Lattice Gauge Theory for $\mathcal{N} = 4$ Super Yang–Mills

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Lattice Gauge Theory for the LHC and Beyond
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with Simon Catterall, Poul Damgaard, Tom DeGrand and Joel Giedt
Plan

- Motivations for lattice supersymmetry in general

- Lattice formulation of $\mathcal{N} = 4$ supersymmetric Yang–Mills
  (new $O(a)$-improved action & public code)

- Latest results for static potential and Konishi anomalous dim.
  (confront with perturbation theory, AdS/CFT, bootstrap)

- Prospects and future directions
  (sign problem; superQCD in two & three dimensions)
Motivation: Why lattice supersymmetry

Much interesting physics in 4D susy gauge theories:
   dualities, holography, confinement, conformality, BSM, . . .

Lattice promises non-perturbative insights from first principles

We can brainstorm many potential susy applications:
- Compute Wilson loops, spectrum, scaling dimensions, etc.,
  going beyond perturbation theory, holography, bootstrap, . . .
- New non-perturbative tests of conjectured dualities
- Predict low-energy constants from dynamical susy breaking
- Validate or refine AdS/CFT-based models for QCD phase diagram, condensed matter systems, . . .
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  for QCD phase diagram, condensed matter systems, . . .

Many ideas probably infeasible; relatively few have been explored
Obstruction: Why not lattice supersymmetry

Recall supersymmetry extends Poincaré symmetry by spinorial generators $Q^I_\alpha$ and $\overline{Q}^I_{\dot{\alpha}}$ with $I = 1, \cdots, N$.

The super-Poincaré algebra includes $\{Q^I_\alpha, \overline{Q}^J_{\dot{\alpha}}\} = 2\delta^{IJ}\sigma^\mu_{\alpha\dot{\alpha}} P_\mu$

but infinitesimal translations don’t exist in discrete space-time

Explicitly broken supersymmetry $\implies$ relevant susy-violating operators

Typically many such operators, especially with scalar fields from matter multiplets or from $\mathcal{N} > 1$

Fine-tuning couplings / counterterms to restore supersymmetry is generally not practical in numerical lattice calculations
Solution: Exact susy on the lattice

Rapid progress in recent years

In certain systems (some subset of) the susy algebra can be exactly preserved at non-zero lattice spacing.

Equivalent constructions obtained from orbifolding / deconstruction and from “topological” twisting — cf. arXiv:0903.4881 for review.

In four dimensions these constructions pick out a unique system: maximally supersymmetric Yang–Mills ($\mathcal{N} = 4$ SYM).

$\mathcal{N} = 4$ SYM is a particularly interesting theory:

—AdS/CFT correspondence
—Testing ground for reformulations of scattering amplitudes
—Arguably simplest non-trivial field theory in four dimensions.
Basic features of $\mathcal{N} = 4$ SYM

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— Testing ground for reformulations of scattering amplitudes
— Arguably simplest non-trivial field theory in four dimensions

- SU($N$) gauge theory with four fermions $\psi^I$ and six scalars $\Phi^{IJ}$, all massless and in adjoint rep.

- Action consists of kinetic, Yukawa and four-scalar terms with coefficients related by symmetries

- Supersymmetric: 16 supercharges $Q^I_\alpha$ and $\overline{Q}^I_{\dot{\alpha}}$ with $I = 1, \cdots, 4$
  Fields and Q’s transform under global $SU(4) \simeq SO(6) \text{ R}$ symmetry

- Conformal: $\beta$ function is zero for any ’t Hooft coupling $\lambda = g^2 N$
An intuitive picture of topological twisting

\[
\begin{pmatrix}
Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\
\bar{Q}^1_{\dot{\alpha}} & \bar{Q}^2_{\dot{\alpha}} & \bar{Q}^3_{\dot{\alpha}} & \bar{Q}^4_{\dot{\alpha}}
\end{pmatrix}
= Q + Q_\mu \gamma_\mu + Q_{\mu\nu} \gamma_\mu \gamma_\nu + \bar{Q}_\mu \gamma_\mu \gamma_5 + \bar{Q} \gamma_5
\]

\[\rightarrow Q + Q_a \gamma_a + Q_{ab} \gamma_a \gamma_b\]

with \(a, b = 1, \ldots, 5\)

