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2i\sigma^\mu_{\alpha\dot\alpha}\nabla_\mu$ for a discrete translation?

Here $\nabla_\mu\phi(x) = \frac{1}{a}\left[\phi(x + a\widehat{\mu}) - \phi(x)\right] = \partial_\mu\phi(x) + \frac{a}{2}\partial_\mu^2\phi(x) + \mathcal{O}(a^2)$

Essential difference between ∂_μ and ∇_μ on the lattice, $a > 0$

$$\begin{aligned}\nabla_\mu\left[\phi(x)\chi(x)\right] &= a^{-1}\left[\phi(x + a\widehat{\mu})\chi(x + a\widehat{\mu}) - \phi(x)\chi(x)\right] \\ &= \left[\nabla_\mu\phi(x)\right]\chi(x) + \phi(x)\nabla_\mu\chi(x) + a\left[\nabla_\mu\phi(x)\right]\nabla_\mu\chi(x)\end{aligned}$$

We only recover the Leibnitz rule $\partial_\mu(fg) = (\partial_\mu f)g + f\partial_\mu g$ when $a \to 0$

$\Longrightarrow$ "Discrete supersymmetry" breaks down on the lattice

(Dondi & Nicolai, "Lattice Supersymmetry", 1977)

Backup: Twisting $\longleftrightarrow$ Kähler–Dirac fermions

The Kähler–Dirac representation is related to the spinor $Q^{\mathrm{I}}_{\alpha}, \overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ by

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu\gamma_\mu + \mathcal{Q}_{\mu\nu}\gamma_\mu\gamma_\nu + \overline{\mathcal{Q}}_\mu\gamma_\mu\gamma_5 + \overline{\mathcal{Q}}\gamma_5$$

$$\longrightarrow \mathcal{Q} + \mathcal{Q}_a\gamma_a + \mathcal{Q}_{ab}\gamma_a\gamma_b$$

with $a, b = 1, \cdots, 5$

The 4×4 matrix involves R symmetry transformations along each row and (euclidean) Lorentz transformations along each column

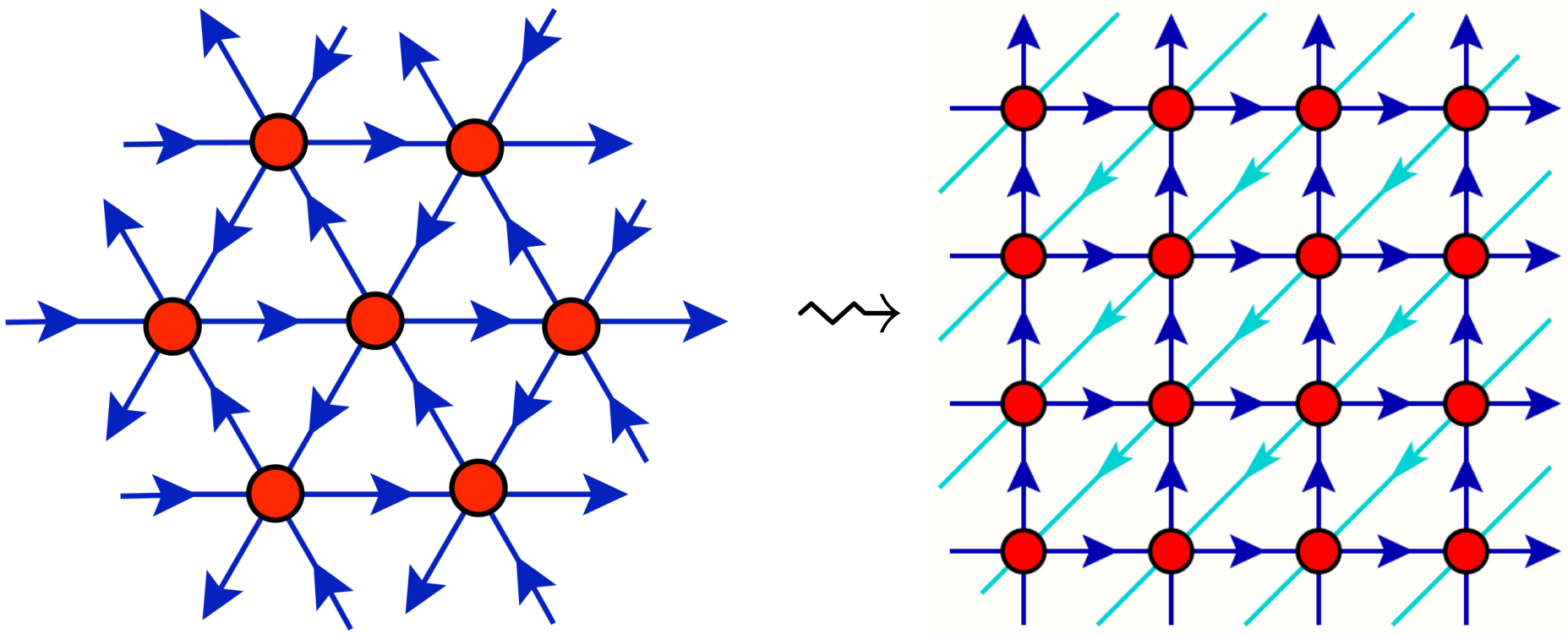
$\Longrightarrow$ Kähler–Dirac components transform under "twisted rotation group"

$$\mathrm{SO}(4)_{tw} \equiv \mathrm{diag}\Big[\mathrm{SO}(4)_{\mathrm{euc}} \otimes \mathrm{SO}(4)_R\Big]$$

$\uparrow$ only $\mathrm{SO}(4)_R \subset \mathrm{SO}(6)_R$

Backup: Hypercubic representation of A_4^* lattice

In the code it is very convenient to represent the A_4^* lattice
as a hypercube plus one backwards diagonal link



