Electroweak Phenomenology and Lattice Strong Dynamics

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and work in progress with the Lattice Strong Dynamics Collaboration
Motivation: Electroweak symmetry breaking

LHC experiments have collected \( \sim 4 \text{ fb}^{-1} \) of data at \( \sqrt{s} = 13 \text{ TeV} \)

Soon we will see new constraints on physics beyond the standard model . . . and possibly new discoveries!

One compelling possibility is new strong dynamics that produces a composite Higgs boson

Protects the electroweak scale from sensitivity to quantum effects (solving the hierarchy / fine-tuning problem)

Lattice gauge theory has a crucial role to play in exploring and understanding new strong dynamics
Motivation: Composite Higgs vs. QCD

Electroweak symmetry breaking through new strong dynamics remains viable but must satisfy stringent experimental constraints.

- The composite Higgs boson must have a mass of 125 GeV and standard-model-like properties.
- Electroweak precision observables (e.g., $S$ parameter) must be consistent with the standard model.

If the new strong dynamics resembled QCD these conditions would not be satisfied.

New strong dynamics different from QCD can be studied non-perturbatively by lattice calculations.
Strategy for lattice studies of new strong dynamics

Systematically depart from familiar ground of lattice QCD

\( N = 3 \) with \( N_F = 2 \) light flavors in fundamental rep

Identify generic features of non-QCD-like strong dynamics

Focus on near-conformal dynamics

Quick orientation:

- \( \beta(\alpha) \)
- \( \alpha \)

Graph:

- AF lost
- Conformal (\( \alpha^* < 1 \))
- CBZ
- SU(Nc) gauge theories, Nf fundamental flavors

(Ethan Neil)
Strategy for lattice studies of new strong dynamics

Systematically depart from familiar ground of lattice QCD

\((N = 3 \text{ with } N_F = 2 \text{ light flavors in fundamental rep})\)

Identify generic features of non-QCD-like strong dynamics

Focus on near-conformal dynamics

—Add more light flavors
  \(\rightarrow N_F = 8 \text{ fundamental}\)

—Enlarge fermion rep
  \(\rightarrow N_F = 2 \text{ two-index symmetric}\)

—Explore \(N = 2 \text{ and } 4\)
  \(\rightarrow (\text{pseudo})\text{real reps for cosets } SU(n)/Sp(n) \text{ and } SU(n)/SO(n)\)
Lattice Strong Dynamics Collaboration

Argonne  Xiao-Yong Jin, James Osborn
Boston  Rich Brower, Claudio Rebbi, Evan Weinberg
Brookhaven  Meifeng Lin
Colorado  Anna Hasenfratz, Ethan Neil
Edinburgh  Oliver Witzel
Livermore  Evan Berkowitz, Enrico Rinaldi, Pavlos Vranas
RBRC  Ethan Neil, Sergey Syritsyn
Syracuse  DS
UC Davis  Joseph Kiskis
Yale  Thomas Appelquist, George Fleming, Andy Gasbarro

Exploring the range of possible phenomena in strongly coupled gauge theories
Results to be shown are from state-of-the-art lattice calculations $\mathcal{O}(100M \text{ core-hours})$ invested overall

Many thanks to DOE, NSF and computing centers!
Plan for this talk

   (Domain wall fermions on $32^3 \times 64$ lattices)

2. Higgs (singlet scalar) mass (arXiv:1510.06771 & ongoing)
   (nHYP-improved staggered fermions up to $64^3 \times 128$)

Common theme: Challenges of chiral extrapolation

3. Chiral condensate and WW scattering parameters (time permitting)

Additional studies can be reviewed by request
($N_F = 8$ phase diagram; discrete $\beta$ function from gradient flow;
effective mass anomalous dimension $\gamma_{\text{eff}}(\lambda)$ from Dirac eigenmodes)
Electroweak precision observables — preliminaries

Good chiral & flavor symmetries important → domain wall fermions

- Add fifth dimension of length $L_s$ (expensive!)
- Exact chiral symmetry at finite lattice spacing in the limit $L_s \to \infty$
- At finite $L_s = 16$, “residual mass” $m_{\text{res}} \ll m_f$; $m = m_f + m_{\text{res}}$
  
  \[ 10^5 m_{\text{res}} = 2.6 \ [2f]; \quad 82 \ [6f]; \quad 268 \ [8f] \]

Compare more directly by approximately matching $m \to 0$ IR scales

\[ M_{V_0} = 0.217(3) \ [2f]; \quad 0.199(3) \ [6f]; \quad 0.171(4) \ [8f] \]
Electroweak precision observable — the $S$ parameter

Constrain the physics of electroweak symmetry breaking from its effects on vacuum polarizations $\Pi(Q)$ of EW gauge bosons

$S$ remains an important constraint on new strong dynamics

Experiment: $S = 0.03 \pm 0.10$

Scaled-up QCD: $S \approx 0.43$

Can also analyze $S$ as a low-energy constant ($\alpha_1$ or $L_{10}$) of electroweak chiral lagrangian $\mathcal{L}_\chi$
The $S$ parameter on the lattice

\[ \mathcal{L}_\chi \supset \frac{\alpha_1}{2} g_1 g_2 B_{\mu\nu} \text{Tr} \left[ U_{\tau 3} U^{\dagger} W^{\mu\nu} \right] \longrightarrow \gamma, Z \quad \text{new} \]

\[
S = -16\pi^2 \alpha_1 = 4\pi N_D \lim_{Q^2 \to 0} \frac{d}{dQ^2} \Pi_{VA}(Q^2) - \Delta S_{SM}(M_H)
\]

- $N_D \geq 1$ is the number of doublets with chiral electroweak couplings
- $\Delta S_{SM}(M_H)$ subtracted so that $S = 0$ in the standard model
  - Removes three eaten Goldstones, depends on Higgs mass
- $\Pi_{VA}(Q^2)$ is transverse component of vacuum polarization tensor

\[
\Pi_{VA}^{\mu\nu}(Q) = Z \sum_x e^{iQ \cdot (x + \vec{\mu}/2)} \text{Tr} \left[ \left\langle \mathcal{V}^{\mu a}(x) \mathcal{V}^{\nu b}(0) \right\rangle - \left\langle \mathcal{A}^{\mu a}(x) \mathcal{A}^{\nu b}(0) \right\rangle \right]
\]
The $S$ parameter on the lattice

\[
\mathcal{L}_\chi \supset \frac{\alpha_1}{2} g_1 g_2 B_{\mu\nu} \text{Tr} \left[ U_{\tau 3} U^\dagger W^{\mu\nu} \right] \longrightarrow \gamma, Z \xrightarrow{\text{new}} Q \to \gamma, Z
\]

\[
S = -16\pi^2 \alpha_1 = 4\pi N_D \lim_{Q^2 \to 0} \frac{d}{dQ^2} \Pi_{V-A}(Q^2) - \Delta S_{SM}(M_H)
\]