\(Q\)'s transform with integer spin under “twisted rotation group”

\[
SO(4)_{tw} \equiv \text{diag} \left[ SO(4)_\text{euc} \otimes SO(4)_R \right]
\]

\(SO(4)_R \subset SO(6)_R\)

This change of variables gives a susy subalgebra \(\{Q, \bar{Q}\} = 2Q^2 = 0\)

This subalgebra can be exactly preserved on the lattice
Twisted $\mathcal{N} = 4$ SYM fields and $\mathcal{Q}$

Everything transforms with **integer spin** under $\text{SO}(4)_{\text{tw}}$ — no spinors

- $Q^I_\alpha$ and $\overline{Q}^I_{\dot{\alpha}} \rightarrow Q$, $Q_a$ and $Q_{ab}$
- $\psi^I$ and $\overline{\psi}^I \rightarrow \eta$, $\psi_a$ and $\chi_{ab}$
- $A_\mu$ and $\phi^{IJ} \rightarrow A_a = (A_\mu, \phi) + i(B_\mu, \phi)$ and $\overline{A}_a$

Complexify gauge fields since scalars $\rightarrow$ vectors under twisting
- (complexification $\implies$ $U(N) = \text{SU}(N) \otimes \text{U}(1)$ gauge invariance)

The motivation is most obvious in five dimensions where

$$\text{SO}(5)_{\text{tw}} \equiv \text{diag} \left[ \text{SO}(5)_{\text{euc}} \otimes \text{SO}(5)_R \right]$$

Then dimensional reduction takes gauge fields $A_a \rightarrow (A_\mu, \phi)$
- and scalar fields $B_a \rightarrow (B_\mu, \phi)$
Twisted $\mathcal{N} = 4$ SYM fields and $Q$

Everything transforms with **integer spin** under $\text{SO}(4)_{tw} — \text{no spinors}$

- $Q^I_\alpha$ and $\overline{Q}^I_{\dot{\alpha}} \rightarrow Q, Q_a$ and $Q_{ab}$
- $\psi^I$ and $\overline{\psi}^I \rightarrow \eta, \psi_a$ and $\chi_{ab}$
- $A_\mu$ and $\Phi^{IJ} \rightarrow A_a = (A_\mu, \phi) + i(B_\mu, \overline{\phi})$ and $\overline{A}_a$

The twisted-scalar supersymmetry $Q$

- correctly interchanges bosonic $\leftrightarrow$ fermionic d.o.f. with $Q^2 = 0$

\[
\begin{align*}
Q A_a &= \psi_a \\
Q \chi_{ab} &= -\overline{F}_{ab} \\
Q \eta &= d \\
Q d &= 0
\end{align*}
\]

- bosonic auxiliary field with e.o.m. $d = \overline{D}_a A_a$
Lattice $\mathcal{N} = 4$ SYM

The lattice theory is nearly a direct transcription, despite breaking the 15 $Q_a$ and $Q_{ab}$

- Covariant derivatives $\rightarrow$ finite difference operators
- Complexified gauge fields $A_a \rightarrow$ gauge links $U_a \in \mathfrak{gl}(N, \mathbb{C})$

$$Q A_a \rightarrow Q U_a = \psi_a \quad Q \psi_a = 0$$

$$Q \chi_{ab} = -\bar{F}_{ab} \quad Q \bar{A}_a \rightarrow Q \bar{U}_a = 0$$

$$Q \eta = d \quad Q d = 0$$

Geometry manifest: $\eta$ and $d$ on sites, $U_a$ and $\psi_a$ on links, etc.