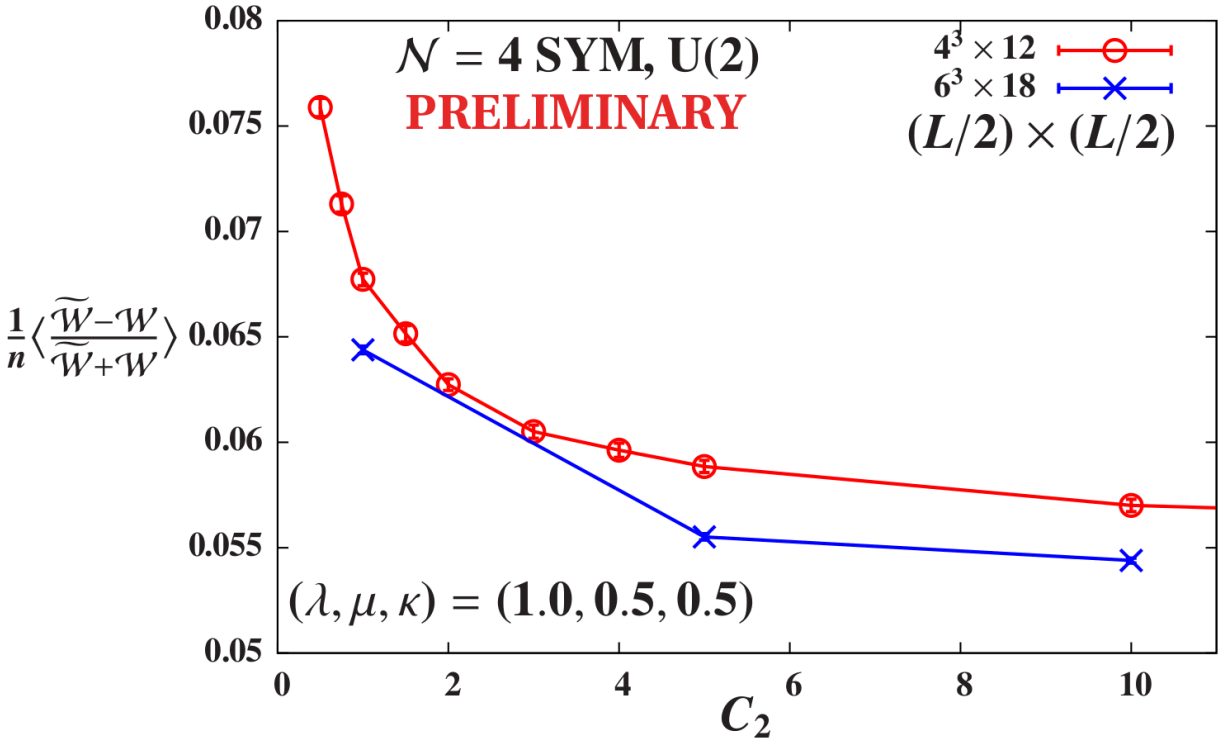
Backup: Restoration of $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supersymmetries

—$\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ from restoration of R symmetry (motivation for A_4^* lattice)

—Modified Wilson loops test R symmetries at non-zero lattice spacing

—Parameter c_2 may need logarithmic tuning in continuum limit

Results from arXiv:1411.0166 to be revisited with the new action...



Backup: More on flat directions

Recall $\mathrm{U}(N) = \mathrm{SU}(N) \otimes \mathrm{U}(1)$ gauge invariance from complexified links

In addition, supersymmetry transformations include $\mathcal{Q}\,\mathcal{U}_a = \psi_a$

$\Longrightarrow$ links must be in algebra, with continuum limit $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$

Flat directions in SU(N) sector are physical,

those in U(1) sector decouple only in continuum limit

Both must be regulated in calculations $\longrightarrow$ two deformations needed:

Scalar potential $\propto \mu^2 \sum_a \left(\mathrm{Tr}\left[\mathcal{U}_a \overline{\mathcal{U}}_a\right] - N\right)^2$ for SU(N) sector

Plaquette determinant $\sim G \sum_{a<b} (\det \mathcal{P}_{ab} - 1)$ for U(1) sector

Scalar potential **softly** breaks $\mathcal{Q}$ supersymmetry

susy-violating operators vanish as $\mu^2 \to 0$

Plaquette determinant can be made $\mathcal{Q}$-invariant $\longrightarrow$ improved action

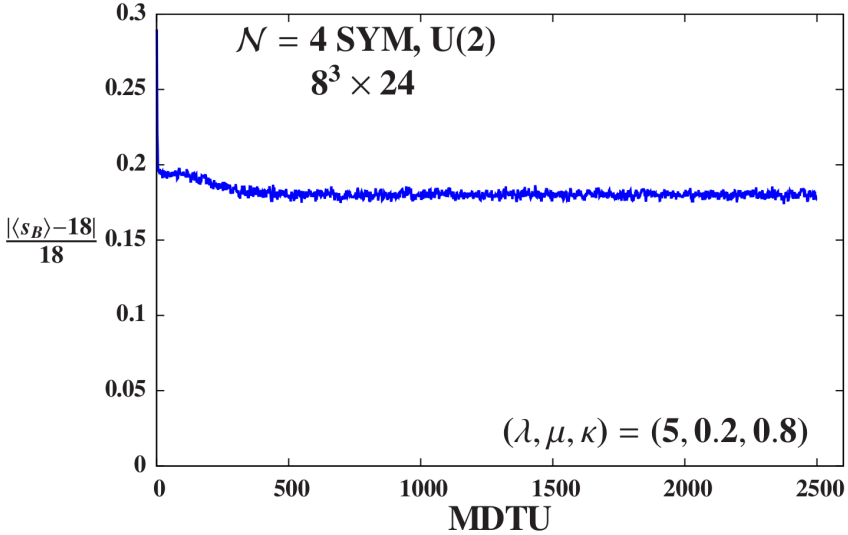
Backup: One problem with flat directions

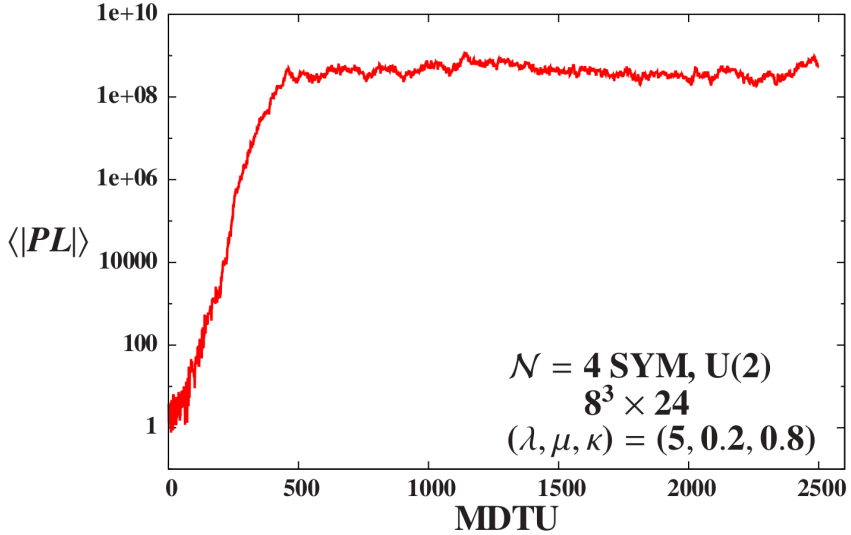
Gauge fields $\mathcal{U}_a$ can move far away from continuum form $\mathbb{I}_N + \mathcal{A}_a$ if $\mu^2/\lambda_{\rm lat}$ becomes too small