$\Pi_{V-A}(Q^2)$ is transverse component of vacuum polarization tensor

\[
\Pi^{\mu\nu}_{V-A}(Q) = Z \sum_x e^{iQ \cdot (x + \mu/2)} \text{Tr} \left[ \langle V^{\mu a}(x)V^{\nu b}(0) \rangle - \langle A^{\mu a}(x)A^{\nu b}(0) \rangle \right]
\]

- Renormalization constant $Z$ evaluated non-perturbatively
- Chiral symmetry of domain wall fermions $\implies Z = Z_A = Z_V$
  \[
  Z = 0.85 [2f]; \quad 0.73 [6f]; \quad 0.70 [8f]
  \]

- **Conserved currents** $V$ and $A$ ensure that lattice artifacts cancel, combined with local currents $V$ and $A$ to reduce costs
Representative polarization function data, $\Pi_{V-A}(Q^2)$

$$S = 4\pi N_D \lim_{Q^2 \to 0} \frac{d}{dQ^2} \Pi_{V-A}(Q^2) - \Delta S_{SM}(M_H)$$

$Q^2 \to 0$ extrapolation via rational function

$$\Pi_{V-A}(Q^2) = \frac{a_0 + a_1 Q^2}{1 + b_1 Q^2 + b_2 Q^4}$$

Motivated by single-pole dominance and sum rules (cf. Aubin et al.)

Can already see contrast between $N_F = 2$ and $N_F = 6$
(may be non-negligible finite-volume effects for lightest $N_F = 6$ point)
Results for polarization function slopes $\Pi'_{V-A}(0)$

Vertical axis: $4\pi \Pi'_{V-A}(0)$

where

$$\Pi'(0) = \lim_{Q^2 \to 0} \frac{d}{dQ^2} \Pi(Q^2)$$

$$S = 4\pi N_D \Pi'_{V-A}(0) - \Delta S_{SM}$$

Horizontal axis: $M_P^2 / M_{V0}^2$ gives a more physical comparison than $m$

$$M_{V0} \equiv \lim_{m \to 0} M_V$$

approximately matched between $N_F = 2$, 6 and 8

$N_F = 6$ and 8 show significant reduction for $M_P \lesssim M_{V0}$,

and expected agreement in the quenched limit $M_P^2 \to \infty$
From slopes to $S$ for $M_H = 125$ GeV

$$S = 4\pi N_D \lim_{Q^2 \to 0} \frac{d}{dQ^2} \Pi_{V-A}(Q^2) - \Delta S_{SM}(M_H)$$

1. $N_D$ doublets with chiral electroweak couplings contribute to $S$
   Scaled-up QCD often considers maximum $N_D = N_F/2$
   but only $N_D \geq 1$ is required for electroweak symmetry breaking

2. $\Delta S_{SM} = \frac{1}{12\pi} \int_{4M_P^2}^\infty \frac{ds}{s} \left[1 - \left(1 - \frac{M_{V0}^2}{s}\right)^3 \Theta(s - M_{V0}^2)\right] - \frac{1}{12\pi} \log \left(\frac{M_{V0}^2}{M_H^2}\right)$

   Integral diverges logarithmically as $M_P^2 \to 0$

   to cancel contribution of three eaten modes

   First term assumes $M_H \sim M_{V0} \sim \text{TeV}$;
   second term corrects for $M_H = 125$ GeV $\ll \text{TeV}$
Results for the $S$ parameter

$N_F = 2$ result

$$\lim_{M_P^2 \to 0} S = 0.42(2)$$

matches scaled-up QCD

Significant reductions as $N_F$ increases

Linear + log fits to light points ($M_P \lesssim M_{V0}$) guide the eye, account for any chiral logs remaining after $\Delta S_{SM}(M_H)$

$$S = A + B \frac{M_P^2}{M_{V0}^2} + \frac{1}{12\pi} \left( \frac{N_F}{2} - 1 \right) \log \left( \frac{M_{V0}^2}{M_P^2} \right) \quad \text{for } N_D = 1$$
Challenges of chiral extrapolation

$N_F = 2$ result

$$\lim_{M_P^2 \to 0} S = 0.42(2)$$

matches scaled-up QCD

Significant reductions as $N_F$ increases

- Lattice calculation involves $N_F^2 - 1$ degenerate pseudoscalars
- **Only three** massless Goldstones eaten by $W$ and $Z$,
  $$N_F^2 - 4$$ PNGBs must acquire non-zero masses

For $N_F = 6$, imagine freezing 32 PNGBs at the blue curve’s minimum, and taking only three to zero mass
Pushing $N_F = 8$ towards the chiral limit

Wish list after domain wall studies

- Want larger physical volumes to avoid finite-volume effects
- Want smaller masses to connect to chiral perturbation theory
- Want more statistics to analyze Higgs (singlet scalar)

Solution: Staggered fermions using nHYP-improved action

- $m = 0.00889$ on $24^3 \times 48$ with $\sim 24,700$ thermalized MDTU
- $m = 0.00750$ on $32^3 \times 64$ with $\sim 24,600$ thermalized MDTU
- $m = 0.00500$ on $32^3 \times 64$ with $\sim 46,600$ thermalized MDTU
- $m = 0.00220$ on $48^3 \times 96$ with $\sim 19,600$ thermalized MDTU
- $m = 0.00125$ on $64^3 \times 128$ with $\sim 2,000$ thermalized MDTU
  (no $64^3 \times 128$ disconnected analyses)
Pushing $N_F = 8$ towards the chiral limit

nHYP improvement reduces discretization artifacts, allows larger lattice spacing $\rightarrow$ larger physical volumes

Enables exploration of smaller masses (larger $M_V/M_P$)

Horizontal axes use mass-dependent gradient flow scale $\sqrt{8t_0}$

Spontaneous chiral symmetry breaking $\Rightarrow M_V/M_P$ on vertical axes diverges in chiral limit
Higgs mass analysis — measurements and correlators