- Supersymmetric lattice action ($QS = 0$)
  follows from $Q^2 \cdot = 0$ and Bianchi identity

$$S = \frac{N}{2\lambda_{lat}} Q \left( \chi_{ab} F_{ab} + \eta \bar{D} a U_a - \frac{1}{2} \eta d \right) - \frac{N}{8\lambda_{lat}} \epsilon_{abcde} \chi_{ab} \bar{D} c \chi_{de}$$

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Five links in four dimensions $\rightarrow A_4^*$ lattice

Revisit dimensional reduction in discrete spacetime, treating all five $U_a$ symmetrically

— Start with hypercubic lattice in 5d momentum space

— **Symmetric** constraint $\sum_a \partial_a = 0$
  projects to 4d momentum space

— Result is $A_4$ lattice
  $\rightarrow$ dual $A_4^*$ lattice in real space
Twisted SO(4) symmetry on the $A_4^*$ lattice

—Can picture $A_4^*$ lattice as 4d analog of 2d triangular lattice

—Basis vectors are linearly dependent and non-orthogonal $\lambda = \lambda_{\text{lat}} / \sqrt{5}$

—Preserves $S_5$ point group symmetry

$S_5$ irreps precisely match onto irreps of twisted SO(4)$_{tw}$

\[
5 = 4 \oplus 1 : \quad \psi_a \rightarrow \psi_\mu, \quad \eta
\]

\[
10 = 6 \oplus 4 : \quad \chi_{ab} \rightarrow \chi_{\mu\nu}, \quad \bar{\psi}_\mu
\]

$S_5 \rightarrow \text{SO(4)}_{tw}$ in continuum limit restores the rest of $Q_a$ and $Q_{ab}$
Twisted $\mathcal{N} = 4$ SYM on the $A_4^*$ lattice

High degree of exact symmetry: gauge invariance + $Q + S_5$

Several important analytic consequences:

- Moduli space preserved to all orders of lattice perturbation theory $\rightarrow$ no scalar potential induced by radiative corrections
- $\beta$ function vanishes at one loop in lattice perturbation theory
- Real-space RG blocking transformations preserve $Q$ and $S_5$
- Only one log. tuning to recover $Q_a$ and $Q_{ab}$ in the continuum

Not quite suitable for numerical calculations

Exact zero modes and flat directions must be regulated, especially important in U(1) sector
\[ S = \frac{N}{2\lambda_{\text{lat}}} Q \left( \chi_{\text{ab}} F_{\text{ab}} + \eta \overline{D}_a U_a - \frac{1}{2} \eta d \right) - \frac{N}{8\lambda_{\text{lat}}} \epsilon_{\text{abcde}} \chi_{\text{ab}} \overline{D}_c \chi_{\text{de}} + \mu^2 V \]

Scalar potential \( V = \frac{1}{2N\lambda_{\text{lat}}} \left( \text{Tr} [U_a \overline{U}_a] - N \right)^2 \) lifts SU(N) flat directions and ensures \( U_a = I_N + A_a \) in continuum limit.

Breaks \( Q \) softly — susy breaking automatically vanishes as \( \mu^2 \to 0 \).

Violations of Ward identities \( \langle QO \rangle = 0 \) show \( Q \) breaking and restoration.

Here considering
\[ Q \left[ \eta U_a \overline{U}_a \right] = dU_a \overline{U}_a - \eta \psi_a \overline{U}_a \]
Full $\mathcal{N} = 4$ SYM lattice action

\[
S = \frac{N}{2\lambda_{\text{lat}}} Q \left( \chi_{ab} \mathcal{F}_{ab} + \frac{1}{2} \eta \mathcal{A} \right) - \frac{N}{8\lambda_{\text{lat}}} \epsilon_{abcde} \chi_{ab} \overline{D}_c \chi_{de} + \mu^2 V
\]

\[
\eta \left( \overline{D}_a U_a + G \sum_{a<b} [\det P_{ab} - 1] I_N \right)
\]