Example for $\mu = 0.2$ and $\lambda_{\rm lat} = 5$ on $8^3 \times 24$ volume

Left: Bosonic action is stable $\sim$18% off its supersymmetric value

Right: Complexified Polyakov ("Maldacena") loop wanders off to $\sim 10^9$

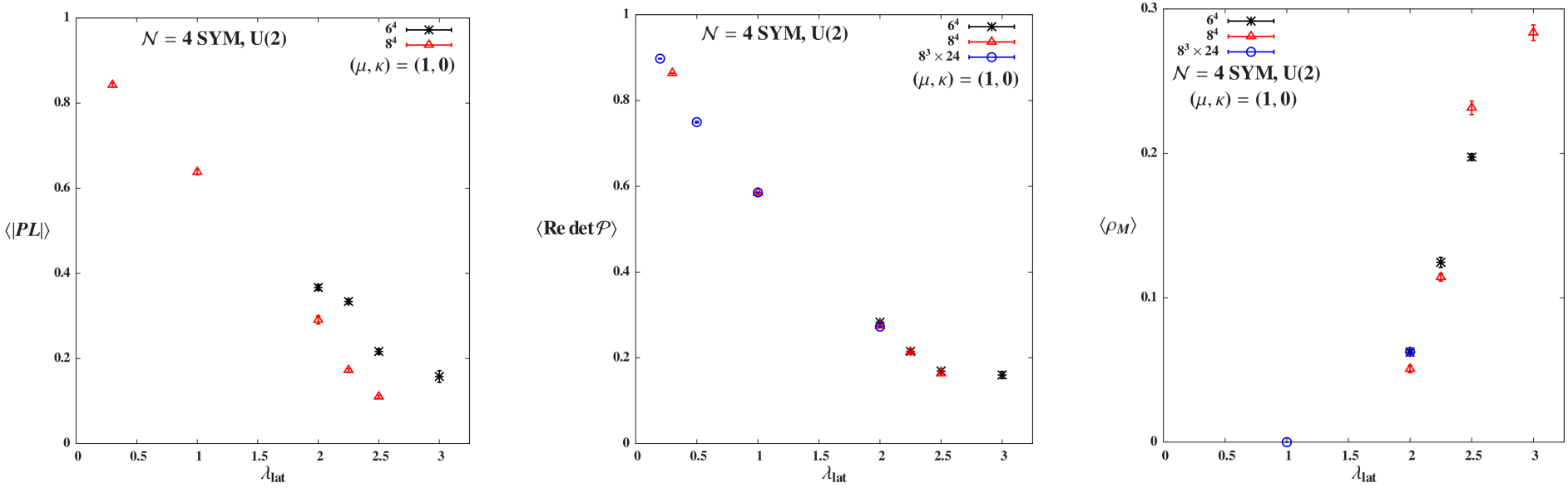




Backup: Another problem with U(1) flat directions

Can induce monopole condensation $\longrightarrow$ transition to confined phase

This lattice phase is not present in continuum $\mathcal{N} = 4$ SYM



Around the same $\lambda_{\rm lat} \approx 2\ldots$

Left: Polyakov loop falls towards zero

Center: Plaquette determinant falls towards zero

Right: Density of U(1) monopole world lines becomes non-zero

Backup: More on soft supersymmetry breaking

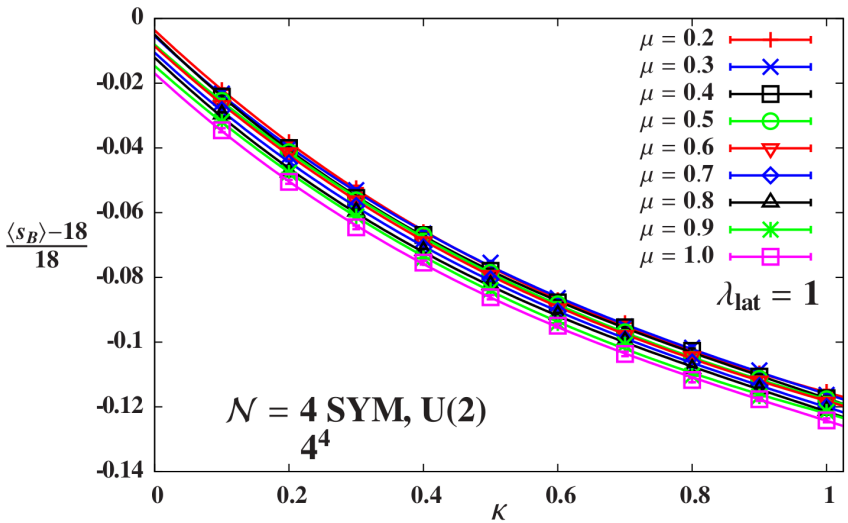
Before 2015 we added $(\det \mathcal{P} - 1)$ as another soft $\mathcal{Q}$-breaking term:

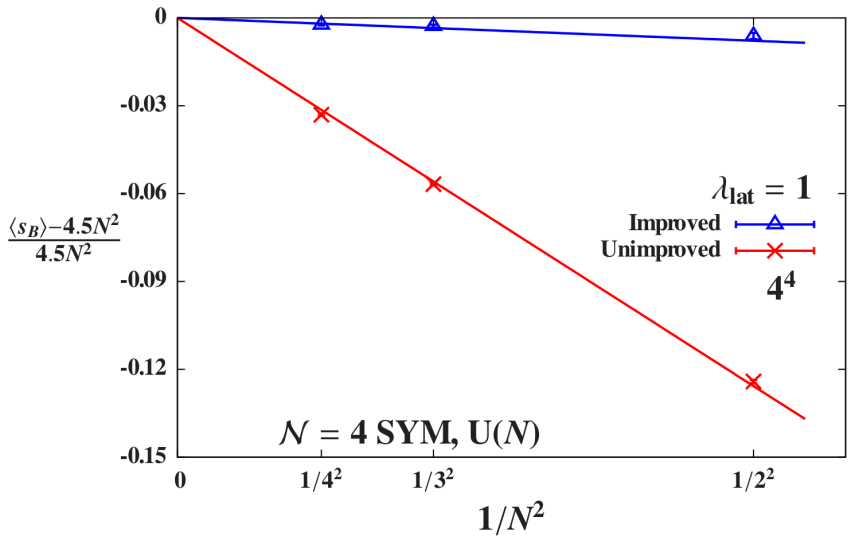
$$S_{\text{soft}} = \frac{N}{2\lambda_{\text{lat}}}\mu^2 \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2 + \kappa \sum_{a<b} |\det \mathcal{P}_{ab} - 1|^2$$

This κ term turned out to dominate the soft $\mathcal{Q}$-breaking effects

Left: The bosonic action provides another Ward identity $\langle s_B \rangle = 9N^2/2$

Right: Soft susy breaking is also suppressed $\propto 1/N^2$





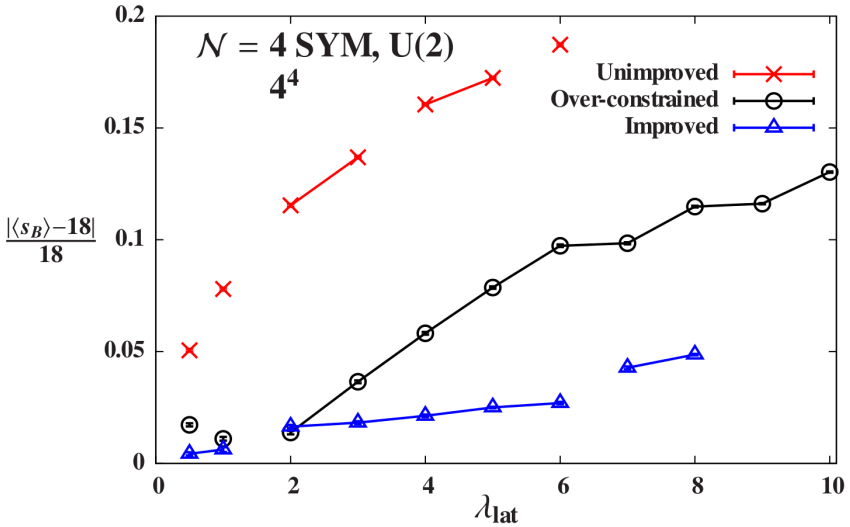
Backup: More on supersymmetric constraints

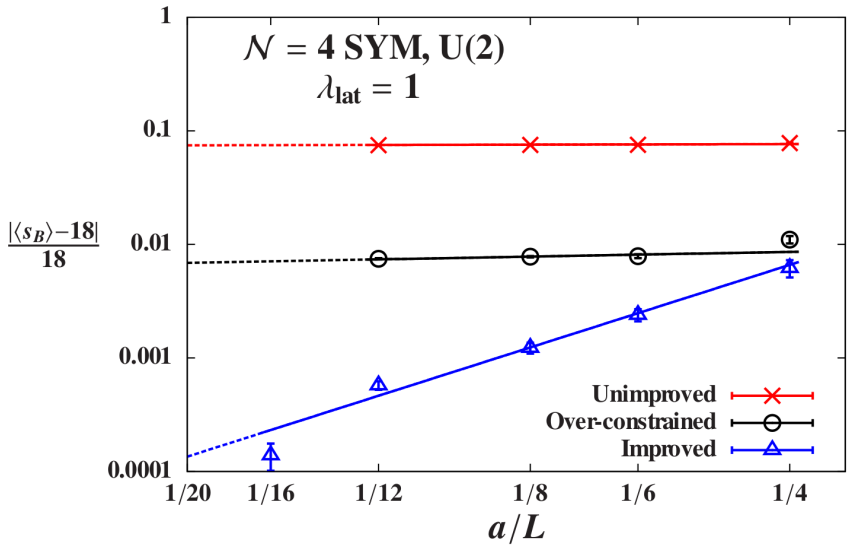
arXiv:1505.03135 introduces method to impose $\mathcal{Q}$-invariant constraints

Basic idea: Modify aux. field equations of motion $\longrightarrow$ moduli space

$$d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) \qquad \longrightarrow \qquad d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) + G\mathcal{O}(n)\mathbb{I}_N$$

Putting both plaquette determinant and scalar potential in $\mathcal{O}(n)$ over-constrains system $\longrightarrow$ sub-optimal Ward identity violations





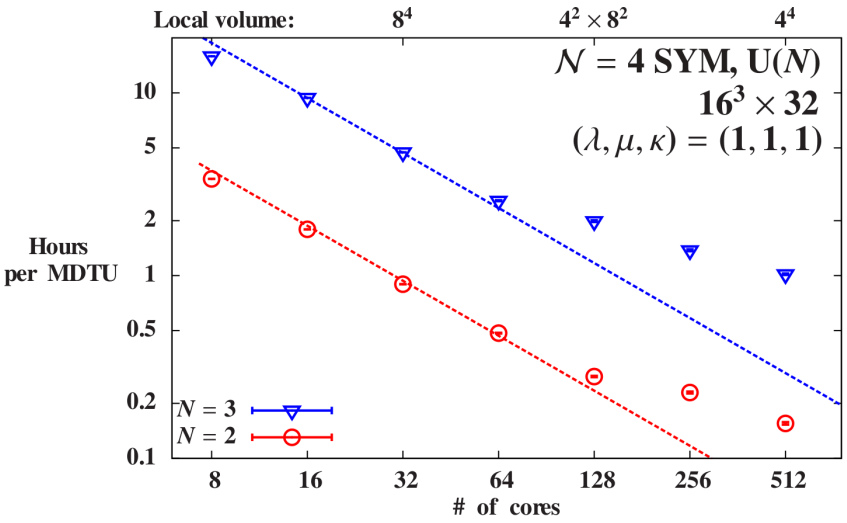
Backup: Code performance—weak and strong scaling