The Higgs is a scalar singlet ($0^{++}$) $\mapsto$ disconnected diagrams

For each disconnected measurement: 6 stochastic-U(1) sources diluted in time, color, and even/odd spatial sites

Full scalar correlator is $S(t) = 2D(t) - C(t)$, combining connected and (vacuum-subtracted) disconnected correlators

However, Higgs appears both in $S(t)$ and in $D(t)$ on its own $\mapsto$ Fit each and include differences in systematic uncertainties [better plateaus in $D(t)$; more excited-state effects in $S(t)$]
Higgs mass analysis — representative fits

For each of $D(t)$ and $S(t)$, carry out correlated fits to

$$A_H \cosh [M_H (t - N_T/2)] + (-1)^t A_1 \cosh [M_1 (t - N_T/2)]$$

$$+ \nu + \text{excited states}$$

- Start with usual staggered state and parity partner
- Add free parameter $\nu$ to control noise in vacuum subtraction [equivalent to fitting $D(t + 1) - D(t)$ or $S(t + 1) - S(t)$]
- Up to two excited states included in fits for $S(t)$
$N_F = 8$ spectrum results

Preliminary results still in lattice units

Scale setting suggests resonance masses $\sim 2$–$3$ TeV

Large separation between Higgs and resonances

Higgs degenerate with pseudo-Goldstones in accessible regime

Dramatically different from QCD-like dynamics, where $M_H \approx 2M_P$ in this regime (dominated by two-pion scattering)

Typical chiral extrapolation integrates out everything except pions, can’t reliably be applied to these data
Challenges of chiral extrapolation

Without reliable chiral extrapolation we can only estimate

\[ M_H \sim \text{few hundred GeV}, \text{with large error bars} \]

Much lighter than scaled-up QCD, still somewhat far from 125 GeV

Of course, we **shouldn’t** get exactly 125 GeV
since we haven’t yet incorporated electroweak & top corrections

These reduce \( M_H \),
but not yet consensus on size of effect. . .
Emerging picture of near-conformal spectrum

Light scalar likely related to near-conformal dynamics
(unconfirmed interpretation as PNGB of approx. scale symmetry)

QCD-like chiral breaking

$M_\rho$
$M_{TT}$
$M_{0^{++}}$

Conformal hyperscaling

Chirally broken, near conformal

$M_{0^{++}}$ light relative to $M_\rho$

(Anna Hasenfratz)
Scale setting & electroweak effective theory

Let’s review the standard approach impeded by the light Higgs

Integrating out resonances around $4\pi v$ scale gives chiral lagrangian

Dynamical d.o.f. are Goldstones $\pi^a$ to be eaten by W and Z, which appear through matrix field $U \equiv \exp \left[ 2iT^a\pi^a/F \right]$

$$\mathcal{L}_{LO} = \frac{F^2}{4} \text{Tr} \left[ D_\mu U^\dagger D^\mu U \right] + \frac{F^2 B}{2} \text{Tr} \left[ m \left( U + U^\dagger \right) \right]$$

Decay constant $F$ sets electroweak scale, W & Z masses

$F = \nu = 246$ GeV in simplest case (one electroweak doublet)

Chiral condensate $\langle \bar{\psi}\psi \rangle \propto F^2 B$ related to fermion mass generation

$\rightarrow$ large $\langle \bar{\psi}\psi \rangle / F^3 \propto B/F$ helps to satisfy FCNC constraints
Chiral condensate enhancement

Three dimensionless ratios all approach \( \langle \bar{\psi} \psi \rangle / F^3 \) in the chiral limit

\[
X^{(FM)} = \frac{M_P^2}{2m F_P} \quad X^{(CM)} = \frac{(M_P^2/2m)^{3/2}}{\langle \bar{\psi} \psi \rangle^{1/2}} \quad X^{(FM)} = \frac{\langle \bar{\psi} \psi \rangle}{F_P^3}
\]

Condensate enhancement relative to \( N_f = 2 \) through “ratios of ratios”

Renormalized \( R_{\text{MS}} \approx 1.6 \) in chiral limit for both \( N_F = 6 \) and \( N_F = 8 \)
Electroweak chiral lagrangian NLO terms

With $X \equiv U \tau_3 U^\dagger$ and $V_\mu \equiv (D_\mu U) U^\dagger$, next-to-leading order includes oblique corrections $S \propto \alpha_1$, $T \propto \beta_1$, $U \propto \alpha_8$

triple gauge vertices and dominant contributions to $WW$ scattering

\[
\begin{align*}
\mathcal{L}_1 &= \frac{\alpha_1}{2} g_1 g_2 B_{\mu\nu} \text{Tr} (X W_{\mu\nu}) \\
\mathcal{L}_3 &= i \alpha_3 g_2 \text{Tr} (W_{\mu\nu} [V^\mu, V^\nu]) \\
\mathcal{L}_5 &= \alpha_5 \{\text{Tr} (V_\mu V^\mu)\}^2 \\
\mathcal{L}_7 &= \alpha_7 \text{Tr} (V_\mu V^\mu) \text{Tr} (X V_\mu) \text{Tr} (X V^\nu) \\
\mathcal{L}_9 &= \frac{i \alpha_9}{2} g_2 \text{Tr} (X W_{\mu\nu}) \text{Tr} (X [V^\mu, V^\nu]) \\
\mathcal{L}_{11} &= \alpha_{11} g_2 \epsilon^{\mu\nu\rho\lambda} \text{Tr} (X V_\mu) \text{Tr} (V_\nu W_{\rho\lambda}) \\
\mathcal{L}_2 &= \frac{i \alpha_2}{2} g_1 B_{\mu\nu} \text{Tr} (X [V^\mu, V^\nu]) \\
\mathcal{L}_4 &= \alpha_4 \{\text{Tr} (V_\mu V^\nu)\}^2 \\
\mathcal{L}_6 &= \alpha_6 \text{Tr} (V_\mu V^\nu) \text{Tr} (X V^\mu) \text{Tr} (X V^\nu) \\
\mathcal{L}_8 &= \frac{\alpha_8}{4} g_2^2 \{\text{Tr} (X W_{\mu\nu})\}^2 \\
\mathcal{L}_{10} &= \frac{\alpha_{10}}{2} \{\text{Tr} (X V_\mu) \text{Tr} (X V^\nu)\}^2 \\
\mathcal{L}'_1 &= \frac{\beta_1}{4} g_2^2 F^2 \{\text{Tr} (X V_\mu)\}^2
\end{align*}
\]

Simplest analysis is for $WW$ scattering parameters $\alpha_4$ and $\alpha_5$
WW scattering from the lattice — The Big Picture

WW scattering is the most direct probe of EWSB dynamics, though not the easiest to study at the LHC.