Constraint on plaquette det. lifts U(1) zero mode & flat directions

$Q$-exact implementation as new moduli space condition

Leads to $\langle QO \rangle \propto (a/L)^2$, much better than naive constraint

Looks like $O(a)$ improvement

($Q$ forbids all dim-5 operators)
so that the full improved action becomes

\[ S_{\text{imp}} = S'_{\text{exact}} + S_{\text{closed}} + S'_{\text{soft}} \]

\[ S'_{\text{exact}} = \frac{N}{2\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ -\bar{F}_{ab}(n)F_{ab}(n) - \chi_{ab}(n)\mathcal{D}_1^{(+)}\psi_b(n) - \eta(n)\mathcal{D}_1^{(-)}\psi_a(n) \right. \]

\[ + \frac{1}{2} \left( \mathcal{D}_1^{(-)}U_a(n) + G \sum_{a \neq b} (\det P_{ab}(n) - 1) \mathbb{1}_N \right)^2 \left. \right] - S_{\text{det}} \]

\[ S_{\text{det}} = \frac{N}{2\lambda_{\text{lat}}} G \sum_n \text{Tr} [\eta(n)] \sum_{a \neq b} [\det P_{ab}(n)] \text{Tr} \left[ U_b^{-1}(n)\psi_b(n) + U_a^{-1}(n + \hat{\mu}_b)\psi_a(n + \hat{\mu}_b) \right] \]

\[ S_{\text{closed}} = -\frac{N}{8\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ \epsilon_{abcd} \chi_{de}(n + \hat{\mu}_a + \hat{\mu}_b + \hat{\mu}_c)\mathcal{D}_1^{(-)}\chi_{ab}(n) \right], \]

\[ S'_{\text{soft}} = \frac{N}{2\lambda_{\text{lat}}} \mu^2 \sum_n \sum_a \left( \frac{1}{N} \text{Tr} \left[ U_a(n)\bar{U}_a(n) \right] - 1 \right)^2 \]

The full $\mathcal{N} = 4$ SYM lattice action is somewhat complicated

($\gtrsim 100$ gathers in the fermion operator)

To reduce barriers to entry our parallel code is publicly developed at

github.com/daschaich/susy

Evolved from MILC lattice QCD code, presented in arXiv:1410.6971
Application: Static potential is Coulombic at all $\lambda$

Static potential $V(r)$ from $r \times T$ Wilson loops: $W(r, T) \propto e^{-V(r)T}$

Fit $V(r)$ to Coulombic or confining form

$V(r) = A - C/r$

$V(r) = A - C/r + \sigma r$

$C$ is Coulomb coefficient

$\sigma$ is string tension

Fits to confining form always produce vanishing string tension $\sigma = 0$

More sophisticated analyses in development using improved action
Coupling dependence of Coulomb coefficient

Continuum perturbation theory predicts \( C(\lambda) = \frac{\lambda}{4\pi} + \mathcal{O}(\lambda^2) \)

AdS/CFT predicts \( C(\lambda) \propto \sqrt{\lambda} \) for \( N \to \infty, \lambda \to \infty, \lambda \ll N \)

\( N = 2 \) results agree with perturbation theory for all \( \lambda \lesssim N \)

\( N = 3 \) results bend down for \( \lambda \gtrsim 1 \) — approaching AdS/CFT?
Application: Konishi operator scaling dimension

$\mathcal{N} = 4$ SYM is conformal at all $\lambda$ → power-law decay for all correlation functions

The Konishi operator is the simplest conformal primary operator

$$\mathcal{O}_K = \sum_I \text{Tr} [\Phi^I \Phi^I]$$

$$C_K(r) \equiv \mathcal{O}_K(x + r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

There are many predictions for its scaling dim. $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$

- From weak-coupling perturbation theory, related to strong coupling by $\frac{4\pi N}{\lambda} \leftrightarrow \frac{\lambda}{4\pi N}$ S duality
- From holography for $N \to \infty$ and $\lambda \to \infty$ but $\lambda \ll N$
- Upper bounds from the conformal bootstrap program

Only lattice gauge theory can access nonperturbative $\lambda$ at moderate $N$
Konishi operator on the lattice

Extract scalar fields from polar decomposition of complexified links

\[ U_a \sim U_a(\mathbb{1}_N + \phi_a) \]

\[ \hat{O}_K = \sum_a \text{Tr} [\phi_a \phi_a] \]

\[ \overline{O}_K = \hat{O}_K - \langle \hat{O}_K \rangle \]

\[ C_K(r) = \overline{O}_K(x + r)\overline{O}_K(x) \propto r^{-2\Delta_K} \]