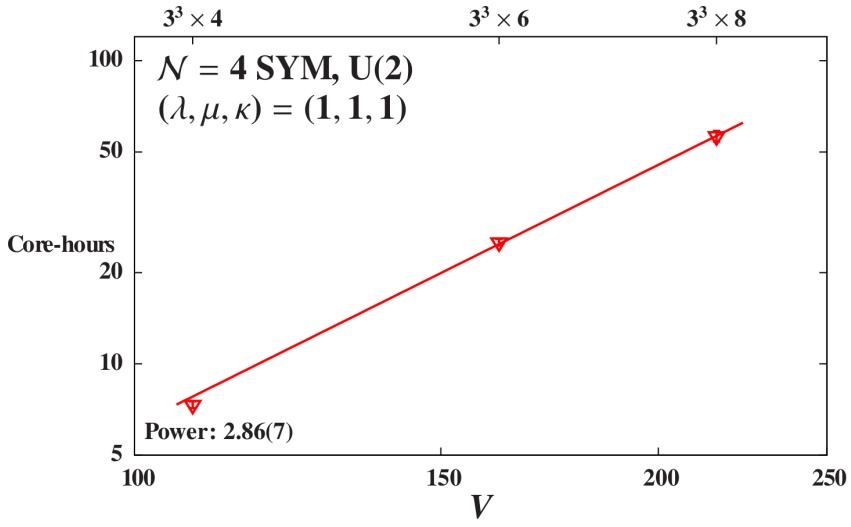
Results from arXiv:1410.6971 for the pre-2015 ("unimproved") action

Left: Strong scaling for U(2) and U(3) $16^3 \times 32$ RHMC

Right: Weak scaling for $\mathcal{O}(n^3)$ pfaffian calculation (fixed local volume)

$n \equiv 16N^2 V$ is number of fermion degrees of freedom





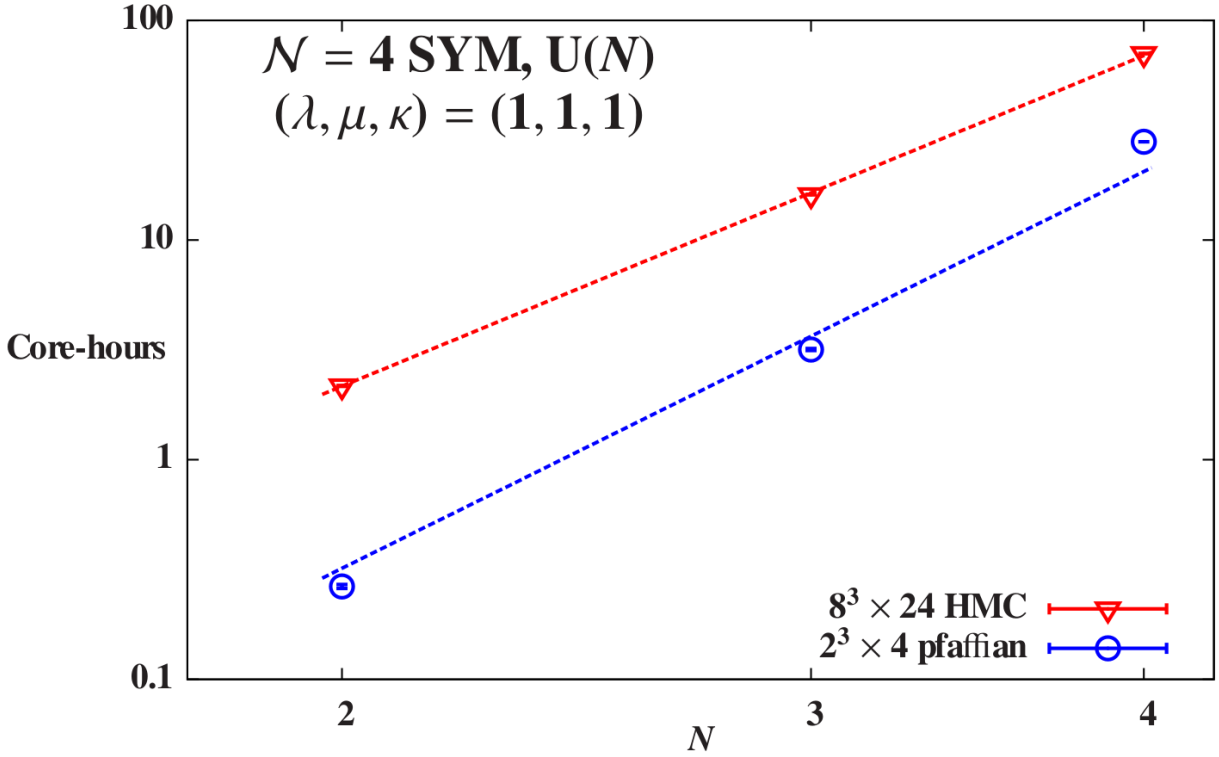
Dashed lines are optimal scaling

Solid line is power-law fit

Backup: Numerical costs for $N = 2$, 3 and 4 colors

Red: Originally found RHMC cost scaling $\sim N^5$
Improved in 2016, plot (from arXiv:1410.6971) yet to be updated