Lattice calculations restricted to low-energy scattering
WW scattering from the lattice — EFT matching

— Hadronic chiral lagrangian has $m > 0$ and $g = 0$
— Electroweak chiral lagrangian has $m = 0$ and $g > 0$
Both reduce to same form in the limit $m \to 0$ and $g \to 0$

\[
\begin{align*}
\text{Hadronic} & \hspace{2cm} \text{EW} \\
EFT & \hspace{2cm} EFT
\end{align*}
\]

\[
m_d \to 0
\]

\[
p^2 \ll M_{ds}^2, M_{ss}^2
\]

\[
\frac{f^2}{4} \text{tr}(\partial_\mu U^\dagger \partial^\mu U) + \alpha_5 \left[ \text{tr}(\partial_\mu U^\dagger \partial^\mu U) \right]^2 + \alpha_4 \left[ \text{tr}(\partial_\mu U^\dagger \partial_\nu U) \right]^2
\]

\[
g, g' \to 0
\]

\[
p^2 \ll M_{ds}^2, M_{ss}^2
\]
Pseudoscalar scattering on the lattice — goal

“Maximal isospin” channel \((I = 2\) for \(N_F = 2\))

Focus on S-wave scattering of identical charged pseudoscalars
\[\rightarrow\] simplest and cleanest scattering process

- Other isospin channels (e.g., \(I = 0\)) involve disconnected diagrams

- Other spin channels (e.g., D-wave) have smaller signals, require higher precision

We want to extract the LECs \(\ell_1\) and \(\ell_2\) related to \(\alpha_4\) and \(\alpha_5\) in \(\mathcal{L}_\chi\)

These hide in the low-energy scattering length \(a_{PP}\)
Pseudoscalar scattering on the lattice — procedure

Recall Maiani & Testa (1990)

No asymptotically non-interacting states in euclidean spacetime

\[ \rightarrow \text{ usual LSZ scattering formalism inapplicable} \]

In a finite volume, measure energy of two-pseudoscalar state $E_{PP}$, projecting each correlator to zero momentum for S-wave scattering

Access scattering phase shift $\delta$ from energy shift $\Delta E_{PP}$ (Lüscher, 1986)

\[
\Delta E_{PP} = E_{PP} - 2M_P = 2\sqrt{|k|^2 + M_P^2} - 2M_P
\]

\[
|k| \cot \delta = \frac{1}{\pi L} \left[ \sum_{j \neq 0} \frac{1}{|j|^2 - |k|^2 L^2 / (4\pi^2)} - 4\pi \Lambda_j \right] \quad \text{(regularized $\zeta$ func.)}
\]

Low-energy **scattering length** from $|k| \cot \delta = \frac{1}{a_{PP}} + \mathcal{O} \left( \frac{|k|^2}{M_P^2} \right)$
Joint chiral fit to $M_P^2$, $F_P$, $\langle \bar{\psi}\psi \rangle$ and $M_P a_{PP}$

- $a_{PP} \approx 1/|\vec{k}| \cot \delta$
- $\langle \bar{\psi}\psi \rangle$ plot in backup slide
- Only $N_F = 2$ fit feasible
- Fit restricted to solid points, $0.01 \leq m_f \leq 0.02$
- $\chi^2/dof = 83/6$
$N_F = 2$ WW scattering parameters from NLO chiral fit

Joint NLO chiral fit predicts sum of hadronic LECs $\ell_1 + \ell_2$

EFT matching discussed above relates this to the sum $\alpha_4 + \alpha_5$
(matching involves one-loop standard model calculation)

$$\alpha_4 + \alpha_5 = \left(3.34 \pm 0.17^{+0.08}_{-0.71}\right) \times 10^{-3} - \frac{1}{128\pi^2} \left[ \log \left(\frac{M^2_H}{v^2}\right) + \mathcal{O}(1)_{SM} \right]$$

(dominant systematic error from chiral fit range)

Context for our $N_F = 2$ result

Unitarity bounds [hep-ph/0604255]:
$$\alpha_4 + \alpha_5 \geq 1.14 \times 10^{-3} \quad \alpha_4 \geq 0.65 \times 10^{-3}$$

Expected LHC bounds [hep-ph/0606118]: (99% CL; 100/fb; 14 TeV)
$$-7.7 < \alpha_4 \times 10^3 < 15 \quad -12 < \alpha_5 \times 10^3 < 10$$
Complications for $N_F > 2$

- As for the $S$ parameter, only charge one chiral doublet $d$.
  Here we take the other $N_F - 2$ to be electroweak singlets $s$,
  leading to $N_F^2 - 4$ pseudoscalars with masses $M_{ds}$ and $M_{ss}$.

- Hadronic chiral perturbation theory ($\chi$PT) now involves 9 LECs
  with more complicated relations to $\alpha_4$ and $\alpha_5$.

- Higher-order terms in $\chi$PT increase with $N_F$.
  Leads to smaller radius of convergence.
Strategy: Reorganize chiral expansion

Replace low energy constants $2mB$ and $F$ by measured $M_P$ and $F_P$

Expansion parameter $\propto M_P^2/F_P^2$, leading order is $M_P a_{PP} = -\frac{M_P^2}{16\pi F_P^2}$

--- An old story in QCD (Weinberg, 1966)

--- Allows direct comparison between $N_F = 2$ and $N_F = 6$ LECs

$N_F = 6$ scattering length only slightly smaller, but chiral logs differ...
Possible enhancement of WW scattering for \( N_F = 6 \)

Combined LEC \( b'_{PP} \) must increase from \( N_F = 2 \) to \( N_F = 6 \), to get similar \( a_{PP} \) despite different chiral logs

\[
b'_{PP} = -256\pi^2 [L_0 + 2L_1 + 2L_2 + L_3 - 2L_4 - L_5 + 2L_6 + L_8]
\]

contains \( \alpha_4 \) and \( \alpha_5 \), but we aren’t able to isolate them

\[
b'_{PP} = -4.67 \pm 0.65^{+1.08}_{-0.05} \quad [2f] ; \quad b'_{PP} = -7.81 \pm 0.46^{+1.23}_{-0.56} \quad [6f]
\]
Recapitulation and outlook

- New strong dynamics related to electroweak symmetry breaking must behave unlike QCD
- Lattice calculations crucial to explore range of possibilities
- Focus on near-conformal gauge theories $\rightarrow$ SU(3) with $N_F = 8$

Effects of increasing $N_F$ compared to scaled-up QCD

- Evidence for dynamical reduction of electroweak $S$ parameter
- Higgs boson is dramatically lighter, degenerate with PNGBs for currently accessible masses
- Chiral condensate ratio $\langle \bar{\psi} \psi \rangle / F^3$ is significantly enhanced
- WW scattering parameters possibly enhanced

Most pressing direction being pursued is to extend chiral effective theory to include a light scalar
Thank you!
Thank you!