Obvious sensitivity to volume as desired for conformal system

Good tools to find \( \Delta_K \):
- Finite-size scaling
- Monte Carlo RG

Need lattice RG blocking scheme to carry out MCRG...
Real-space RG for lattice $\mathcal{N} = 4$ SYM

Lattice RG blocking transformation must preserve symmetries $\mathcal{Q}$ and $S_5 \leftrightarrow$ geometric structure of the system

Simple scheme constructed in arXiv:1408.7067:

\[
\begin{align*}
U'_a(x') &= \xi U_a(x)U_a(x + \hat{\mu}_a) \\
\psi'_a(x') &= \xi [\psi_a(x)U_a(x + \hat{\mu}_a) + U_a(x)\psi_a(x + \hat{\mu}_a)] \\
\eta'(x') &= \eta(x)
\end{align*}
\]

etc.

Doubles lattice spacing $a \rightarrow a' = 2a$, with $\xi$ a tunable rescaling factor

Set $\xi$ by equating plaquette on $n$-times-blocked $L^4$ ensemble with that on independent $(n-1)$-times-blocked $(L/2)^4$ ensemble

$\mathcal{Q}$-preserving RG blocking is necessary ingredient in derivation that only one log. tuning needed to recover $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ in the continuum
Scaling dimensions from MCRG stability matrix

Write system as (infinite) sum of operators, \( H = \sum_i c_i \mathcal{O}_i \)
with couplings \( c_i \) that flow under RG blocking transformation \( R_b \)

\( n \)-times-blocked system is \( H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)} \)

Fixed point defined by \( H^* = R_b H^* \) with couplings \( c_i^* \)

Linear expansion around fixed point defines stability matrix \( T_{ij}^* \)

\[
c_i^{(n)} - c_i^* = \sum_j \frac{\partial c_i^{(n)}}{\partial c_j^{(n-1)}} \bigg|_{H^*} \left( c_j^{(n-1)} - c_j^* \right) \equiv \sum_j T_{ij}^* \left( c_j^{(n-1)} - c_j^* \right)
\]

Correlators of \( \mathcal{O}_i, \mathcal{O}_j \) \( \rightarrow \) elements of stability matrix \( \) (Swendsen, 1979)

Eigenvalues of \( T_{ij}^* \) \( \rightarrow \) scaling dimensions of corresponding operators
Preliminary $\Delta K$ results from Monte Carlo RG

Far from bootstrap bounds

Aim to distinguish perturbative vs. free theory

Rough agreement between $N = 2, 3, 4$

Only statistical uncertainties so far, averaged over
- 1 & 2 RG blocking steps
- Blocked volumes $3^4$ through $8^4$
- 1–5 operators in stability matrix

More sophisticated analyses in development, while running larger volumes at stronger couplings
Practical question: Potential sign problem

In lattice gauge theory we compute operator expectation values

\[
\langle O \rangle = \frac{1}{Z} \int [dU][d\overline{U}] O \, e^{-S_B[U,\overline{U}]} \, pfD[U,\overline{U}]
\]

Pfaffian can be complex for lattice $\mathcal{N} = 4$ SYM, $pfD = |pfD| e^{i\alpha}$

Complicates interpretation of $\{ e^{-S_B} \, pfD \}$ as Boltzmann weight

We carry out phase-quenched RHMC, $pfD \rightarrow |pfD|$ 

In principle need to reweight phase-quenched (pq) observables:

\[
\langle O \rangle = \frac{\langle O e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}} \quad \text{with} \quad \langle O e^{i\alpha} \rangle_{pq} = \frac{1}{Z_{pq}} \int [dU][d\overline{U}] O e^{i\alpha} \, e^{-S_B} \, |pfD|
\]

\[\Rightarrow\] Monitor $\langle e^{i\alpha} \rangle_{pq}$ as function of volume, coupling, $N$
Pfaffian phase dependence on volume and coupling

**Left:** \( 1 - \langle \cos(\alpha) \rangle_{pq} \ll 1 \) independent of volume and \( N \) at \( \lambda_{\text{lat}} = 1 \)

**Right:** New \( 4^4 \) results at \( 4 \leq \lambda_{\text{lat}} \leq 8 \) show much larger fluctuations

Currently filling in more volumes and \( N \) for improved action

Extremely expensive analysis despite new parallel algorithm:

\[ O(n^3) \text{ scaling} \rightarrow \sim 50 \text{ hours for single } 4^4 \text{ measurement} \]
Two puzzles posed by the sign problem

- With **periodic temporal boundary conditions** for the fermions, we have an obvious sign problem, $\langle e^{i\alpha} \rangle_{pq}$ consistent with zero.