Blue: Pfaffian cost scaling consistent with expected N^6

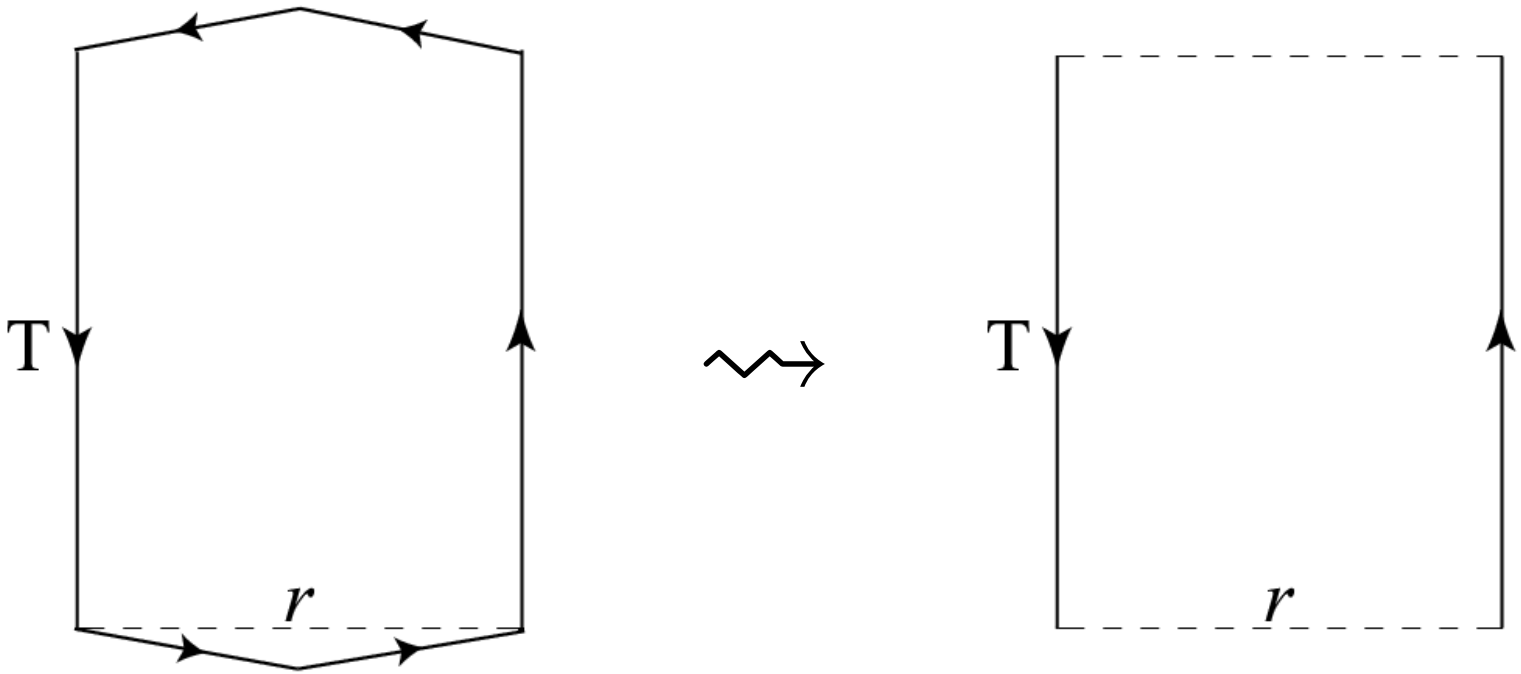


Backup: $\mathcal{N} = 4$ SYM static potential from Wilson loops

Extract static potential $V(r)$ from $r \times T$ Wilson loops

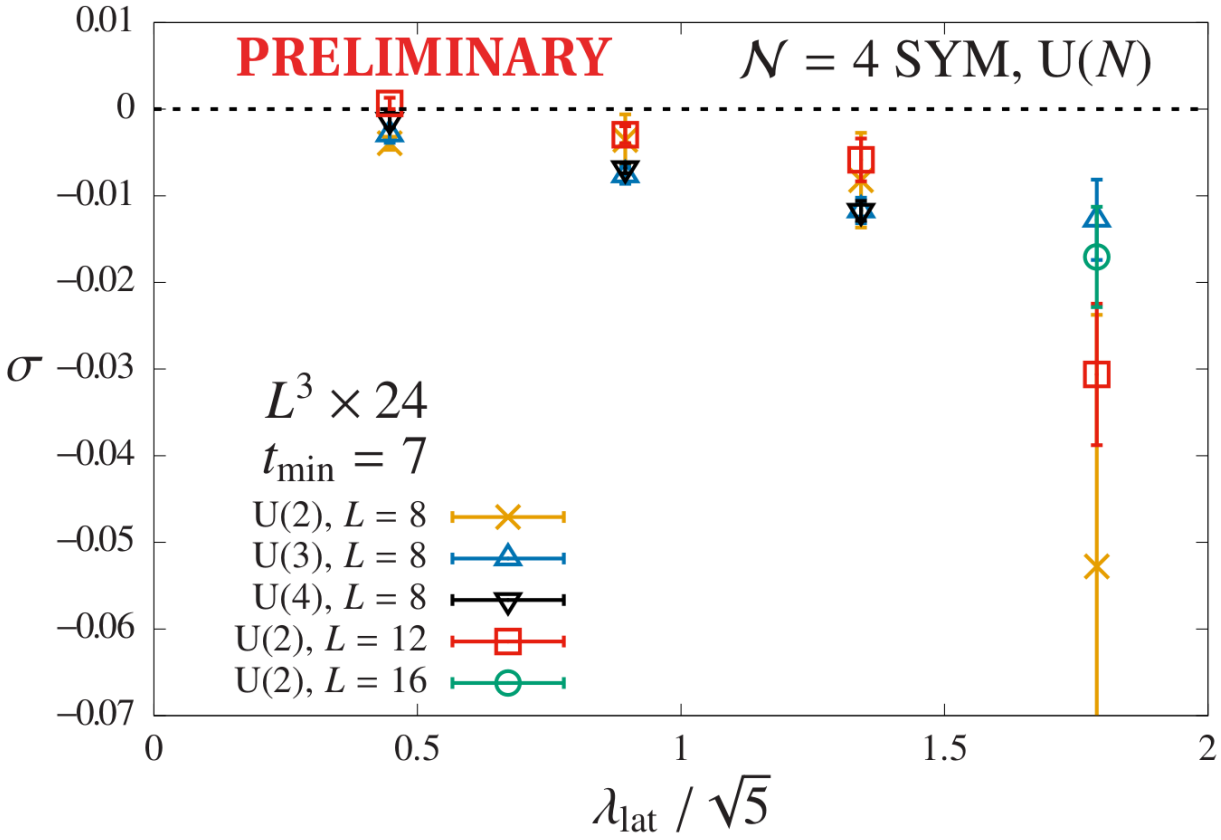
$$W(r, T) \propto e^{-V(r)\,T} \qquad\qquad V(r) = A - C/r + \sigma r$$