Collaborators

Funding and computing resources

[Logos of funding and computing resources]
Backup: Lattice gauge theory in a nutshell

Lattice gauge theory is a fully non-perturbative and gauge-invariant regularization of quantum field theory (QFT).

Any QFT observable is formally

$$\langle O \rangle = \frac{1}{Z} \int D\Phi \ O(\Phi) \ e^{-S[\Phi]}$$

...but this is an infinite-dimensional integral.

Regularize the theory by formulating it in a finite, discrete spacetime → **the lattice**

- Work in Euclidean spacetime (Wick rotation)
- Spacing between lattice sites ("a") introduces UV cutoff scale $1/a$
Any QFT observable is formally
\[ \langle O \rangle = \frac{1}{Z} \int \mathcal{D}\Phi \ O(\Phi) \ e^{-S[\Phi]} \]

... but this is an infinite-dimensional integral

Regularize the theory by formulating it in a finite, discrete spacetime \( \rightarrow \text{the lattice} \)

Work in Euclidean spacetime (Wick rotation)

Spacing between lattice sites ("a") introduces UV cutoff scale \(1/a\)

Lattice cutoff preserves hypercubic subgroup of full Lorentz symmetry

Remove cutoff by taking continuum limit \(a \rightarrow 0\) (with \(L/a \rightarrow \infty\))
Backup: Numerical lattice gauge theory calculations

\[ \langle O \rangle = \frac{1}{Z} \int D\Phi \ O(\Phi) \ e^{-S[\Phi]} \]

Finite-dimensional integral \(\implies\) we can compute \(\langle O \rangle\) numerically

**Importance sampling Monte Carlo**

Approximate integral with a finite ensemble of field configurations \(\{\Phi_i\}\)

Algorithms choose each configuration \(\Phi_i\) with probability \(\frac{1}{Z} e^{-S[\Phi_i]}\)

to find those that make the most important contributions

Then \(\langle O \rangle = \frac{1}{N} \sum_{i=1}^{N} O(\Phi_i)\) with statistical uncertainty \(\propto \sqrt{\frac{1}{N}}\)

Generating ensembles \(\{\Phi_i\}\) often dominates computational costs

These saved data can be reused to investigate many observables
Backup: Hybrid Monte Carlo (HMC) algorithm

Recall goal: Sample field configurations $\Phi_i$ with probability $\frac{1}{Z} e^{-S[\Phi_i]}$

HMC is a Markov process, based on Metropolis–Rosenbluth–Teller (MRT)

Fermions $\rightarrow$ extensive action computation, so best to update entire system at once

Use fictitious molecular dynamics evolution

1. Introduce a fictitious fifth dimension (“MD time” $\tau$) and stochastic canonical momenta for all field variables

2. Run inexact MD evolution along a trajectory in $\tau$ to generate new four-dimensional field configuration

3. Apply MRT accept/reject test to MD discretization error
Backup: Gradient flow scale setting

Gradient flow scale $\sqrt{8t_0}$ defined by condition $t^2 \langle E(t) \rangle \bigg|_{t=t_0} = c$

For both $N_F = 8$ domain wall (left) and staggered (right)

- $c \lesssim 0.3$ may be affected by discretization artifacts
- $c \gtrsim 0.3$ leads to significant mass dependence
Backup: A bit about the Wilson flow

Evolution of gauge links $U(x, \mu)$ in a “flow time” $t$:

$$
\frac{d}{dt} V_t(x, \mu) = -g_0^2 \left[ \frac{\delta}{\delta V_t(x, \mu)} S_W(V_t) \right] V_t(x, \mu),
$$

where $V_{t=0}(x, \mu) = U(x, \mu)$ and $S_W$ is the Wilson gauge action

$$
S_W(U) = \frac{2N}{g_0^2} \sum_{\{P\}} \text{ReTr} [1 - P(U)]
$$

Solution:

$$
V_t(x, \mu) = \exp \left[ -t g_0^2 \frac{\delta}{\delta U(x, \mu)} S_W(U) \right] U(x, \mu)
$$

$\Rightarrow$ numerical integration of infinitesimal stout smearing steps
Backup: Electroweak vacuum polarization functions

\[ \gamma \sim \sim \sim \gamma = i g_1 g_2 \cos \theta_w \sin \theta_w \Pi_{ee} \delta_{\mu\nu} + \ldots \]

\[ Z \sim \sim \sim \gamma = i g_1 g_2 \left( \Pi_{3e} - \sin^2 \theta_w \Pi_{ee} \right) \delta_{\mu\nu} + \ldots \]

\[ Z \sim \sim \sim Z = \frac{i g_1 g_2}{\cos \theta_w \sin \theta_w} \left( \Pi_{33} - 2 \sin^2 \theta_w \Pi_{3e} + \sin^4 \theta_w \Pi_{ee} \right) \delta_{\mu\nu} + \ldots \]

\[ W \sim \sim \sim W = i g_2^2 \Pi_{11} \delta_{\mu\nu} + \ldots \]

\[ \Pi_{VV} = 2\Pi_{3e} \]

\[ \Pi_{AA} = 4\Pi_{33} - 2\Pi_{3e} \]

\[ S = 4\pi N_D \lim_{Q^2 \to 0} \frac{d}{dQ^2} \left[ \Pi_{VV}(Q^2) - \Pi_{AA}(Q^2) \right] - \Delta S_{SM}(M_H) \]
Backup: Scaling up QCD gives $S \gtrsim 0.4$