- With **anti-periodic BCs** and all else the same $e^{i\alpha} \approx 1$, phase reweighting has negligible effect.

Why such sensitivity to the BCs?

Also, other observables are nearly identical for these two ensembles.

Why doesn’t the sign problem affect other observables?
Recapitulation and outlook

Rapid recent progress in lattice supersymmetry
- Lattice promises non-perturbative insights from first principles
- Lattice $\mathcal{N} = 4$ SYM is practical thanks to exact $Q$ susy
- Public code to reduce barriers to entry

Latest results from ongoing calculations
- Static potential is Coulombic at all couplings, $C(\lambda)$ confronted with perturbation theory and AdS/CFT
- Promising initial Konishi anomalous dimension at weak coupling

Many more directions are being — or can be — pursued
- Understanding the (absence of a) sign problem
- Systems with less supersymmetry, in lower dimensions, including matter fields, exhibiting spontaneous susy breaking, ...
Thank you!
Thank you!

Collaborators
Simon Catterall, Poul Damgaard, Tom DeGrand and Joel Giedt

Funding and computing resources
Supplement: \((d - 1)\)-dimensional lattice superQCD

Method to add fundamental matter multiplets without breaking \(Q^2 = 0\)


Consider 2-slice lattice with \(U(N) \times U(F)\) gauge group:
—(Adj, 1) fields on one slice
—(1, Adj) fields on the other
—Bi-fund. \((\square, \square)\) in between

Set \(U(F)\) gauge coupling to zero
\(\rightarrow\) \(U(N)\) in \(d - 1\) dims.

with \(F\) fund. hypermultiplets

(Periodic BC \(\rightarrow\) anti-fund.)

\(U(N_c)\) SYM Adjoint Model
\([\mu, \bar{\mu}, (\eta, \psi_\mu, \chi_{\mu\nu})]\)

\(U(N_F)\) SYM Adjoint Model
\((\phi, \bar{\phi}, \lambda, \lambda_\mu, \lambda_{\mu
\nu})\)

\(\text{Frozen (Non-dynamical)}\)

arXiv:1505.00467
Spontaneous supersymmetry breaking

Can add $Q$-exact moduli space condition (Fayet–Iliopoulos $D$ term),

$$
S \ni \eta \left( \overline{D}_\mu U_\mu + \sum_{i=1}^{F} \phi_i \overline{\phi}_i \right) \longrightarrow \eta \left( \overline{D}_\mu U_\mu + \sum_{i=1}^{F} \phi_i \overline{\phi}_i + r\Pi_N \right)
$$

$$
\langle d \rangle = \langle \sum_{i=1}^{F} \phi_i \overline{\phi}_i + r\Pi_N \rangle \neq 0 \implies \text{spontaneous susy breaking}
$$

Effectively $N \times N$ conditions imposed on $N \times F$ degrees of freedom.

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arXiv:1505.00467
Backup: Failure of Leibnitz rule in discrete space-time

Given that \( \{ Q_\alpha, \overline{Q}_{\dot{\alpha}} \} = 2 \sigma^\mu_{\alpha \dot{\alpha}} P_\mu = 2 i \sigma^\mu_{\alpha \dot{\alpha}} \partial_\mu \) is problematic, why not try \( \{ Q_\alpha, \overline{Q}_{\dot{\alpha}} \} = 2 i \sigma^\mu_{\alpha \dot{\alpha}} \nabla_\mu \) for a discrete translation?