Coulomb gauge trick from lattice QCD reduces A_4^* lattice complications



Backup: Static potential is Coulombic at all λ

String tension σ from fits to confining form $V(r) = A - C/r + \sigma r$

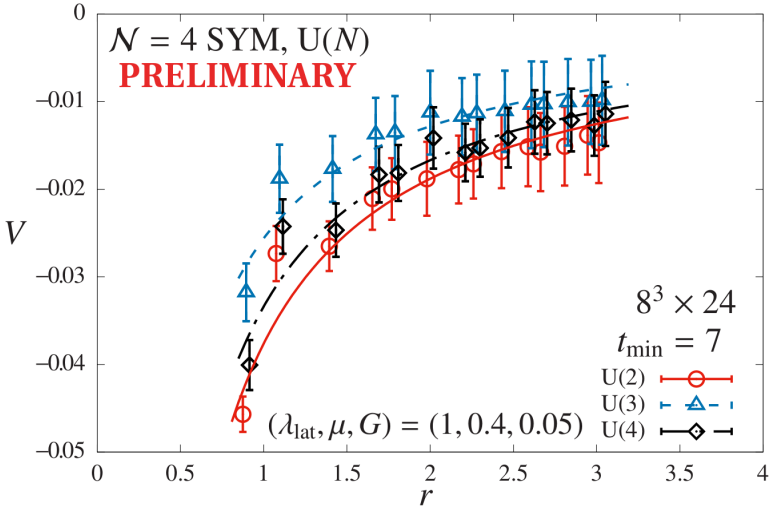


Slightly negative values flatten $V(r_l)$ for $r_l \lesssim L/2$

$\sigma \to 0$ as accessible range of r_l increases on larger volumes

Backup: Tree-level improvement for static potential

Discretization artifacts visible in naive static potential analyses



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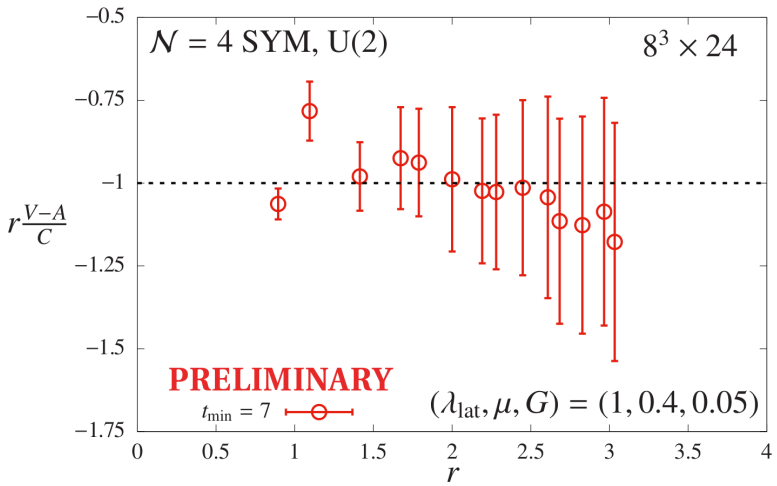


Improve by applying tree-level lattice perturbation theory for the $\mathcal{N} = 4$ SYM bosonic propagator on the A_4^* lattice:

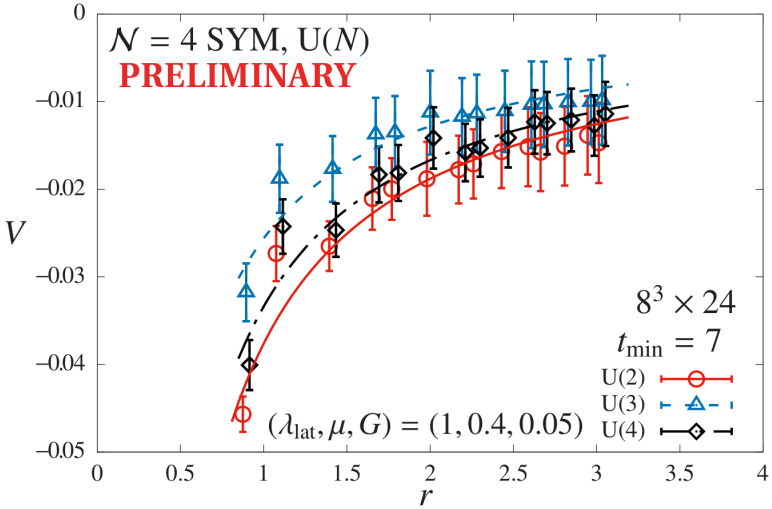
$$V(r) \longrightarrow V(r_l) \quad \text{where} \quad \frac{1}{r_l^2} \equiv 4\pi^2 \int \frac{d^4k}{(2\pi)^4} \frac{\cos\left[ir \cdot k\right]}{4\sum_{\mu=1}^4 \sin^2\left(k \cdot \widehat{e}_\mu \,/\, 2\right)}$$

$\widehat{e}_\mu$ are A_4^* lattice basis vectors (arXiv:1102.1725)

Momenta $k = \frac{2\pi}{L} \sum_{\mu=1}^4 n_\mu \widehat{g}_\mu$ depend on dual basis vectors

Backup: Tree-level improvement for static potential

Discretization artifacts visible in naive static potential analyses



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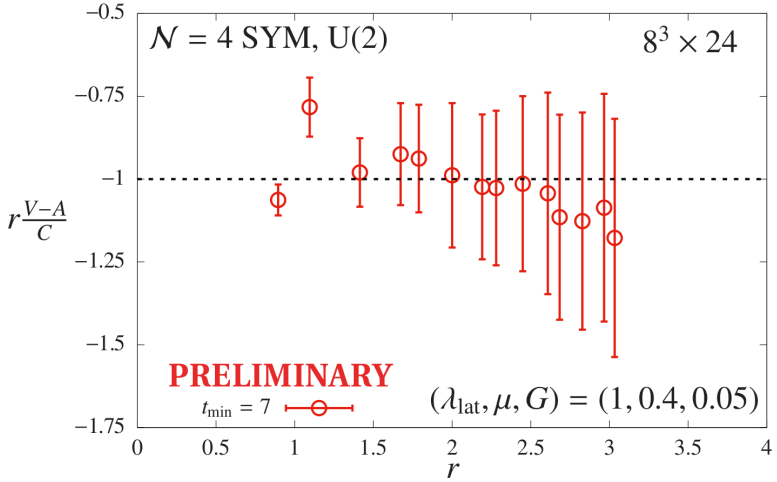
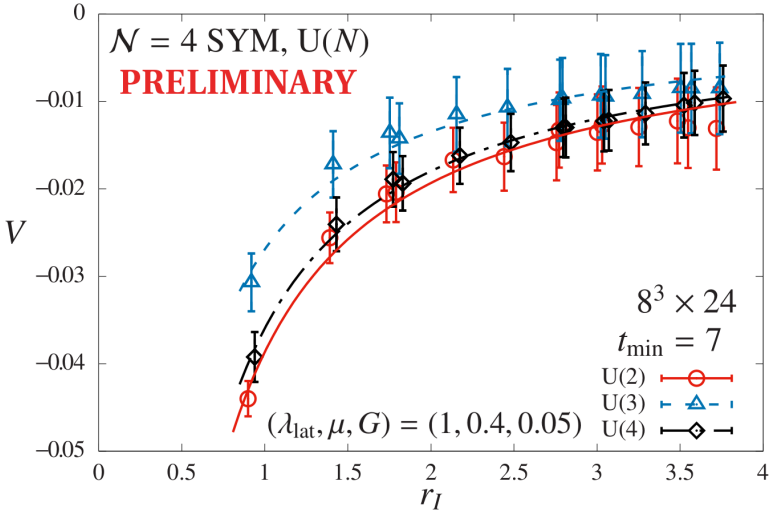