$N_F \geq 2$ fermions in fundamental rep of SU($N$) for $N \geq 3$, with $1 \leq N_D \leq N_F/2$ doublets given chiral electroweak charges

$$S \approx 0.3 \frac{N_F N}{2} \frac{N}{3} + \frac{N_D - 1}{12\pi} \log \left( \frac{M_V^2}{M_P^2} \right) + \frac{1}{12\pi} \log \left( \frac{\sim \text{TeV}^2}{M_H^2} \right)$$

1. **Resonance contribution** uses QCD phenomenology to model $R(s)$

$$4\pi \lim_{Q^2 \to 0} \frac{d}{dQ^2} \Pi_{VA}(Q^2) = \frac{1}{3\pi} \int_0^\infty ds \frac{ds}{s} [R_V(s) - R_A(s)]$$

(essentially single-pole dominance with large-$N$ scaling)

2. **Chiral-log contribution** based on leading-order chiral pert. theory

3. 125 GeV Higgs contributes $\sim 0.1$ (leading-order estimate)

There is some subtlety regarding $M_H$ (cf. arXiv:1211.1083) for strong sector in isolation (no EW or radiative corrections)
Backup: Conserved and local domain wall currents

Conserved currents are point-split and summed over fifth dimension:

\[ V_{\mu}^a(x) = \sum_{s=0}^{L_s-1} j_{\mu}^a(x, s) \]
\[ A_{\mu}^a(x) = \sum_{s=0}^{L_s-1} \text{sign} \left( s - \frac{L_s - 1}{2} \right) j_{\mu}^a(x, s) \]

\[ j_{\mu}^a(x, s) = \overline{\Psi}(x + \hat{\mu}, s) P_{+\mu} \tau^a U^\dagger_{x,\mu} \Psi(x, s) - \overline{\Psi}(x, s) P_{-\mu} \tau^a U_{x,\mu} \Psi(x + \hat{\mu}, s) \]

where \( P_{\pm\mu} \equiv \frac{1 \pm \gamma_\mu}{2} \)

Local currents are constructed from boundaries of fifth dimension:

\[ V_{\mu}^a(x) = \overline{q}(x) \gamma_{\mu} \tau^a q(x) \]
\[ A_{\mu}^a(x) = \overline{q}(x) \gamma_{\mu} \gamma_5 \tau^a q(x) \]

\[ q(x) = \frac{1 - \gamma_5}{2} \psi(x, 0) + \frac{1 + \gamma_5}{2} \psi(x, L_s - 1) \]
Backup: Non-conservation of local currents

\[ \Pi_{V-A}^{\mu\nu}(Q) = Z \sum_x e^{iQ \cdot (x + \mu/2)} \text{Tr} \left[ \left\langle V^{\mu a}(x) V^{\nu b}(0) \right\rangle - \left\langle A^{\mu a}(x) A^{\nu b}(0) \right\rangle \right] \]

Local currents are simply \( \bar{q} \gamma_\mu q \) defined on the domain walls

No Ward identity: \( \hat{Q}_\mu \left[ \sum_x e^{iQ \cdot x} \left\langle V^{a}(x) V^{a}(0) \right\rangle \right] \neq 0 \)
Backup: Ward identity for conserved currents

$$\Pi_{V-A}^{\mu\nu}(Q) = Z \sum_x e^{iQ \cdot (x + \mu/2)} \text{Tr} \left[ \left< V^a(x) V^{\nu b}(0) \right> - \left< A^a(x) A^{\nu b}(0) \right> \right]$$

Conserved currents are point-split, summed over fifth dimension

Obey Ward identity, PCAC:

$$\hat{Q}_\mu \left[ \sum_x e^{iQ \cdot (x + \mu/2)} \left< V^a(x) V^{a}(0) \right> \right] = 0$$
Backup: Lattice artifacts cancel in mixed correlators

Plot shows divergence of local current in each correlator,

\[
\sum_x e^{iQ \cdot (x + \vec{\mu}/2)} \langle \mathcal{V}_\mu^a(x) \mathcal{V}_\nu^a(0) \rangle \cdot \hat{Q}_\nu \neq 0 \text{ for each } \nu
\]

Cancellation seems due to conserved currents forming exact multiplet, also possible with overlap — even staggered (Y. Aoki @ Lattice 2013)
Backup: Finite-volume diagnostic plot

Arrows show direction of decreasing mass

Expect finite-volume effects to push points up and to the right

Finite-volume effects may be significant for lightest $N_F = 6$ point
Backup: Spurious $S \rightarrow 0$ from finite volume effects

Compare $N_F = 6$ results on $16^3 \times 32$ and $32^3 \times 64$ lattice volumes

$L = 16$ slopes $4\pi \Pi'_{V-A}(0)$ crash to zero as $m \rightarrow 0$
attributable to spurious parity doubling from finite-volume effects

Simultaneously finite-volume effects freeze $M_P L \approx 5.5$

$L = 32$ results show no such effects in $M_P L$,
even for lightest $N_F = 6$ point where $M_P / F_P$ increases
**Backup: Padé fit $Q^2$-range dependence**

Uncorrelated fits to “Padé-(1, 2)” rational function with $\chi^2$/dof $\ll 1$

\[
\Pi_{V-A}(Q^2) = \frac{a_0 + a_1 Q^2}{1 + b_1 Q^2 + b_2 Q^4} = \frac{\sum_{m=0}^{1} a_m Q^{2m}}{1 + \sum_{n=1}^{2} b_n Q^{2n}}
\]

Results reported above use $Q_{\text{Max}}^2 = 0.4$
Twisted boundary conditions (TwBCs)

- Introduce external abelian field (add phase at lattice boundaries)
- Allows access to arbitrary $Q^2$, not just lattice modes $2\pi n/L$

Correlations $\implies$ TwBCs do not improve Padé fit results for slope

May help connect to chiral perturbation theory, where we need both small $M_P$ and small $Q^2$
Backup: Chiral perturbation theory for $\Pi_{V-A}(Q^2)$

$\Pi_{V-A}(Q^2)$ in NLO hadronic $\chi$PT:

$$
\Pi_{V-A}(M_{dd}^2, Q^2) = - F_P^2 - Q^2 \left[ 8 L_{10}^r(\mu) + \frac{1}{24\pi^2} \left\{ \log \left[ \frac{M_{dd}^2}{\mu^2} \right] + \frac{1}{3} \right. \right. \\
\left. \left. - H \left( \frac{4M_{dd}^2}{Q^2} \right) \right\} \right]
$$

$$H(x) = (1 + x) \left[ \sqrt{1 + x} \log \left( \frac{\sqrt{1 + x} - 1}{\sqrt{1 + x} + 1 + 2} \right) \right]
$$