Here \( \nabla_\mu \phi(x) = \frac{1}{a} [\phi(x + a \hat{\mu}) - \phi(x)] = \partial_\mu \phi(x) + \frac{a}{2} \partial_\mu^2 \phi(x) + \mathcal{O}(a^2) \)

**Essential difference between \( \partial_\mu \) and \( \nabla_\mu \) on the lattice, \( a > 0 \)**

\[
\nabla_\mu [\phi(x) \chi(x)] = a^{-1} [\phi(x + a \hat{\mu}) \chi(x + a \hat{\mu}) - \phi(x) \chi(x)] \\
= [\nabla_\mu \phi(x)] \chi(x) + \phi(x) \nabla_\mu \chi(x) + a [\nabla_\mu \phi(x)] \nabla_\mu \chi(x)
\]

We only recover the Leibnitz rule \( \partial_\mu (fg) = (\partial_\mu f)g + f \partial_\mu g \) when \( a \to 0 \)

\( \implies \) “Discrete supersymmetry” breaks down on the lattice

(Dondi & Nicolai, “Lattice Supersymmetry”, 1977)
The Kähler–Dirac representation is related to the spinor \( Q^I_\alpha, \overline{Q}^I_{\dot{\alpha}} \) by

\[
\begin{pmatrix}
Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\
\overline{Q}^1_{\dot{\alpha}} & \overline{Q}^2_{\dot{\alpha}} & \overline{Q}^3_{\dot{\alpha}} & \overline{Q}^4_{\dot{\alpha}}
\end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu \gamma_\mu + \mathcal{Q}_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{\mathcal{Q}}_\mu \gamma_\mu \gamma_5 + \overline{\mathcal{Q}} \gamma_5
\]

\[
\rightarrow \mathcal{Q} + \mathcal{Q}_a \gamma_a + \mathcal{Q}_{ab} \gamma_a \gamma_b
\]

with \( a, b = 1, \ldots, 5 \)

The 4 \times 4 matrix involves R symmetry transformations along each row and (euclidean) Lorentz transformations along each column.

\[\Rightarrow\text{ Kähler–Dirac components transform under “twisted rotation group”}\]

\[\text{SO}(4)_{tw} \equiv \text{diag} \left[ \text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R \right] \]

Only \( \text{SO}(4)_R \subset \text{SO}(6)_R \)
In the code it is very convenient to represent the $A_4^*$ lattice as a hypercube with a backwards diagonal.
Complex gauge field $\Longrightarrow U(N) = SU(N) \otimes U(1)$ gauge invariance

$U(1)$ sector decouples only in continuum limit

$\mathcal{Q} U_a = \psi_a \Longrightarrow$ gauge links must be elements of algebra

Resulting flat directions required by supersymmetric construction

but must be lifted to ensure $U_a = I_N + A_a$ in continuum limit

---

We need to add two deformations to regulate flat directions

$SU(N)$ scalar potential $\propto \mu^2 \sum_a \left( \text{Tr} \left[ U_a \overline{U}_a \right] - N \right)^2$

$U(1)$ plaquette determinant $\sim G \sum_{a < b} (\det P_{ab} - 1)$

Scalar potential **softly** breaks $\mathcal{Q}$ supersymmetry

susy-violating operators vanish as $\mu^2 \rightarrow 0$

Plaquette determinant can be made $\mathcal{Q}$-invariant [arXiv:1505.03135]
Backup: One problem with flat directions

Gauge fields $U_a$ can move far away from continuum form $\mathbb{I}_N + A_a$ if $\mu^2/\lambda_{\text{lat}}$ becomes too small.

Example for $\mu = 0.2$ and $\lambda_{\text{lat}} = 5$ on $8^3 \times 24$ volume

**Left:** Bosonic action is stable $\sim 18\%$ off its supersymmetric value

**Right:** Polyakov loop wanders off to $\sim 10^9$
Backup: Another problem with $U(1)$ flat directions

Can induce monopole condensation $\rightarrow$ transition to confined phase

This lattice phase is not present in continuum $\mathcal{N} = 4$ SYM

Around the same $\lambda_{\text{lat}} \approx 2\ldots$

**Left:** Polyakov loop falls towards zero

**Center:** Plaquette determinant falls towards zero

**Right:** Density of $U(1)$ monopole world lines becomes non-zero
Backup: More on soft susy breaking

Before 2015 we used a more naive constraint on plaquette det.:

\[
S_{\text{soft}} = \frac{N}{2\lambda_{\text{lat}}} \mu^2 \sum_a \left( \frac{1}{N} \text{Tr} [U_a \overline{U}_a] - 1 \right)^2 + \kappa \sum_{a< b} |\text{det} \mathcal{P}_{ab} - 1|^2
\]