Tree-level improvement significantly reduces discretization artifacts



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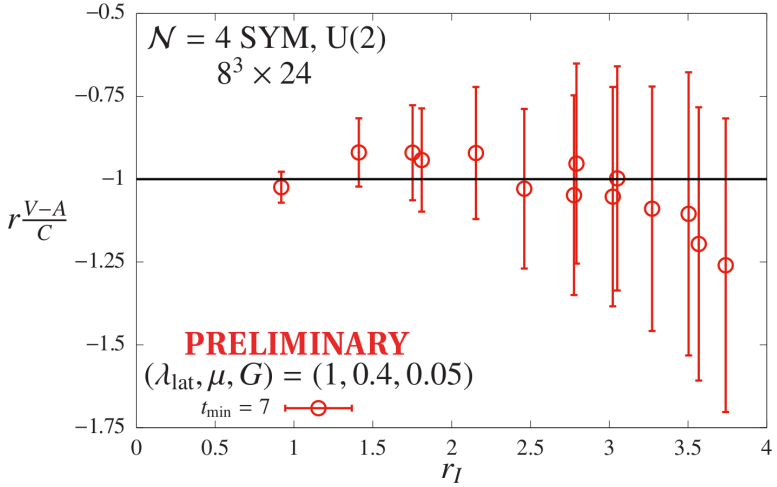
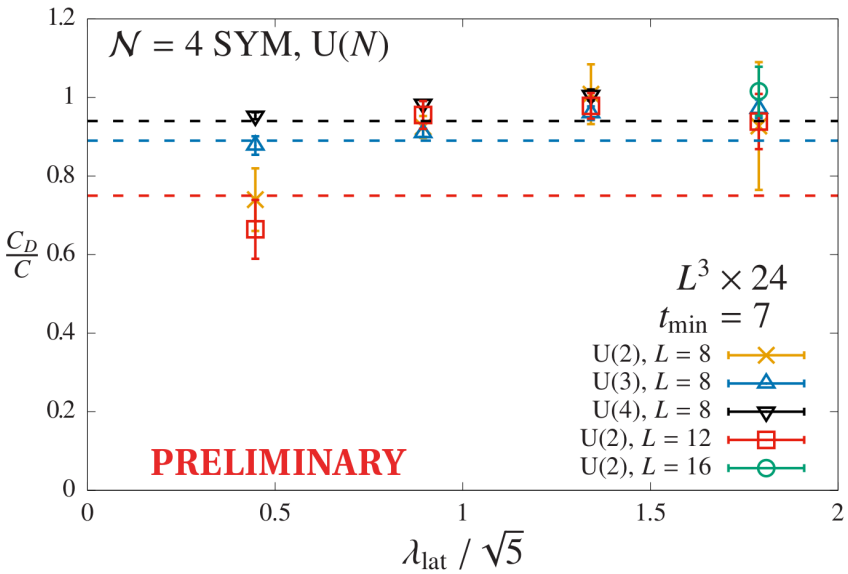
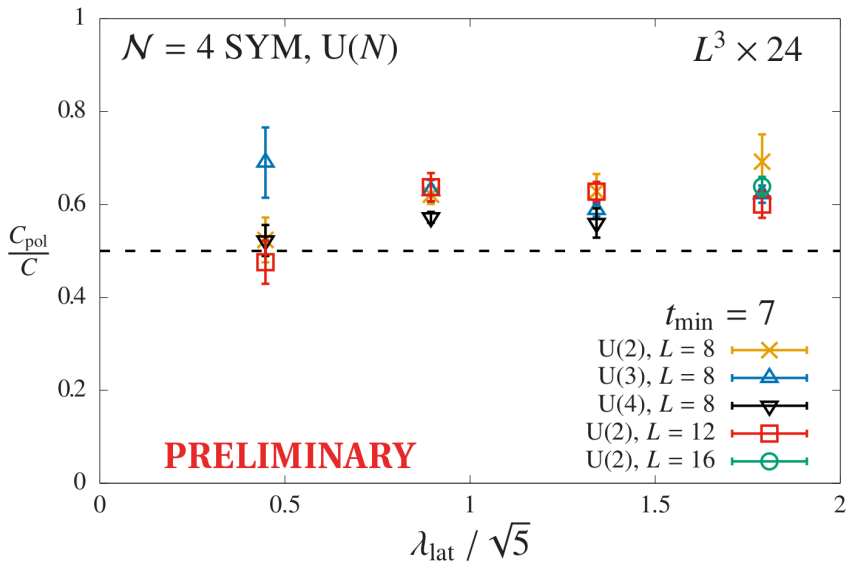


Backup: More tests of the static potential

Left: Projecting Wilson loops from U(N) $\longrightarrow$ SU(N) $\Longrightarrow$ factor of $\frac{N^2-1}{N^2}$

Right: Unitarizing links removes scalars $\Longrightarrow$ factor of $1/2$





Several ratios end up above expected values

Cause not clear — seems insensitive to lattice volume and μ

Backup: Smearing for Konishi analyses

As in glueball analyses, use smearing to enlarge operator basis

Using APE-like smearing: $— \quad \longrightarrow \quad (1-\alpha)— \quad + \quad \frac{\alpha}{8}\sum \sqcap,$

with staples built from unitary parts of links but no final unitarization

(unitarized smearing — e.g. stout — doesn't affect Konishi)

Average plaquette is stable upon smearing (**right**)

while minimum plaquette steadily increases (**left**)