Match with $S = -16\pi^2\alpha_1$ in electroweak chiral lagrangian:

$$S(\mu, M_{ds}) = \frac{1}{12\pi} \left[ -192\pi^2 \left( L_{10}^r(\mu) + \frac{1}{384\pi^2} \left\{ \log \left[ \frac{M_{ds}^2}{\mu^2} \right] + 1 \right\} \right) \right. \right. \\
\left. \left. + \log \left[ \frac{\mu^2}{M_H} \right] - \frac{1}{6} \right] \right.$$. 
Backup: Eight-flavor spectrum in dimensionless ratios

Preliminary results still in lattice units

Scale setting suggests resonance masses $\sim 2$–3 TeV

Large separation between Higgs and resonances

Higgs degenerate with pseudo-Goldstones in accessible regime

Dramatically different from QCD-like dynamics, where $M_H \approx 2M_P$ in this regime (dominated by two-pion scattering)

Typical chiral extrapolation integrates out everything except pions, can’t reliably be applied to these data
Backup: NLO chiral expansions

For general $N_F$, $A = 2 - N_F + 2N_F^2 + N_F^3$

\[
M_Pa_{PP} = -\frac{2mB}{16\pi F^2} \left\{ 1 + \frac{2mB}{(4\pi F)^2} \left[ b_{PP} - 2\frac{N_F - 1}{N_F^2} + \frac{A}{N_F^2} \log \left( \frac{2mB}{\mu^2} \right) \right] \right\}
\]

\[
M_P^2 = 2mB \left\{ 1 + \frac{2mB}{(4\pi F)^2} \left[ b_M + \frac{1}{N_F} \log \left( \frac{2mB}{\mu^2} \right) \right] \right\}
\]

\[
F_P = F \left\{ 1 + \frac{2mB}{(4\pi F)^2} \left[ b_F - \frac{N_F}{2} \log \left( \frac{2mB}{\mu^2} \right) \right] \right\}
\]

\[
\langle \bar{\psi}\psi \rangle = \frac{F^22mB}{2m} \left\{ 1 + \frac{2mB}{(4\pi F)^2} \left[ b_C - \frac{N_F^2 - 1}{N_F} \log \left( \frac{2mB}{\mu^2} \right) \right] \right\}
\]

- LECs $b$ are all linear combinations of low-energy constants $L_i$
- LECs’ dependence on scale $\mu$ cancels the corresponding logs
- $b_C$ includes “contact term” $\sim m/a^2$
- NNLO $M_P^2$ coefficients enhanced by $N_F^2$

(arXiv:0910.5424)
“Contact term” \( \sim m/a^2 \) clearly dominant (straight lines)

\( N_F = 2 \) joint NNLO\( \chi \)PT fit including \( F_P, \ M_P^2 \) and \( \langle \bar{\psi}\psi \rangle \)
Backup: $\langle \bar{\psi} \psi \rangle$ in three ways for $N_F = 12$

The chiral condensate directly probes chiral symmetry, but this is explicitly broken by non-zero fermion mass on lattice

"Direct" $\langle \bar{\psi} \psi \rangle$
uses $m_{valence} = m_{sea}$

$\Sigma$ measured from $m_v = 0$ eigenmodes

Partially quenched with $m_v \to 0$

"Contact term" $\sim m_v / a^2$ clearly dominates,
may lead to poorly controlled chiral extrapolation
Backup: Fermion mass dependence of $\langle \bar{\psi} \psi \rangle$

$\langle \bar{\psi} \psi \rangle$ depends on both valence mass $m_v$ and sea mass $m_s$

Eigenspectrum $\rho(\lambda)$ of massless Dirac operator depends only on $m_s$

\[
\langle \bar{\psi} \psi \rangle_{m_v; m_s} = m_v \int \frac{\rho(\lambda, m_s)}{\lambda^2 + m_v^2} d\lambda + m_v^5 \int \frac{\rho(\lambda, m_s)}{(\lambda^2 + m_v^2)^2} \lambda^4 d\lambda
\]
\[+ \gamma_1 m_v \Lambda^2 + \gamma_2 m_v + O(1/\Lambda)\]

where $\Lambda = a^{-1}$ is the UV cutoff

(Leutwyler & Smilga)

Quadratic UV divergence complicates chiral extrapolation

Can address with partially-quenched ($m_v \neq m_s$) measurements, to extrapolate $m_v \rightarrow 0$ with fixed $m_s$

Can also remove $m_v$ dependence via $\Sigma m_s = \pi \rho(0, m_s) = \langle \bar{\psi} \psi \rangle_{m_v=0; m_s}$

It is a good check that these two approaches agree!
Backup: Dependence on gauge coupling for $N_F = 12$

Look at simple ratio $M_V/M_P$

plotted against relevant parameter (fermion mass $m \sim M_P$)

Even though $\beta_F$ is formally irrelevant

it has significant effects for $M_P \gtrsim 0.2a^{-1}$
Backup: Thermal transitions to identify $S\chi_{SB}$

May distinguish between chirally broken and IR-conformal cases from scaling $\Delta \beta_F$ of finite-temperature transitions as $N_T$ increases.

Plots show transitions and some RG flow lines in space of fermion mass $m$ and gauge coupling $\beta_F$.

Contrast only clear near critical surface at $m = 0$.
Search for $N_F = 8$ spontaneous $\chi$SB

QCD-like scaling at large $m \gtrsim 0.01$ does not persist as $m$ decreases.

Thermal transitions run into lattice phase before reaching chiral limit.

Even large lattice volumes up to $48^3 \times 24$ are insufficient to establish spontaneous chiral symmetry breaking.
Backup: Search for $N_F = 8$ spontaneous $\chi$SB

Extrapolating $m \to 0$ at fixed $\beta_F = 4.7$ suggests $N_T \gtrsim 48$ needed to establish spontaneous chiral symmetry breaking.

This behavior is extremely different from QCD but does not necessarily imply IR conformality.
Backup: Sample $N_F = 8$ transition signals

Need $N_T = 20$ to observe chirally broken phase at $m = 0.005$
Backup: Order parameters for $S^4$ phase

Staggered lattice actions possess exact single-site shift symmetry which is spontaneously broken in a novel lattice phase we encountered.