Both terms softly break \( Q \) but \( \text{det} \mathcal{P}_{ab} \) effects dominate

**Left:** The bosonic action provides another Ward identity \( \langle s_B \rangle = 9N^2/2 \)

**Right:** Soft susy breaking is also suppressed \( \propto 1/N^2 \)
arXiv:1505.03135 introduces method to impose $Q$-invariant constraints

Basic idea: Modify aux. field equations of motion $\longrightarrow$ moduli space

\[ d(n) = D_a U_a(n) \longrightarrow \overline{D}_a U_a(n) + GO(n)\Pi_N \]

Putting both plaquette determinant and scalar potential in $O(n)$ over-constrains system $\longrightarrow$ sub-optimal Ward identity violations
Backup: Code performance—weak and strong scaling

Results from arXiv:1410.6971 for the pre-2015 ("unimproved") action

**Left:** Strong scaling for $U(2)$ and $U(3)$ $16^3 \times 32$ RHMC

**Right:** Weak scaling for $O(n^3)$ pfaffian calculation (fixed local volume)

$n \equiv 16N^2L^3N_T$ is number of fermion degrees of freedom

Dashed lines are optimal scaling, solid line is power-law fit
Backup: Numerical costs for $N = 2, 3$ and $4$ colors

**Red:** Find RHMC cost scaling $\sim N^5$ — recall adjoint fermion d.o.f. $\propto N^2$

**Blue:** Pfaffian cost scaling consistent with expected $N^6$

Additional factor of $\sim 2 \times$ from new improved action
Backup: Restoration of $Q_a$ and $Q_{ab}$ supersymmetries

Results from arXiv:1411.0166 to be revisited with the new action...

$Q_a$ and $Q_{ab}$ from restoration of R symmetry (motivation for $A_4^*$ lattice)

Modified Wilson loops test R symmetries at non-zero lattice spacing

Parameter $c_2$ may need log. tuning in continuum limit
Backup: $\mathcal{N} = 4$ SYM static potential from Wilson loops

Extract static potential $V(r)$ from $r \times T$ Wilson loops

$$W(r, T) \propto e^{-V(r)} T$$

$$V(r) = A - C/r + \sigma r$$

Coulomb gauge trick from lattice QCD reduces $A_4^*$ lattice complications
For range of couplings currently being studied, (continuum) perturbation theory for $C(\lambda)$ is well behaved.
Backup: More tests of the $U(2)$ static potential

**Left:** Projecting Wilson loops from $U(2) \rightarrow SU(2)$

$$\implies \text{factor of } \frac{N^2-1}{N^2} = \frac{3}{4}$$

**Right:** Unitarizing links removes scalars $\implies \text{factor of } \frac{1}{2}$

Some results slightly above expected factors, may be related to non-zero auxiliary couplings $\mu$ and $\kappa / G$
Backup: More tests of the U(3) static potential

**Left:** Projecting Wilson loops from $U(3) \rightarrow SU(3)$

$$\Rightarrow \text{factor of } \frac{N^2-1}{N^2} = \frac{8}{9}$$

**Right:** Unitarizing links removes scalars $\Rightarrow$ factor of $1/2$

Some results slightly above expected factors, may be related to non-zero auxiliary couplings $\mu$ and $\kappa/G$
Backup: Smearing for Konishi analyses

As in glueball analyses, smearing enlarges operator basis

Using APE-like smearing: $$(1 - \alpha) - + \frac{\alpha}{g} \sum \cap,$$

with staples built from unitary parts of links but no final unitarization

(unitarized smearing — e.g. stout — doesn’t affect Konishi)

Average plaquette is stable upon smearing (right)
while minimum plaquette steadily increases (left)