Order parameters (any or all $\mu$)

\[
\Delta P_\mu = \langle \text{ReTr} \, \square n - \text{ReTr} \, \square n+\mu \rangle_{n_{\mu} \text{ even}}
\]

\[
\Delta L_\mu = \langle \alpha_{\mu,n} \overline{\chi}_n U_{\mu,n} \chi_{n+\mu} - \alpha_{\mu,n+\mu} \overline{\chi}_{n+\mu} U_{\mu,n+\mu} \chi_{n+2\mu} \rangle_{n_{\mu} \text{ even}}
\]

$S^4$ likely non-universal, though other groups see same phase structure.
Backup: Thermal transitions for $N_F = 12$

Behave as expected for an IR-conformal system

Accumulate at zero-temperature bulk transition for small enough $m$

$N_T = 12$ and $N_T = 16$ transitions are indistinguishable
In addition to a scale \( \sqrt{8t_0} \),
the gradient flow defines a scale-dependent running coupling \( g_c^2(L; a) \).

Recall: The gradient flow integrates an infinitesimal smoothing operation.
Local observables measured after “flow time” \( t \) depend on original fields within \( r \sim \sqrt{8t} \).

Perturbatively \( g_{\text{MS}}^2(\mu) \propto t^2 E(t) \) with \( \mu = 1/\sqrt{8t} \).
where \( E = -\frac{1}{2} \text{Tr} [G_{\mu\nu} G^{\mu\nu}] \) is the energy density.

Define running coupling \( g_c^2(L; a) \) by fixing \( c = L/\sqrt{8t} \).
Backup: Discrete $\beta$ function for $N_F = 8$

Continuum extrapolated $\beta_s(g_c^2)$ with scale change $s = 3/2$
 increases monotonically for $g_c^2 \lesssim 14$

Although $\beta_s$ is even smaller than IR-conformal four-loop $\overline{\text{MS}}$ prediction
 any IR fixed point must be at stronger coupling

![Graph showing $N_f = 8$, $c = 0.25$, $\beta_{3/2}$, and $u$ values]
Backup: Scale-dependent $\gamma_{\text{eff}}(\lambda)$ from eigenmodes

$\lambda$ defines an energy scale $\implies$ fitting $\nu(\lambda)$ predicts effective anomalous dimension $\gamma_{\text{eff}}(\lambda)$ at that scale.

For IR-conformal systems

**UV:** Asymp. freedom $\implies \gamma_{\text{eff}}(\lambda) \rightarrow 0$

corresponding to $\alpha(\lambda) \rightarrow 3$

**IR:** Fixed point $\implies \gamma_{\text{eff}}(\lambda) \rightarrow \gamma^*_m$

$\gamma^*_m$ scheme-independent
typically expect $\gamma^*_m \lesssim 1$

Ideally monitor evolution from perturbative UV to strongly coupled IR

$\lambda$
Backup: $\gamma_{\text{eff}}(\lambda)$ from eigenmodes for $N_F = 8$

Fit $\nu(\lambda) \propto \lambda^{1+\alpha}$ in a limited range of $\lambda$ to find $1 + \gamma_{\text{eff}}(\lambda) = \frac{4}{1 + \alpha(\lambda)}$

$\nu(\lambda)$ computed stochastically

Fit ranges included in error bands

Only retain regions where all volumes overlap

All systems have $m = 0$ and $\rho(0) = 0$

Behaves very differently compared to either $N_F = 12$ or QCD

$\gamma_{\text{eff}}$ appears to evolve very slowly across a wide range of scales
Backup: $\gamma_{\text{eff}}(\lambda)$ for chirally broken systems

$\lambda$ defines an energy scale

Fitting $\nu(\lambda) \propto \lambda^{1+\alpha(\lambda)}$ accesses $1 + \gamma_{\text{eff}}(\lambda) = \frac{4}{1+\alpha(\lambda)}$ at that scale

For chirally broken systems

**UV:** Asymp. freedom $\Rightarrow \gamma_{\text{eff}}(\lambda) \to 0$

Corresponding to $\alpha(\lambda) \to 3$

**IR:** $\langle \bar{\psi} \psi \rangle \propto \rho(0) > 0 \Rightarrow \alpha(\lambda) \to 0$

Would produce $\gamma_{\text{eff}}(\lambda) \to 3$

But $\rho(\lambda)$ no longer $\propto \lambda^\alpha$

Ideally monitor evolution from perturbative UV to chirally broken IR
As discussed above, $\langle \bar{\psi} \psi \rangle \propto \rho(\lambda \to 0) > 0 \implies \gamma_{\text{eff}} \uparrow 3$, but scaling $\rho(\lambda) \propto \lambda^\alpha$ breaks down in this situation.

Finite-volume effects can produce a “gap” with $\rho(0) = 0$.
This is a different breakdown of the scaling, leading to $\gamma_{\text{eff}} \downarrow 0$.

Both of these effects are unphysical and we remove the finite-volume transients from most $\gamma_{\text{eff}}$ plots.
Backup: $\gamma_{\text{eff}}(\lambda)$ for QCD-like $N_F = 4$

Fit $\nu(\lambda) \propto \lambda^{1+\alpha}$ in a limited range of $\lambda$ to find $1 + \gamma_{\text{eff}}(\lambda) = \frac{4}{1 + \alpha(\lambda)}$

1000 eigenvalues on each volume

Fit ranges included in error bands

Only retain results free from finite-volume effects

$m = 0$ except for chirally broken systems at $\beta_F = 6.6$ and 6.4

where $\gamma_{\text{eff}} \nearrow 2$ becomes unphysically large
Backup: Rescaled $\gamma_{\text{eff}}(\lambda)$ for QCD-like $N_F = 4$

- Rescale $\lambda \rightarrow \left(\frac{a_{7.4}}{a}\right)^{1+\gamma_{\text{eff}}(\lambda)}\lambda$ to plot with constant lattice spacing
- Relative lattice spacings from gradient flow & MCRG matching
- Match to one-loop perturbation theory at $\lambda \cdot a_{7.4} = 0.8$

![Graph showing universality of $\gamma_m$ vs $\lambda \cdot a_{7.4}$ for $N_f = 4$.]

Universal curve from $\chi_{\text{SB}}$ to asymptotic freedom

Strong test of method & control over systematics